THE ABHYANKAR-MOH THEOREM FOR PLANE VALUATIONS AT INFINITY

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Abstract. We introduce the class of plane valuations at infinity and prove an analogue to the Abhyankar-Moh (semigroup) Theorem for it.

1. Introduction

Between 1940 and 1960, Zariski and Abhyankar developed the theory of valuations in the context of the theory of singularities with the aim of proving resolution for algebraic schemes. Some important references are [21, 22, 23, 1, 2]. In the last years, there has been a resurgence of interest in valuations within this context. Valuations of fields of quotients of (Noetherian) regular two-dimensional local rings (R, m) centered at R, which are named plane valuations, are one of the best known classes of valuations. These valuations were classified by Zariski and a refinement of that classification in terms of the dual graphs associated with the valuations can be seen in [20] (see also [15]). This shows that plane valuations can be classified in a similar way to analytically irreducible plane curve singularities.

A particular type among these singularities is that of the singularities at infinity of projective plane curves with only one place at infinity. An important result that concerns this type of curves is the Abhyankar-Moh (semigroup) Theorem [4, 3, 5]. It proves, under certain condition on the characteristic of the ground field, the existence of a finite set of positive integers satisfying certain properties that generates the so-called semigroup at infinity and that we call a δ-sequence in $\mathbb{N}_{>0}$. A converse result is also true: given a δ-sequence in $\mathbb{N}_{>0}$, Δ, there exists a curve with only one place at infinity whose semigroup at infinity is spanned by Δ. One can see a proof in [19] for the complex field and, for any field, it can be deduced from [6] (see [14] and references therein).

The proximity between plane valuations and curve singularities and the fact that the singularity at infinity of a curve with only one place at infinity can be approached by approximates (see Definition 2.3) suggest the possibility of defining *plane valuations at infinity* (Definition 3.4), suitable (generalized) *δ-sequences* and to prove for them an analogue to the Abhyankar-Moh Theorem and its converse. We only assume that the field $k := R/m$ is perfect. If one restricts to valuations where k is the field $\mathbb{C}$ of complex numbers and the value group is included in the set of real numbers, then plane valuations at infinity form part of the so-called valuations *centered* at infinity introduced in [10] to study the dynamics of polynomial mappings from $\mathbb{C}^2$ to $\mathbb{C}^2$ (see also [11]).

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Coding Theory is an applied matter for which the theory of plane curves with only one place at infinity is useful (see [7] for instance). Also, good codes can be obtained if concepts as plane valuations at infinity and attached δ-sequences are used. This was done by the authors in [14]; however the nature of that paper (focused on Coding Theory) did not allow us to completely show the extension to plane valuations of the above mentioned results concerning plane curves with only one place at infinity. In fact, we only proved the converse of the Abhyankar-Moh Theorem for those types of valuations that were useful for our purposes. This paper is devoted to develop the mentioned suggestion in a self-contained manner, proving an analogue to the Abhyankar-Moh Theorem for plane valuations at infinity as they were defined in [14] and completing the details, concerning the definition of δ-sequence and the converse of the mentioned theorem, that were not given in [14].

In Section 2, we recall the concepts of plane curve with only one place at infinity C, δ-sequence in $\mathbb{N}_{>0}$ and family of approximates for curves as C. We also state the Abhyankar-Moh Theorem and show the existing relation between δ-sequences in $\mathbb{N}_{>0}$ and maximal contact values of the singularity at infinity of curves C as above, fact that will be useful in the paper. Section 3 summarizes some basic results on plane valuations, introduces the class of plane valuations which we are interested in, plane valuations at infinity, and describes the value semigroup of these valuations according their type. We normalize these semigroups to adapt them to the concept of δ-sequence for our class of valuations, concept that is provided in Section 4. In this section, we prove our main result, Theorem 4.1, that we call Abhyankar-Moh Theorem for plane valuations at infinity, whose statement goes parallel with the classical one. We also give in Remark 4.2 a nice geometrical interpretation, for divisorial valuations at infinity, of the well-ordering of the semigroup spanned by their associated δ-sequences.

2. Curves having only one place at infinity and approximates

Along this paper, k will be a perfect field and $\mathbb{P}^2_k$ (or $\mathbb{P}^2$ for short) will stand for the projective plane over k.

Definition 2.1. Let L be the line at infinity in the compactification of the affine plane to $\mathbb{P}^2$. Let C be a projective absolutely irreducible curve of $\mathbb{P}^2$ (i.e., irreducible as a curve in $\mathbb{P}^2_{\overline{k}}$, $\overline{k}$ being the algebraic closure of k). We will say that C *has only one place at infinity* if the intersection $C \cap L$ is a single point p (the one at infinity) and C has only one analytic branch at p (notice that the branch is defined over k because k is perfect).

Let C be a curve with only one place at infinity. Denote by K the quotient field of the local ring $\mathcal{O}_{C,p}$; the germ of C at p defines a discrete valuation on K, that we denote by $\nu_{C,p}$, which allows us to state the following

Definition 2.2. Let C be a curve with only one place at infinity given by p. The *semigroup at infinity* of C is the following sub-semigroup of the set of non-negative integers $\mathbb{N}$

$$S_{C,\infty} := \{-\nu_{C,p}(h) | h \in T\},$$

where T is the k-algebra $\mathcal{O}_C(C \setminus \{p\})$.

Given a curve C with only one place at infinity, consider the sequence of point blowing-ups

$$X_n \to \cdots \to X_1 \to X_0 := \mathbb{P}^2 \tag{1}$$

that provides the minimal embedded resolution of the singularity of C at infinity. Recall that the dual graph Γ associated with this singularity is a tree with n vertices such that each vertex represents the strict transform in X_n of one of the irreducible exceptional divisor appearing in the above sequence of blowing-ups and two vertices are joined by an edge whenever the corresponding divisors intersect; additionally each vertex is labelled with the number of blowing-ups needed to create its corresponding exceptional divisor. The dual graph has the shape depicted in Figure 1. It is the union of g subgraphs Γ_i, where g is the number of characteristic pairs of the singularity.

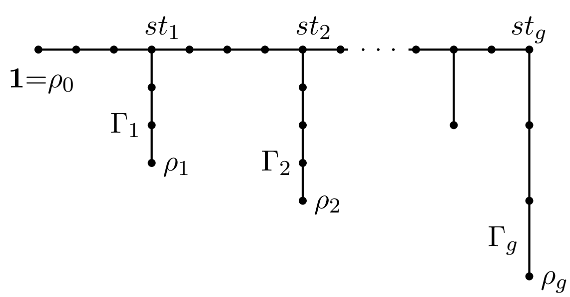

FIGURE 1. The dual graph

Let E_{s_i} $(1 \leq i \leq g)$ be the exceptional divisor obtained after blowing-up the last free point (see the end of Section 3.1) corresponding to the subgraph Γ_i. It corresponds to the vertex ρ_i in the dual graph. An analytically irreducible germ of curve at p, ψ, is said to be an *i-curvette* of the germ of C at p if the strict transform of ψ in the surface that contains E_{s_i} is not singular and meets transversely E_{s_i} and no other exceptional curves.

From now on, for convenience, we fix homogeneous coordinates $(X : Y : Z)$ on $\mathbb{P}^2$. $Z = 0$ will be the line at infinity, $p = (1 : 0 : 0)$ and all the singularities "at infinity" we will consider will be located at p. Set (x, y) coordinates in the affine chart $Z \neq 0$ and $(u = y/x, v = 1/x)$ coordinates around p. Note that for each polynomial $g(x, y) \in k[x, y]$, the following equality holds:

$$g(x, y) = v^{-\deg(g)}\overline{g}(u, v), \tag{2}$$

where $\overline{g}(u, v)$ is the expression of $g(x, y)$ in the local coordinates (u, v). Moreover, if q is either in $k[x, y]$ or in $k[u, v]$, in this paper by an abuse of notation, $\nu_{C,p}(q)$ will mean $\nu_{C,p}(q')$, where q' is the element in the fraction field of $\mathcal{O}_{C,p}$ that q defines.

A useful concept will be given in the following definition.

Definition 2.3. Let C be a curve with only one place at infinity defined by a polynomial in $k[x, y]$ of the form:

$$f(x, y) = y^m + a_1(x)y^{m-1} + \cdots + a_m(x),$$

where m is the total degree of f, $a_i(x) \in k[x]$ $(1 \leq i \leq m)$ and $\mathrm{char}(k)$ does not divide m. Let g be the number of characteristic pairs of the singularity of C at p. A sequence of polynomials in $k[x,y]$, $\{q_0(x,y), q_1(x,y), \ldots, q_g(x,y)\}$, will be named a *family of approximates* for the curve C whenever the following conditions happen:

- **a:** $q_0(x,y) = x$, $q_1(x,y) = y$, $\delta_0 := -\nu_{C,p}(q_0)$ and $\delta_1 := -\nu_{C,p}(q_1)$.
- **b:** $q_i(x,y)$ $(1 < i \leq g)$ is a monic polynomial in the indeterminate y and $\deg(q_i) = \deg_y(q_i) = \delta_0/d_i$, where $d_i = \gcd(\delta_0, \delta_1, \ldots, \delta_{i-1})$, being $\delta_i := -\nu_{C,p}(q_i)$.
- **c:** If $q_i(x,y) = v^{-\deg(q_i)}\bar{q}_i(u,v)$ (see (2)), then the germ of curve at p defined by $\bar{q}_i(u,v)$ (for $1 < i \leq g$) is an i-curvette (respectively, $(i-1)$-curvette) of the germ of C at p when $\delta_0 - \delta_1$ does not divide (respectively, divides) δ_0.

The following result is usually known as Abhyankar-Moh (semigroup) Theorem. (See [4, 3, 5, 18, 17] as references).

Theorem 2.1. *Let C be a curve having only one place at infinity and assume that* $\mathrm{char}(k)$ *does not divide* $\gcd(-\nu_{C,p}(x), -\nu_{C,p}(y))$. *Then, there exists a finite sequence of positive integers $\Delta = \{\delta_0, \delta_1, \ldots, \delta_g\}$ that generates the semigroup $S_{C,\infty}$ and satisfies the following conditions:*

1. *If $d_i = \gcd(\delta_0, \delta_1, \ldots, \delta_{i-1})$, for $1 \leq i \leq g+1$, and $n_i = d_i/d_{i+1}$, $1 \leq i \leq g$, then $d_{g+1} = 1$ and $n_i > 1$ for $1 \leq i \leq g$.*
2. *For $1 \leq i \leq g$, $n_i\delta_i$ belongs to the subsemigroup of the non-negative integers, $\mathbb{N}$, generated by $\delta_0, \delta_1, \ldots, \delta_{i-1}$, that we usually denote $\langle \delta_0, \delta_1, \ldots, \delta_{i-1}\rangle$.*
3. *$\delta_0 > \delta_1$ and $\delta_i < \delta_{i-1}n_{i-1}$ for $i = 2, 3, \ldots, g$.*

Throughout this paper, a sequence $\Delta = \{\delta_0, \delta_1, \ldots, \delta_g\}$ of positive integers that satisfies the conditions (1), (2) and (3) of Theorem 2.1 will be named a *δ-sequence in* $\mathbb{N}_{>0}$, where $\mathbb{N}_{>0}$ denotes the set of strictly positive integers.

Remark 2.1. The proof of Theorem 2.1 is based on the fact that, under the assumptions of Definition 2.3 on the polynomial $f(x,y) \in k[x,y]$ defining C, a specific family of approximates for the curve (called *approximate roots*) can be obtained. We point out here that, given an equation $f(x,y) = 0$ for the curve C, the mentioned assumptions on f can be attained by means of a change of variables, obtaining an isomorphic curve C' satisfying all the requirements. Then, the generators of $S_{C',\infty} = S_{C,\infty}$ mentioned in the statement of Theorem 2.1 are just the values $\delta_0, \ldots, \delta_g$ associated with the family of approximates given by the approximate roots of C' (see Definition 2.3). Notice also that δ_0 coincides with $\deg(C')$.

From Equality (2), Remark 2.1 and the properties that define the families of approximates, it is easy to deduce the following result, which shows how the maximal contact values of the singularity at infinity of a curve C as in Theorem 2.1 (see [6, Ch. 4], for instance) can be expressed in terms of a δ-sequence in $\mathbb{N}_{>0}$ attached to the semigroup $S_{C,\infty}$.

Proposition 2.1. *Let C be a curve satisfying the hypotheses of Theorem 2.1 and let $\{\bar{\beta}_i\}_{i=0}^{g}$ be the set of maximal contact values of the germ of C at p. Then, there exists a δ-sequence in $\mathbb{N}_{>0}$, $\{\delta_i\}_{i=0}^{s}$, generating the semigroup $S_{C,\infty}$, such that if $\nu_{C,p}(u)$ does not* *divide (respectively, divides) $\nu_{C,p}(v)$, then $s = g$, $\bar{\beta}_0 = \delta_0 - \delta_1$, $\bar{\beta}_1 = \delta_0$ and $\bar{\beta}_i = \frac{(\delta_0)^2}{d_i} - \delta_i$ for $1 < i \leq g$ (respectively, $s = g+1$, $\bar{\beta}_0 = \delta_0 - \delta_1$, and $\bar{\beta}_{i-1} = \frac{(\delta_0)^2}{d_i} - \delta_i$ for $1 < i \leq g+1$).*

The following result relates δ-sequences in $\mathbb{N}_{>0}$ corresponding to two curves with only one place at infinity such that the minimal embedded resolution of one of them is a resolution of the other one.

Corollary 2.1. *Let C and C' be two curves satisfying the hypotheses of Theorem 2.1 and such that the sequence of morphisms (1) that provides a minimal embedded resolution of the singularity of C at p is also an embedded resolution of the one of C'. Let $\{\bar{\beta}_i\}_{i=0}^{g}$ (respectively, $\{\bar{\beta}'_i\}_{i=0}^{g'}\}$) be the sequence of maximal contact values of the germ of C (respectively, C') at p. If $\Delta = \{\delta_0, \ldots, \delta_s\}$ and $\Delta' = \{\delta'_0, \ldots, \delta'_{s'}\}$ are δ-sequences in $\mathbb{N}_{>0}$ associated, respectively, with the curves C and C' and satisfying the equalities given in Proposition 2.1, then $s' \leq s$ and $\delta_i/\delta'_i = \bar{\beta}_0/\bar{\beta}'_0$ for $1 \leq i \leq s'-1$.*

3. Plane valuations at infinity

3.1. **Generalities on plane valuations.** To begin with, we recall the general concept of *valuation*.

Definition 3.1. A *valuation* of a field K is a mapping

$$\nu : K^*(:= K \setminus \{0\}) \to G,$$

where G is a totally ordered group, such that it satisfies

- $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$ and
- $\nu(fg) = \nu(f) + \nu(g)$,

f, g being elements in K^*. The subring of K, $R_\nu := \{f \in K^* | \nu(f) \geq 0\} \cup \{0\}$, is called the *valuation ring* of ν. R_ν is a local ring whose maximal ideal is $m_\nu := \{f \in K^* | \nu(f) > 0\} \cup \{0\}$.

Given a local regular domain (R, m), we will say that a valuation ν of the quotient field of R *is centered* at R if $R \subseteq R_\nu$ and $R \cap m_\nu = m$. The subset of G given by $S_\nu := \nu(R \setminus \{0\})$ is called *semigroup* of the valuation ν (relative to R). In the rest of the paper we will only consider *plane valuations*, that is, valuations of the quotient field of a local regular domain (R, m) of dimension two which are centered at R. Assume for a while that the field $k := R/m$ is algebraically closed. In this case, plane valuations are the algebraic version of a sequence of point blowing-ups (see [20, 15] for details). In fact, attached to a plane valuation ν, there is a unique sequence of point blowing-ups

$$\cdots \longrightarrow X_{N+1} \xrightarrow{\pi_{N+1}} X_N \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = \operatorname{Spec} R, \tag{3}$$

where π_1 is the blowing-up of X_0 centered at its closed point p_0 and, for each $i \geq 1$, π_{i+1} is the blowing-up of X_i at the unique closed point p_i of the exceptional divisor E_i (obtained after the blowing-up π_i) satisfying that ν is centered at the local ring $\mathcal{O}_{X_i,p_i}$ $(:= R_i)$. Conversely, each sequence as in (3) provides a unique plane valuation. We will denote by $\mathcal{C}_\nu = \{p_i\}_{i \geq 0}$ the sequence (finite or infinite) of closed points involved in the blowing-ups of (3). When $\mathcal{C}_\nu$ is finite, ν is called the *divisorial valuation* corresponding to the last exceptional divisor obtained in (3); this is so since if π_{N+1} is the last blowing-up in the sequence (3) given by ν, then ν is the m_N-adic valuation, m_N being the maximal ideal of the ring R_N. Otherwise (when $\mathcal{C}_\nu$ is not finite), the plane valuation ν can be regarded as the limit of the sequence of divisorial valuations $\{\nu_i\}_{i\geq 0}$, ν_i being the divisorial valuation corresponding to the divisor E_i.

With the above notation, let p_i and p_j be points in $\mathcal{C}_\nu = \{p_i\}_{i\geq 0}$. We will say that p_i *is proximate* to p_j (and it will be denoted by $p_i \rightarrow p_j$) if $i > j$ and p_i belongs to the strict transform (by the corresponding sequence of blowing-ups given in (3)) of E_{j+1}. This binary relation among the points of $\mathcal{C}_\nu$ will be called *proximity relation* and it induces a binary relation $\mathcal{P}_\nu$ in the set of natural numbers ($i \rightarrow j$ if $p_i \rightarrow p_j$). Also, the point p_i is said to be *satellite* if there exists $j < i-1$ such that $p_i \rightarrow p_j$ (in other words, if p_i belongs to the intersection of the strict transforms of two exceptional divisors); otherwise, p_i is said to be a *free* point. It is worth pointing out that the semigroup S_ν of a plane valuation depends only on the relation $\mathcal{P}_\nu$. According with this relation, a plane valuation ν (with associated sequence $\mathcal{C}_\nu = \{p_i\}_{i\geq 0}$) belongs to one of the following five types (see [20] and [13]):

– **TYPE A**: if $\mathcal{C}_\nu$ is finite.
– **TYPE B**: if there exists $i_0 \in \mathbb{N}_{>0}$ such that the point p_i is free for all $i > i_0$.
– **TYPE C**: if there exists $i_0 \in \mathbb{N}_{>0}$ such that $p_i \rightarrow p_{i_0}$ for all $i > i_0$.
– **TYPE D**: if there exists $i_0 \in \mathbb{N}_{>0}$ such that p_i is a satellite point for all $i > i_0$ but ν is not a type C valuation. This means that the sequence (3) ends with infinitely many blowing-ups at satellite points, but they are not ever centered at some point of the strict transforms of the same divisor.
– **TYPE E**: if the sequence $\mathcal{C}_\nu$ alternates indefinitely blocks of free and satellite points.

In [9], valuations of types B, C, D and E are named, respectively, curve, exceptional curve, irrational and infinitely singular valuations.

3.2. **Valuations at infinity.** Now, we are going to define the particular type of plane valuations that this paper deals with: *plane valuations at infinity*. In the rest of the paper *we will not assume that the ground field k is algebraically closed*. It is important to notice that, in spite of this, the procedure and concepts above explained will work similarly because, due to the especial nature of the valuations that we will consider, the centers of the associated blowing-ups will be defined over k. We recall that the field k is perfect.

We start by stating the concept of general element of a divisorial valuation.

Definition 3.2. Let ν be a divisorial valuation. An element f in the maximal ideal of R is named to be a *general element of* ν if the germ of curve given by f is analytically irreducible, its strict transform in the last variety obtained by the sequence (3) attached to ν, X_{N+1}, is smooth and meets E_{N+1} transversely at a non-singular point of the exceptional divisor of the sequence (3).

Remark 3.1. General elements are useful to compute plane divisorial valuations. Indeed, if $f \in R$, then

$$\nu(f) = \min \{(f,g) | g \text{ is a general element of } \nu\},$$

where (f, g) stands for the intersection multiplicity of the germs of curve given by f and g. The above minimum is attained when the strict transforms of the germs defined by f and g do not meet E_{N+1} at the same point.

Remark 3.2. Assume that $\{u, v\}$ is a regular system of parameters of the ring R. Attached to a divisorial valuation ν and with respect to the system $\{u, v\}$, there exists an expression, named Hamburger-Noether expansion of ν, that provides regular system of parameters for the rings R_i determined by ν. From this Hamburger-Noether expansion, it can be obtained elements $u(t, s), v(t, s)$ in the formal power series ring in two indeterminates $k[[t, s]]$ such that, for any $f \in R$, $\nu(f) = \mathrm{ord}_t f\,(u(t, s), v(t, s))$ (see, [12, Sect. 3.1]). Moreover, if $\bar{q}_i := \bar{q}_i(u, v)$, $1 \leq i \leq g$, is a i-curvette of any general element of ν, then $\bar{q}_i\,((u(t, s), v(t, s)) = \lambda_i s^{a_i} t^{\nu(\bar{q}_i)} + r_i$, where $0 \neq \lambda_i \in k$, $a_i \geq 0$ and r_i is an element in $k[[t, s]]$ with exponents of t larger than $\nu(\bar{q}_i)$. Even more, for general elements of ν, $\bar{q}_{g+1} := \bar{q}_{g+1}(u, v)$, it happens that $\bar{q}_{g+1}\,((u(t, s), v(t, s)) = p(s) s^a t^{\nu(\bar{q}_{g+1})} + r_{g+1}$, $a \geq 0$ and, here, $p(s) \in k[s] \setminus k$, $p(0) \neq 0$ and the exponents of t in r_{g+1} are always larger than $\nu(\bar{q}_{g+1})$. The mentioned non-negative integer values a_i and a vanish when the defining divisor of ν is given by a free point. As a reference one can see the proof of [12, Th. 1]. There, it is considered the free case; the behavior in the non-free case was pointed out to the first author by A. Nuñez when, together with F. Delgado, they were carrying out a joint work.

Let p be a closed point of $\mathbb{P}^2$ on the line of infinity and assume, from now on, that $R = \mathcal{O}_{\mathbb{P}^2,p}$ and K is the quotient field of R.

Definition 3.3. A *plane divisorial valuation at infinity* is a plane divisorial valuation of K centered at R that admits, as a general element, an element in R providing the germ at p of some curve with only one place at infinity (p being its point at infinity).

Definition 3.4. A plane valuation ν of K centered at R is said to be *at infinity* whenever it is a limit of plane divisorial valuations at infinity. More explicitly, ν will be at infinity if there exists a sequence of divisorial valuations at infinity $\{\nu_i\}_{i=1}^{\infty}$ such that $\mathcal{C}_{\nu_i} \subseteq \mathcal{C}_{\nu_{i+1}}$ for all $i \in \mathbb{N}_{>0}$ and $\mathcal{C}_\nu = \bigcup_{i \geq 1} \mathcal{C}_{\nu_i}$.

There exist plane valuations at infinity of all types above described. The concept of valuation at infinity of type A is equivalent to the one of plane divisorial valuation at infinity; such a valuation is obtained whenever the sequence $\{\nu_i\}_{i=1}^{\infty}$ given in the above definition satisfies that $\nu_i = \nu_{i+1}$ for every index larger than or equal to a fixed index $i_0 \in \mathbb{N}_{>0}$ (in fact, it can be taken constant for all i). It is obtained a valuation at infinity of type B if there exists $i_0 \in \mathbb{N}_{>0}$ such that $\mathcal{C}_{\nu_{i_0}}$ is the set of centers of the blowing-ups corresponding with the minimal embedded resolution of the germ at p of a curve having only one place at infinity and, for all $i \geq i_0$, the strict transform of this germ meets transversely the exceptional divisor associated with ν_i. Explicit constructions of plane valuations at infinity of types C, D, and E are described in [14].

The semigroup S_ν of a *valuation of type A* is, up to equivalence of valuations, minimally generated by the maximal contact values of the germ at p defined by a general element of ν, $\{\bar{\beta}_i\}_{i=0}^{g}$; g is the number of characteristic pairs of a general element of ν (see Sections 6 and 8 of [20]). These maximal contact values can be computed by evaluating by ν the

$g+1$ first elements of any *generating sequence* $\{\mathfrak{q}_i\}_{i=0}^{g+1}$ of ν [20]. Each element $\mathfrak{q}_i \in R$ $(0 \leq i \leq g+1)$ defines an analytically irreducible germ of curve ψ_i at p, $\mathfrak{q}_{g+1}$ is a general element of ν, and, for each $i \leq g$, ψ_i is an i-curvette of ψ_{g+1}. Notice that, due to Remark 3.1, S_ν coincides with the semigroup of values of ψ_{g+1}. Usually, the elements in the set $\{\nu(\mathfrak{q}_i)\}_{i=0}^{g+1}$ are named the maximal contact values of the valuation ν.

Plane valuations at infinity ν of type B correspond to valuations at infinity in the *Case 4.1.a* of [20, Sect. 9]. Here we can identify the group G with $\mathbb{Z}^2$ (with the lexicographical ordering) and, with this identification, S_ν is minimally generated by the set $\{\bar{\beta}_0 := (0, \bar{\beta}'_0), \bar{\beta}_1 := (0, \bar{\beta}'_1), \ldots, \bar{\beta}_g := (0, \bar{\beta}'_g), \bar{\beta}_{g+1} := (1, a)\}$, with $a \in \mathbb{Z}$ (notice that $(1, a)$ is the minimum element of S_ν with non-zero first coordinate), where $\{\bar{\beta}'_i\}_{i=0}^{g+1}$ are the maximal contact values associated with the divisorial valuation corresponding to the last satellite point of $\mathcal{C}_\nu$ [8, 1.10]. Since we have freedom to choose the value of a, we will take, for convention, $a = 0$.

Plane valuations at infinity ν of type C correspond to valuations at infinity in the *Case 3* of [20, Sect. 9]. The semigroup S_ν can be identified with a sub-semigroup of $\mathbb{Z}^2$ (with the lexicographical ordering) whose minimal set of generators $\{\bar{\beta}_i\}_{i=0}^{g}$ can be represented, as in the previous case, in a particular form and with some degree of freedom (see [20, Sect. 9] and also [8, 1.10]). We fix, for convention, the computation of these generators in the way that we describe below. Let i_0 be the maximum among the integers j corresponding to points $p_j \in \mathcal{C}_\nu$ that admit more than one point proximate to it (that is, p_{i_0} has infinitely many proximate points in $\mathcal{C}_\nu$). Since a valuation is determined by its minimum values at the maximal ideals m_j of the local rings $R_j = \mathcal{O}_{X_j, p_j}$ (denoted by $\nu(m_j)$), we set $\nu(m_j) := (0,1)$ for all $j > i_0$, $\nu(m_{i_0}) := (1,0)$ and $\nu(m_j) := \sum_{k \to j} \nu(m_k)$ for $j < i_0$. Now, given a generating sequence $\{\mathfrak{q}_i\}_{i=0}^{g+1}$ of the divisorial valuation associated to whichever divisor E_j with $j > i_0 + 2$, we have that $\bar{\beta}_i = \nu(\mathfrak{q}_i) = \sum_{p_k \in \mathcal{C}_\nu} e_k(\mathfrak{q}_i) \nu(m_k)$, $0 \leq i \leq g$, where $e_k(\mathfrak{q}_i)$ denotes the multiplicity of the strict transform of the germ given by $\mathfrak{q}_i$ at the point p_k and the product $e_k(\mathfrak{q}_i)\nu(m_k)$ is the usual one between elements in $\mathbb{Z}$ and $\mathbb{Z}^2$.

Plane valuations at infinity ν of type D correspond to valuations at infinity in the *Case 2* of [20, Sect. 9]. The semigroup S_ν can be identified with a sub-semigroup of $\mathbb{R}$. If $\{\nu_i\}_{i=1}^{\infty}$ is a sequence of divisorial valuations at infinity defining ν, then there exists $i_0 \in \mathbb{N}$ such that, for all $i \geq i_0$, the sets of minimal generators $\{\bar{\beta}^i_j\}_{j=0}^{g}$ of S_{ν_i} have the same cardinality $g+1$. For our future development, it is convenient to normalize the valuations ν_i by dividing by $\nu_i(v) - \bar{\beta}^i_0$. If $g \geq 2$, set

$$\frac{\bar{\beta}^i_j}{\nu_i(v) - \bar{\beta}^i_0} = \frac{\bar{\beta}^k_j}{\nu_k(v) - \bar{\beta}^k_0} =: \bar{\beta}_j \in \mathbb{Q} \tag{4}$$

for $j < g$ and $i, k > i_0$, and $\bar{\beta}_g := \lim_{i\to\infty} \frac{\bar{\beta}^i_g}{\nu_i(v) - \bar{\beta}^i_0}$ (that belongs to $\mathbb{R} \setminus \mathbb{Q}$). Otherwise, we must set $\bar{\beta}_j := \lim_{i\to\infty} \frac{\bar{\beta}^i_j}{\nu_i(v) - \bar{\beta}^i_j} \in \mathbb{R} \setminus \mathbb{Q}$ $(j = 0, 1)$. Then the semigroup S_ν is generated by $\{\bar{\beta}_i\}_{i=0}^{g}$.

Notice that the value semigroup of this type of valuations is spanned by a set of finitely many positive rational numbers plus a positive real non-rational number, fact that also happens when $g = 1$ because one can consider the isomorphic normalized semigroup generated by 1 and $\bar{\beta}_1/\bar{\beta}_0$.

Plane valuations at infinity of type E correspond to valuations at infinity in *Case 1* of [20, Sect. 9]. The semigroup S_ν can be identified with a sub-semigroup of $\mathbb{Q}$. If $\{\nu_i\}_{i=1}^{\infty}$ is a sequence of divisorial valuations at infinity defining ν, then S_ν is generated by an infinite set $\Gamma := \{\bar{\beta}_0, \bar{\beta}_1, \bar{\beta}_2, \ldots\}$. We can pick a suitable infinite sub-sequence $\{\nu_{i_j}\}_{j=1}^{\infty}$ of $\{\nu_i\}_{i=1}^{\infty}$ that also converges to ν and satisfies that for each $j \geq 1$, there exists a subset of Γ, $\{\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_g\}$, with $g = g(j)$ and $g(j) > g(j')$ if $j > j'$, such that $\bar{\beta}_l = \frac{\bar{\beta}'_l}{\nu_{i_j}(v) - \bar{\beta}'_0}$, $0 \leq l \leq g$, $\{\bar{\beta}'_l\}_{l=0}^{g}$ being the minimal set of generators of $S_{\nu_{i_j}}$ (notice that, for different indices j, the mentioned quotients must coincide for the common indices l and that this fact allows us to normalize the valuations ν_{i_j} by dividing by $\nu_{i_j}(v) - \bar{\beta}'_0$). For simplicity's sake we will only consider sequences as $\{\nu_{i_j}\}_{j=1}^{\infty}$ to define this type of valuations.

Remark 3.3. In the next section, we will define the concept the δ-sequence, which is suitable to treat the generation of semigroups at infinity of valuations at infinity. For this purpose, we will need, in some cases, to normalize δ-sequences in $\mathbb{N}_{>0}$ and also their corresponding maximal contact values. This fact explains why we have normalized the above sets of maximal contact values in the described manner.

4. Semigroup at infinity and δ-sequences.

Firstly, we introduce the concept of semigroup at infinity of a plane valuation at infinity.

Definition 4.1. Let $\nu : K^* \to G$ be a plane valuation at infinity. The *semigroup at infinity* of ν is defined to be the following sub-semigroup of G:

$$S_{\nu,\infty} := \{-\nu(f) \mid f \in k[x, y] \setminus \{0\}\}.$$

Our objective is to prove an analogue to the Abhyankar-Moh Theorem for semigroups at infinity as above. The natural first step is to consider, in the new stage, the analogue of the δ-sequences in $\mathbb{N}_{>0}$ that appear in the Abhyankar-Moh Theorem (we will call them simply *δ-sequences*).

As a previous concept, a *normalized δ-sequence in* $\mathbb{N}_{>0}$ will be an ordered finite set of rational numbers $\overline{\Delta} = \{\overline{\delta}_0, \overline{\delta}_1, \ldots, \overline{\delta}_g\}$ such that there is a δ-sequence in $\mathbb{N}_{>0}$, $\Delta = \{\delta_0, \delta_1, \ldots, \delta_g\}$, satisfying $\overline{\delta}_i = \delta_i/\delta_1$ for $0 \leq i \leq g$. Also, for any δ-sequence in $\mathbb{N}_{>0}$, $\Delta = \{\delta_0, \delta_1, \ldots, \delta_g\}$, we define the following associated positive integers: if $\delta_0 - \delta_1$ does not divide δ_0, then

$$e_0 := \delta_0 - \delta_1, \quad e_i := d_{i+1}$$
$$m_0 := \delta_0, \quad m_i := n_i\delta_i - \delta_{i+1}$$

for $1 \leq i \leq g - 1$. Otherwise,

$$e_0 := d_2 = \delta_0 - \delta_1, \quad e_i := d_{i+2}$$
$$m_0 := \delta_0 + n_1\delta_1 - \delta_2, \quad m_i := n_{i+1}\delta_{i+1} - \delta_{i+2}$$

for $1 \leq i \leq g - 2$.

Now, we are going to give the definition of δ-sequence for the different types of plane valuations at infinity. This definition will be followed by some examples and an explanatory remark.

Definition 4.2. A *δ-sequence of* **TYPE A** (respectively, **B**, **C**, **D**, **E**) is a sequence $\Delta = \{\delta_0, \delta_1, \ldots, \delta_i, \ldots\}$ of elements in $\mathbb{Z}$ (respectively, $\mathbb{Z}^2$, $\mathbb{Z}^2$, $\mathbb{R}$, $\mathbb{Q}$) such that

TYPE A: $\Delta = \{\delta_0, \delta_1, \ldots, \delta_g, \delta_{g+1}\} \subset \mathbb{Z}$ is finite, the elements of the set $\{\delta_0, \ldots, \delta_g\}$ satisfy the conditions (1), (2) and (3) of the Abhyankar-Moh Theorem and $\delta_{g+1} \leq n_g\delta_g$.

TYPE B: There exists a δ-sequence in $\mathbb{N}_{>0}$, $\Delta^* = \{\delta_0^*, \delta_1^*, \ldots, \delta_g^*\}$, such that $\Delta = \{(0, \delta_0^*), (0, \delta_1^*), \ldots, (0, \delta_g^*), (-1, (\delta_0^*)^2)\}$.

TYPE C: $\Delta = \{\delta_0, \delta_1, \ldots, \delta_g\} \subset \mathbb{Z}^2$ is finite, $g \geq 2$ (respectively, ≥ 3) and there exists a δ-sequence in $\mathbb{N}_{>0}$, $\Delta^* = \{\delta_0^*, \delta_1^*, \ldots, \delta_g^*\}$, such that $\delta_0^* - \delta_1^*$ does not divide (respectively, divides) δ_0^* and

$$\delta_i = \frac{\delta_i^*}{Aa_t + B}(A, B) \quad (0 \leq i \leq g-1) \quad \text{and}$$

$$\delta_g = \frac{\delta_g^* + A'a_t + B'}{Aa_t + B}(A, B) - (A', B'),$$

where $\langle a_1; a_2, \ldots, a_t\rangle$, $a_t \geq 2$, is the continued fraction expansion of the quotient m_{g-1}/e_{g-1} (respectively, m_{g-2}/e_{g-2}) given by Δ^* and, considering the finite recurrence relation $\underline{y}_i = a_{t-i}\underline{y}_{i-1} + \underline{y}_{i-2}$, $\underline{y}_{-1} = (0, 1)$, $\underline{y}_0 = (1, 0)$, then $(A, B) := \underline{y}_{t-2}$ and $(A', B') := \underline{y}_{t-3}$. We complete this definition by adding that $\Delta = \{\delta_0, \delta_1\}$ (respectively, $\Delta = \{\delta_0, \delta_1, \delta_2\}$) is a δ-sequence of type C whenever $\delta_0 = \underline{y}_{t-1}$ and $\delta_0 - \delta_1 = \underline{y}_{t-2}$ (respectively, $\delta_0 = j\underline{y}_{t-2}$, $\delta_0 - \delta_1 = \underline{y}_{t-2}$ and $\delta_0 + n_1\delta_1 - \delta_2 = \underline{y}_{t-1}$) for the above recurrence attached to a δ-sequence in $\mathbb{N}_{>0}$, $\Delta^* = \{\delta_0^*, \delta_1^*\}$ (respectively, $\Delta^* = \{\delta_0^*, \delta_1^*, \delta_2^*\}$, such that $j := \delta_0^*/(\delta_0^* - \delta_1^*) \in \mathbb{N}_{\geq 0}$ and $n_1 := \delta_0^*/\gcd(\delta_0^*, \delta_1^*)$).

TYPE D: $\Delta = \{\delta_0, \delta_1, \ldots, \delta_g\} \subset \mathbb{R}$ is finite, $g \geq 2$, δ_i is a positive rational number for $0 \leq i \leq g-1$, δ_g is non-rational, and there exists a sequence

$$\left\{\overline{\Delta}_j = \{\delta_0^j, \delta_1^j, \ldots, \delta_g^j\}\right\}_{j \geq 1}$$

of normalized δ-sequences in $\mathbb{N}_{>0}$ such that $\delta_i^j = \delta_i$ for $0 \leq i \leq g-1$ and any j and $\delta_g = \lim_{j\to\infty} \delta_g^j$. We complete this definition by adding that $\Delta = \{\tau, 1\}$, $\tau > 1$ being a non-rational number, is also a δ-sequence of type D.

TYPE E: $\Delta = \{\delta_0, \delta_1, \ldots, \delta_i, \ldots\} \subset \mathbb{Q}$ is infinite and any ordered subset $\Delta_j = \{\delta_0, \delta_1, \ldots, \delta_j\}$ is a normalized δ-sequence in $\mathbb{N}_{>0}$.

Examples. Next, we show examples of δ-sequences of types from A to E: $\{18, 12, 33, 4, -5\}$ is of type A, $\{(0, 18), (0, 12), (0, 33), (0, 4), (-1, 18^2)\}$ of type B, $\{(6, 6), (4, 4), (11, 11), (1, 2)\}$ of type C, $\{3/2, 1, 33/12, 4/12, (33 + 14\sqrt{2})/6(7 + 3\sqrt{2})\}$ of type D and the first terms of a δ-sequence of type E are $\{3/2, 1, 33/12, 1/3, 15/4, \ldots\}$.

Remark 4.1. The proximity relation attached to the minimal embedded resolution of the singularity at infinity of a curve with only one place at infinity that satisfies the Abhyankar-Moh Theorem can be recovered from the finite sequence of positive integers satisfying (1), (2) and (3) of that theorem by using the formulae before Definition 4.2. To do it, one needs to consider the continued fractions of quotients of the type m_l/e_l, m_l

and e_l being the values defined before Definition 4.2 (see Section 2 in [14] and references therein).

δ-sequences in Definition 4.2 are defined in such a way that the same property happens for any type of valuation, although we need, for that purpose, to use an extended version of the Euclidian Algorithm that can also involve values either in $\mathbb{Z}^2$ or in $\mathbb{R}$. We must add that in the case of a δ-sequence of type A, $\{\delta_0, \delta_1, \ldots, \delta_g, \delta_{g+1}\}$, the quotients m_l/e_l (for $0 \leq l \leq g-1$ if $\delta_0 - \delta_1$ does not divide δ_0, and for $0 \leq l \leq g-2$ otherwise) determine the proximity relation of the points of $\mathcal{C}_\nu = \{p_i\}_{i=0}^n$ corresponding to the minimal embedded resolution of the germ given by a general element of ν; that is, the proximity relation among those points p_i such that $i \leq i_0$, i_0 being the maximal value such that p_{i_0} is a satellite point. To recover the dual graph of ν, in addition to the above information, we need to know the number $\mathfrak{f} = n - i_0$ of last free points. This number is given by the element δ_{g+1} of the δ-sequence; in fact, $\mathfrak{f} = n_g\delta_g - \delta_{g+1}$ (see the proof of the forthcoming Theorem 4.1).

Since any plane valuation ν can be regarded as a limit of divisorial ones and since, in this last case, the continued fractions of the quotients m_l/e_l (together with $\mathfrak{f}$) provide the corresponding proximity relations, the definition of δ-sequence for the remaining types of valuations is made to extend at infinity the previous behavior. So, when one can obtain infinitely many quotients m_l/e_l, the corresponding valuation is of type E. Otherwise, only finitely many quotients m_l/e_l appear and the limit of divisorial valuations can be seen in the last one m_r/e_r. In these cases, the quotients m_l/e_l of the divisorial valuations ν_i converging ν, $i \gg 0$, are the same that those for ν whenever $l < r$ and the continued fractions $\langle a_1; a_2, \ldots, a_s\rangle$ corresponding to the last pair (m_r, e_r) of the valuations ν_i must be taken "at infinity". This can be done either increasing the value s indefinitely or increasing a_s indefinitely. In the first case m_r/e_r converges to a real non-rational number and we get type D valuations. In the second case, we obtain type C valuations when $s \neq 1$ and type B ones, otherwise. Note that to take a_1 at infinity is the same thing that to do so with the value $\mathfrak{f}$.

Next, we make explicit for the above examples what we have said in the previous paragraphs. The proximity relation provided by the type A valuation is given by the following pairs (m_l, e_l): $(21,6), (62,3), (2,1)$ with attached continued fractions $\langle 3;2\rangle$, $\langle 20;1,2\rangle$ and $\langle 2\rangle$; in addition, the equality $n_g\delta_g - \delta_{g+1} = 17$ indicates that the last 17 points of $\mathcal{C}_\nu$ are free. For the type B valuation, we get pairs $((0,21),(0,6)), ((0,63),(0,3))$ and $((1,0),(0,1))$, with associated continued fractions $\langle 3;2\rangle$, $\langle 20;1,2\rangle$ and $\langle\infty\rangle$; this last one corresponds to blowing-up at infinitely many free points. In the type C case, we obtain $((7,7),(2,2))$ and $((21,20),(1,1))$, with continued fraction $\langle 3;2\rangle$ and $\langle 20;1,\infty\rangle$ (indeed, for the last pair, performing the (generalized) Euclidian Algorithm we have $(21,0) = \mathbf{20}(1,1) + (1,0)$; $(1,1) = \mathbf{1}(1,0) + (0,1)$ and $(1,0) = \infty(0,1)$). Notice that here, ∞ indicates the existence of infinitely many satellite point blowing-ups. In case D, the pairs are $(21/12, 1/2)$, $(62/12, 3/12)$ and $((9+4\sqrt{2})/6(7+3\sqrt{2}), 1/6)$, and the associated continued fractions are $\langle 3;2\rangle$, $\langle 20;1,2\rangle$ and $\langle 1;3,2,\sqrt{2}\rangle$. Finally, the first pairs of the type E valuation reproduce the behavior of the one of type A.

To end this paper, we state and prove the announced analogue of the Abhyankar-Moh Theorem for plane valuations at infinity.

Theorem 4.1. *Let ν be a plane valuation at infinity under the assumption that it can be defined by a sequence of plane divisorial valuations at infinity $\{\nu_i\}_{i=1}^{\infty}$ such that the characteristic of the field k, $\mathrm{char}(k)$, does not divide $\gcd(-\nu_i(x), -\nu_i(y))$ for all $i \in \mathbb{N}_{>0}$. Then, there exists a δ-sequence Δ of the same type as ν such that the semigroup at infinity $S_{\nu,\infty}$ is generated by Δ.*

Proof. We will prove the result by showing it for each type of valuation. First, assume that ν *is a valuation of type* A and consider a curve C having only one place at infinity whose germ at p defines a general element of ν. Since $\nu(x) = \nu_{C,p}(x)$ and $\nu(y) = \nu_{C,p}(y)$, applying Theorem 2.1, the existence of a δ-sequence $\Gamma = \{\delta_0, \delta_1, \ldots, \delta_g\}$ in $\mathbb{N}_{>0}$ generating the semigroup $S_{C,\infty}$ is proved. If $q_{g+1}(x, y) = 0$ is the affine equation (in the chart $Z \neq 0$) of the curve C, without loss of generality, we can assume that the polynomial q_{g+1} satisfies the requirements of Definition 2.3. Let $\{q_i(x, y)\}_{i=0}^{g}$ be a family of approximates for C such that $\delta_i = -\nu_{C,p}(q_i)$, $0 \leq i \leq g$. As usual, we will identify polynomials in $k[x, y]$ with their images in $\mathcal{O}_C(C \setminus \{p\})$.

Notice that $\nu(\bar{q}_i(u, v)) = \nu_{C,p}(\bar{q}_i(u, v))$ when $0 \leq i \leq g$ and, as a consequence, the inclusion $\Gamma \subseteq S_{\nu,\infty}$ holds. Let i_0 be the largest index such that the point $p_{i_0} \in \mathcal{C}_\nu = \{p_i\}_{i=0}^{n}$ is satellite, then

$$-\nu(q_{g+1}(x, y)) = \delta_0 \nu(v) - \nu(\bar{q}_{g+1}(u, v)) = \delta_0^2 - n_g \nu(\bar{q}_g(u, v)) - (n - i_0) =$$

$$\delta_0^2 - n_g \left(\frac{\delta_0^2}{n_g} + \nu(q_g(x, y)) \right) - (n - i_0) = n_g \delta_g - (n - i_0).$$

If we set $\delta_{g+1} := n_g \delta_g - (n - i_0)$, it is clear that $\Delta := \{\delta_0, \delta_1, \ldots, \delta_g, \delta_{g+1}\}$ is a δ-sequence of type A and that it is contained in $S_{\nu,\infty}$. Now, we are going to prove that the semigroup $S_{\nu,\infty}$ is generated by Δ.

Let us consider $f = f(x, y) \in k[x, y] \setminus \{0\}$ and set $f(x, y) = v^{-\deg(f)} \bar{f}(u, v)$. To compute $\nu(f)$, one can use the procedure described in the Remark 3.2 and write f as an element of the Laurent power series ring in the indeterminate t with coefficients in the field $k(s)$, $\mathfrak{L}_s(t)$. Indeed, by the proof of Theorem 1 in [12], it holds that

$$\bar{f}(u(t, s), v(t, s)) = (\mathfrak{q}_0 + \mathfrak{q}_1 p(s) + \cdots + \mathfrak{q}_r p(s)^r) t^{\nu(\bar{f})} + \mathfrak{r},$$

where $\mathfrak{q}_i = \varrho_i s^{b_i}$, $b_i \geq 0$ and $\varrho_i \in k$, some $\mathfrak{q}_i$ does not vanish and also the exponents of t in $\mathfrak{r} \in \mathfrak{L}_s(t)$ are always larger than $\nu(\bar{f})$. On the other hand, since v is a curvette of any general element of ν, $v(t, s) = \lambda_1 s^{a_1} t^{\nu(v)} + r_1$, $\lambda_1 \neq 0$, as we have said in the Remark 3.2. As a consequence, $f(x, y)$ can be written as an element in $\mathfrak{L}_s(t)$ whose first jet in t is

$$(\mathfrak{m}_0 + \mathfrak{m}_1 p(s) + \cdots + \mathfrak{m}_r p(s)^r) t^{\nu(f)}, \tag{5}$$

$\mathfrak{m}_i = s^{c_i} \mathfrak{q}_i$ for some $c_i \in \mathbb{Z}$, $0 \leq i \leq r$, which allows us to get the value $\nu(f)$. An analogous situation happens for the polynomials $q_i = q_i(x, y)$, whose first jets are

$$\gamma_i s^{d_i} t^{\nu(q_i)},\ 0 \neq \gamma_i \in k, d_i \in \mathbb{Z},\ \text{for } q_i,\ 0 \leq i \leq g, \quad \text{and } s^d p(s) t^{\nu(q_{g+1})}, d \in \mathbb{Z},\ \text{for } q_{g+1}. \tag{6}$$

Now, using [4, Sect. 7] and considering the class given by f in $k[x, y]/(q_{g+1})$, we get that $f + (q_{g+1}) = \sum_{k=0}^{d} \xi_k \prod_{i=0}^{g} q_i^{s_{ik}} + (q_{g+1})$, $\xi_k \in k$ and the exponents satisfy the conditions

$0 \leq s_{ik} < n_i$, $(1 \leq i \leq g,\ 0 \leq k \leq d)$. So,

$$f = \sum_{k=0}^{d} \left(\xi_k \prod_{i=0}^{g} q_i^{s_{ik}} \right) + q_{g+1}(x,y) f_1(x,y), \tag{7}$$

where $f_1(x,y) \in k[x,y]$.

Due to that the semigroup spanned by Γ is telescopic [7, Rem. 3.8], the values $\nu(\prod_{i=0}^{g} q_i^{s_{ik}})$ are different, and this fact, together with the one expressed in (6), proves that $\nu(f)$ is the minimum of the values by ν of the summands involved in the expression (7).

Finally, if $\nu(f) = \nu(\prod_{i=0}^{g} q_i^{s_{ik}})$ for some k, $1 \leq k \leq d$, then the result is proved since $\nu(q_i) = -\delta_i$. If not, $\nu(f) = \nu(q_{g+1}(x,y) f_1(x,y))$ and we could repeat for f_1 the same procedure made with f at most r times because otherwise using (6) and (5) we would obtain different expressions of f in $\mathfrak{L}_s(t)$ which is not possible. Thus, it must happen that $\nu(f) = \nu(\prod_{i=0}^{g} q_i^{p_i} q_{g+1}^{p})$ for some non-negative integers p_i, $0 \leq i \leq g$ and p, what concludes the proof in this case.

Assume now that ν is a valuation of type B. Let C be the plane curve having only one place at infinity whose successive strict transforms (of the germ of C at p) pass through all points in $\mathcal{C}_\nu$. As in the previous case, we can consider a family of approximates $\{q_i(x,y)\}_{i=0}^{g}$ attached to the curve C and an associated δ-sequence in $\mathbb{N}_{>0}$, $\Gamma = \{\delta_0, \delta_1, \ldots, \delta_g\}$, such that $S_{C,\infty}$ is generated by Γ and the numbers δ_i satisfy the equalities given in Definition 2.3. Let $f(x,y) \in k[x,y] \setminus \{0\}$ and suppose that $q(x,y) \in k[x,y]$ gives an equation for C. With the notation as in (2), set $f(x,y) = v^{-\deg(f)} \bar{f}(u,v)$ and analogously $q(x,y) = v^{-\deg(q)} \bar{q}(u,v)$. Now, $\bar{f}(u,v) = \bar{q}(u,v)^s \bar{l}(u,v)$, where $s \geq 0$, $\bar{l}(u,v) \in K[u,v]$ and $\bar{q}(u,v)$ does not divide $\bar{l}(u,v)$. Recall that that for any element $h \in R$ such that $h(u,v) = \bar{q}(u,v)^a g(u,v)$, where $\bar{q}(u,v)$ does not divide $g(u,v)$, it holds that $\nu(h(u,v)) = (a, \nu_{C,p}(g(u,v))$. Consider the polynomial $l(x,y) = v^{\deg(l)} \bar{l}(u,v)$, then it happens that $\nu(l(x,y)) = \left(0, \nu_{C,p}(v^{\deg(l)} \bar{l}(u,v))\right) = (0, \nu_{C,p}(l(x,y)))$. On the other hand, $\nu(q(x,y)) = \nu\left(v^{\deg(q)} \bar{q}(u,v)\right) = (0, -\delta_0^2) + (1,0)$, and this proves

$$\nu(f(x,y)) = \nu(q^s(x,y)) + \nu(l(x,y)) = -s(-1, \delta_0^2) - \sum_{i=0}^{g} s_i (0, \delta_i)$$

for some $s_i \geq 0$. That is $\Delta := \{(0,\delta_0), (0,\delta_1), \ldots, (0,\delta_g), (-1,\delta_0^2)\}$ is a δ-sequence of type B that spans $S_{\nu,\infty}$.

The proof for valuations of type C runs as follows. Set $\{\nu_j\}_{j=1}^{\infty}$ a family of divisorial valuations that define ν. Consider $f(x,y) \in k[x,y] \setminus \{0\}$ and the corresponding polynomial $\overline{f}(u,v)$ defined in (2). Pick a valuation ν_{j_0} such that the intersection set between the sequence of infinitely near points of any branch of the germ given by $\overline{f}(u,v)$ and the set $\mathcal{C}_\nu$ coincides with the one with $\mathcal{C}_{\nu_{j_0}}$. Let C be a curve having only one place at infinity and such that its germ at p gives a general element of ν_{j_0}. Consider a family of approximates $\{q_i(x,y)\}_{i=0}^{g}$ for C. As above, we can suppose that $S_{C,\infty}$ is spanned by a δ-sequence in $\mathbb{N}_{>0}$, $\{\delta_i\}_{i=0}^{g}$, which is related with the approximates as in Definition 2.3. Remark 3.1 proves that $\nu_{j_0}(f(x,y)) = \nu_{C,p}(f(x,y))$ and so, there exist positive integers $s_0, s_1, \ldots, s_g$ such that $\nu_{j_0}(f) = \nu_{j_0}(\prod_{i=0}^{g} q_i^{s_i}(x,y))$. For analytically irreducible elements h in R, divisorial valuations ν_{j_0} satisfy a Noether formula, that is $\nu_{j_0}(h) = \sum_{j=0}^{r} \nu_{j_0}(m_j) e(p_j)$, where r gives the number of common points $\{p_j\}_{j=0}^r$ between the sequences of infinitely near points relative to h and to ν and $e(p_j)$ is the multiplicity of the strict transform of the germ given by h at the point p_j. The same Noether formula is also true for type C valuations with the values described for ν in Section 3.2 (see [14, Sect. 3]) and this proves that $\nu(f) = \nu(\prod_{i=0}^g q_i^{s_i}(x,y))$. Therefore $S_{\nu,\infty}$ is spanned by the values $-\nu(q_i(x,y)) \in \mathbb{Z}^2$, $0 \leq i \leq g$, which constitute a δ-sequence Δ of type C that can be computed from the above δ-sequence in $\mathbb{N}_{>0}$ as is described in Definition 4.2.

Suppose now that ν is a valuation of type D. Let $\{\nu_j\}_{j=1}^\infty$ be a family of divisorial valuations at infinity that define ν. Without loss of generality, we can assume that, for all j, ν_j is the divisorial valuation defined by the exceptional divisor associated with a blowing-up centered at a satellite point and, moreover, all the free points in $\mathcal{C}_\nu$ also belong to $\mathcal{C}_{\nu_j}$. Set $\{\delta_i^j\}_{i=0}^g$ a normalized δ-sequence in $\mathbb{N}_{>0}$ associated with a curve having only one place at infinity which provides a general element of ν_j. As we have said, for $f(x,y) \in k[x,y]\setminus\{0\}$ and if j is large enough, the value of f by the valuation ν_j (normalized as we explained at the end of Section 3.2) is $\nu_j(f) = \sum_{i=0}^g s_i\delta_i^j$, $s_0, s_1, \ldots, s_g$ being non-negative integers. Thus,

$$\nu(f(x,y)) = \lim_{j\to\infty} \nu_j(f) = \lim_{j\to\infty} \sum_{i=0}^g s_i\delta_i^j.$$

For simplicity assume that $g \geq 2$ (all works similarly when $g = 1$). By Corollary 2.1, one has that when $i < g$, $\delta_i^j = \delta_i^{j'}$ for whichever pair j, j' of indices of the divisorial valuations. Moreover, from the paragraph in Section 3.2 corresponding to this type of valuations, it is straightforward to deduce that $\delta_g := \lim_{j\to\infty} \delta_g^j$ is a non-rational number. Therefore, $S_{\nu,\infty}$ is spanned by the δ-sequence of type D given by $\Delta := \{\delta_0^j, \delta_1^j, \ldots, \delta_{g-1}^j, \delta_g\}$.

Finally, *whenever ν is a valuation of type* E, taking $f(x,y)$, $\{\nu_j\}_{j=1}^\infty$ and ν_{j_0} as in the case of valuations of type C, it holds that $\nu(f) = \nu_{j_0}(f)$ (here, as in the above paragraph, the valuations ν_j are also normalized). From this fact and arguing in a similar way as above it follows immediately that $S_{\nu,\infty}$ is spanned by a δ-sequence Δ of type E, whose first elements coincide with the normalized δ-sequence of the δ-sequence in $\mathbb{N}_{>0}$ defined by a curve with only one place at infinity whose germ in p gives a general element of ν_{j_0}.□

Remark 4.2. Suppose that $\mathrm{char}(k) = 0$ and let ν be a valuation at infinity of type A (notice that ν satisfies the hypothesis of Theorem 4.1). In this case the sign of δ_{g+1} has a nice geometrical interpretation. Indeed, set $\Delta = \{\delta_0, \delta_1, \ldots, \delta_g, \delta_{g+1}\}$ a δ-sequence of type A generating the semigroup $S_{\nu,\infty}$ and let $f(x,y) = 0$ be an equation defining a curve C having only one place at infinity whose germ at p is a general element for ν. In addition, assume that $f(x,y)$ and $\{\delta_0, \delta_1, \ldots, \delta_g\}$ are as in Definition 2.3 (in particular, δ_0 is the degree of C). Let us consider the infinite sequence

$$\cdots \to X_i \to X_{i-1} \to \cdots \to X_1 \to X_0 := \mathbb{P}^2$$

such that $X_1 \to \mathbb{P}^2$ is the blowing-up of $\mathbb{P}^2$ at $p_0 := p$ and, for each $i \geq 2$, $X_i \to X_{i-1}$ is the blowing-up at the intersection point p_{i-1} of the strict transform of C in X_{i-1} and the exceptional divisor of the preceding blowing-up. Assume that the first $r+1$ blowing-ups induce the minimal embedded resolution of the singularity of C at p (as in (1)). Let Λ be the pencil of plane curves of $\mathbb{P}^2$ whose equations in the affine chart $Z \neq 0$ are $f(x, y) = \lambda$, where $\lambda \in k$. By a result of Moh [16], all the curves in Λ have only one place at infinity and there exists an integer $\mathbf{m} \geq r$ such that their strict transforms pass through p_i for all $i \leq \mathbf{m}$, $\mathbf{m}$ being the least integer such that the composition of the first $\mathbf{m}+1$ blowing-ups eliminates the base points of the pencil Λ. Thus $\delta_0^2 = \sum_{i=0}^{\mathbf{m}} m_{p_i}(C)^2$, where $m_{p_i}(C)$ denotes the multiplicity at p_i of the strict transform of C in X_i. By the proof of Theorem 4.1, it happens

$$\delta_{g+1} = -\nu(q_{g+1}(x, y)) = \delta_0^2 - \nu(\bar{q}_{g+1}(u, v)) = \delta_0^2 - \sum_{i=0}^{\mathbf{n}} m_{p_i}(C)^2,$$

where $\mathcal{C}_\nu = \{p_i\}_{i=0}^{\mathbf{n}}$.

Therefore, if $\mathbf{n} \leq \mathbf{m}$ (respectively, $\mathbf{n} > \mathbf{m}$), then $\delta_{g+1} \geq 0$ (respectively, < 0) and so $S_{\nu,\infty}$ is (respectively, is not) a well-ordered semigroup (with the natural ordering).

Remark 4.3. Theorem 4.1 holds under certain conditions on the characteristic of the field k. When the plane valuation at infinity ν is either of type A or B, it has no difficulty to check that condition. Otherwise and when $\operatorname{char}(k) \neq 0$, it cannot be easy to decide whether the cited condition happens. Notwithstanding, it is not difficult to find examples where to check that condition is simple. For instance, following [14, Sect. 4.3.4], a valuation of type E can be obtained considering, as a δ-sequence, the limit of the normalized δ-sequences of the δ-sequences in $\mathbb{N}_{>0}$: $\{5, 3\}$, $\{5 \cdot 2, 3 \cdot 2, 3 \cdot 3\}$, $\{5 \cdot 2^2, 3 \cdot 2^2, 3^2 \cdot 2, 3^3\}$, ... (we have used $\{5, 3\}$ to start and the value $z = 2$ with the notation in [14, Sect. 4.3.4]). Then, Theorem 4.1 happens when the characteristic of the field is different from 2. Let us see another example corresponding to a type C valuation. Let ν be the valuation of type C whose δ-sequence is $\{(2, 1), (1, 0)\}$. Then the sequence of values $(-\nu_i(x), -\nu_i(y))$ described in the Theorem 4.1 is an infinite subset of the set $\{(2j + 1, j)\}_{j \geq 2}$. Since $\gcd(2j + 1, j) = 1$ for any index j, our result happens for any characteristic of the field k.

Remark 4.4. The converse of Abhyankar-Moh Theorem for plane valuations at infinity, without any restriction on the characteristic of the field, is true. It is proved for valuations of types C, D and E in [14, Th. 4.9]. Let us see it for the types A and B. Assume that Δ is a δ-sequence either of type A or B. Set $\Delta' = \{\delta_0, \delta_1, \ldots, \delta_g\}$ the δ-sequence in $\mathbb{N}_{>0}$ that allows us to provide Δ. Consider the unique expression

$$n_i \delta_i = \sum_{j=0}^{i-1} a_{ij} \delta_j,$$

where $a_{i0} \geq 0$ and $0 \leq a_{ij} < n_j$, for $1 \leq j \leq i-1$. Let us define the following polynomials: $q_0 := x$, $q_1 := y$ and, for $1 \leq i \leq g$,

$$q_{i+1} := q_i^{n_i} - t_i \left(\prod_{j=0}^{i-1} q_j^{a_{ij}} \right),$$

where $t_i \in k \setminus \{0\}$ are arbitrary. The curve $C_{\Delta'}$ defined by $q_{g+1}(x, y) = 0$ has only one place at infinity, the set $\{q_i\}_{i=0}^{g}$ is a family of approximates for it and $S_{C_{\Delta'},\infty}$ is generated by Δ' (see [14] and references therein for more details).

Then the valuation ν defined by the set $\{p_i\}_{i=1}^{\infty}$ of infinitely near points attached to the resolution of $C_{\Delta'}$ is a valuation at infinity of type B whose semigroup at infinity is spanned by the δ-sequence Δ of type B. In addition, if we take the valuation ν defined by the finite subset $\{p_i\}_{i=1}^{n}$ of the above set $\{p_i\}_{i=1}^{\infty}$ such that $n = n_g\delta_g - \delta_{g+1} + i_0$, δ_{g+1} being the last element in Δ and i_0 the largest index such that p_{i_0} is a satellite point, then the semigroup at infinity of ν is spanned by the δ-sequence Δ of type A.

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