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On topological groups via a-local functions

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Abstract

An ideal on a set X is a nonempty collection of subsets of X which satisfies the following conditions (1)$A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) an ideal $\mathcal{I}$ on X and $A \subset X$, $\Re_a(A)$ is defined as $\cup\{U \in \tau^a : U - A \in \mathcal{I}\}$, where the family of all a-open sets of X forms a topology [5, 6], denoted by τ^a. A topology, denoted τ^{a^*}, finer than τ^a is generated by the basis $\beta(\mathcal{I}, \tau) = \{V - I : V \in \tau^a(x), I \in \mathcal{I}\}$, and a topology, denoted $\langle \Re_a(\tau) \rangle$ coarser than τ^a is generated by the basis $\Re_a(\tau) = \{\Re_a(U) : U \in \tau^a\}$. In this paper A bijection $f : (X, \tau, \mathcal{I}) \to (X, \sigma, \mathcal{J})$ is called a $\mathcal{A}*$-homeomorphism if $f : (X, \tau^{a^*}) \to (Y, \sigma^{a^*})$ is a homeomorphism, $\Re_a$-homeomorphism if $f : (X, \Re_a(\tau)) \to (Y, \Re_a(\sigma))$ is a homeomorphism. Properties preserved by $\mathcal{A}*$-homeomorphism are studied as well as necessary and sufficient conditions for a $\Re_a$-homeomorphism to be a $\mathcal{A}*$-homeomorphism.

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1. Introduction and Preliminaries

Ideals in topological spaces have been considered since 1930. The subject of ideals in topological spaces has been studied by Kuratowski [11] and

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Vaidyanathaswamy [18]. Jankovic and Hamlett [10] investigated further properties of ideal space. In this paper, we investigate a-local functions and its properties in ideals in topological space [1]. Also, the relationships among local functions such as local function [19, 10] and semi-local function [7] are investigated.

A subset of a space (X, τ) is said to be regular open (resp. regular closed) [12] if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A is called δ-open [20] if for each $x \in A$, there exists a regular open set G such that $x \in G \subset A$. The complement of δ-open set is called δ-closed. A point $x \in X$ is called a δ-cluster point of A if $int(Cl(U)) \cap A \neq \phi$ for each open set V containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $Cl_\delta(A)$ [20]. The δ-interior of A is the union of all regular open sets of X contained in A and its denoted by $Int_\delta(A)$ [20]. A is δ-open if $Int_\delta(A) = A$. δ-open sets forms a topology τ^δ.

A subset A of a space (X, τ) is said to be a-open (resp. a-closed) [5] if $A \subset Int(Cl(Int_\delta(A)))$ (resp. $Cl(Int(Cl_\delta(A))) \subset A$, or $A \subset Int(Cl(Int_\delta(A)))$ (resp. $Cl(Int(Cl_\delta(A))) \subset A$. The family of a-open sets of X forms a topology, denoted by τ^a [6]. The intersection of all a-closed sets contained A is called the a-closure of A and is denoted by $aCl(A)$. The a-interior of A, denoted by $aInt(A)$, is defined by the union of all a-open sets contained in A [5].

An ideal $\mathcal{I}$ on a topological space $(X, \mathcal{I})$ is a nonempty collection of subsets of X which satisfies the following conditions:
(1) $A \in \mathcal{I}$ *and* $B \subset A$ *implies* $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ *and* $B \in \mathcal{I}$ *implies* $A \cup B \in \mathcal{I}$. Applications to various fields were further investigated by Jankovic and Hamlett [10] Dontchev et al. [4]; Mukherjee et al. [13]; Arenas et al. [3]; Navaneethakrishnan et al. [14]; Nasef and Mahmoud [15], etc. Given a topological space $(X, \mathcal{I})$ with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [11, 10] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$,

$$A^*(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I}, \text{ for every U} \in \tau(x)\}$$

where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(\tau, \mathcal{I})$, called the $*$-topology, which is finer than τ is defined by $Cl^*(A) = A \cup A^*(\tau, \mathcal{I})$, when there is no chance of confusion. $A^*(\mathcal{I})$ is denoted by A^* and τ^* for $\tau^*(\mathcal{I}, \tau)$. X^* is often a proper subset of X. The hypothesis $X = X^*$ [7] is equivalent to the hypothesis $\tau \cap \mathcal{I} = \phi$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space. N is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal space $(X, \tau, \mathcal{I})$ is $\star$-closed [4] (resp. $\star$-dense in itself [7]) if $A^* \subseteq A$ (resp $A \subseteq A^*$). A subset A of an ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}_g-$closed [20] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open. For every ideal topological space there exists a topology $\tau^*(\mathcal{I})$ finer than τ generated by $\beta(\mathcal{I}, \tau) = \{U - A \mid U \in \tau \text{ and } A \in \mathcal{I}\}$, but in general $\beta(\mathcal{I}, \tau)$ is not always topology [10]. Let $(X, \mathcal{I}, \tau)$ ba an ideal topological space. We say that the topology τ is *compatible* with the $\mathcal{I}$, denoted $\tau \sim \mathcal{I}$, if the following holds for every $A \subset X$, if for every $x \in A$ there exists a $U \in \tau$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Given a space $(X, \tau, \mathcal{I})$, $(Y, \sigma, \mathcal{J})$, and a function $f : (X, \tau, \mathcal{I}) \to (Y, \tau, \mathcal{J})$, we call f a $*$-homomorphism with respect to τ, $\mathcal{I}$, σ, and $\mathcal{J}$ if $f : (X, \tau^*) \to (Y, \sigma^*)$ is a homomorphism, where a homomorphism is a continuous injective function between two topological spaces, that is invertible with continuous inverse. We first prove some preliminary lemmas which lead to a theorem extending the theorem in [17] and apply the theorem to topological groups. Quite recently, in [2], the present authors defined and investigated the notions $\Re_a : \wp(X) \to \tau$ as follows, $\Re_a(A) = \{x \in X :$ there exists $U_x \in \tau^a$ containing x such that $U_x - A \in \mathcal{I}\}$, for every $A \in \wp(X)$. In [16], Newcomb defined $A = B[mod\,\mathcal{I}]$ if $(A - B) \cup (B - A) \in \mathcal{I}$ and observe that $= [mod\,\mathcal{I}]$ is an equivalence relation. In this paper a bijection $f : (X, \tau, \mathcal{I}) \to (X, \sigma, \mathcal{J})$ is called a $\mathcal{A}*$-homeomorphism if $f : (X, \tau^{a^*}) \to (Y, \sigma^{a^*})$ is a homeomorphism, $\Re_a$-homeomorphism if $f : (X, \Re_a(\tau)) \to (Y, \Re_a(\sigma))$ is a homeomorphism. Properties preserved by $\mathcal{A}*$-homeomorphism are studied as well as necessary and sufficient conditions for a $\Re_a$-homeomorphism to be a $\mathcal{A}*$-homeomorphism.

2. a-Local Function and $\Re_a$- operator

Let $(X, \tau, \mathcal{I})$ an ideal topological space and A a subset of X. Then $A^{a^*}(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$, for every $U \in \tau^a(x)\}$ is called a-local function of A [1] with respect to $\mathcal{I}$ and τ, where $\tau^a(x) = \{U \in \tau^a : x \in U\}$. We denote simply A^{a^*} for $A^{a^*}(\mathcal{I}, \tau)$.

Remark 2.1 ([1]).

(1) The minimal ideal is considered $\{\varnothing\}$ in any topological space (X, τ) and the maximal ideal is considered $P(X)$. It can be deduced that $A^{a^*}(\{\varnothing\}) = Cl_a(A) \neq Cl(A)$ and $A^{a^*}(P(X)) = \varnothing$ for every $A \subset$ X.
(2) If $A \in \mathcal{I}$, then $A^{a^*} = \varnothing$.
(3) $A \not\subseteq A^{a^*}$ and $A^{a^*} \not\subseteq A$ in general.

Theorem 2.2 ([1]). *Let $(X, \tau, \mathcal{I})$ an ideal in topological space and A, B subsets of X. Then for a-local functions the following properties hold:*

(1) *If $A \subset B$, then $A^{a^*} \subset B^{a^*}$,*
(2) *For another ideal $J \supset \mathcal{I}$ on X, $A^{a^*}(J) \subset A^{a^*}(\mathcal{I})$,*
(3) $A^{a^*} \subset aCl(A)$,
(4) $A^{a^*}(\mathcal{I}) = aCl(A^{a^*}) \subset aCl(A)$ *(i.e A^{a^*} is an a-closed subset of $aCl(A)$),*
(5) $(A^{a^*})^{a^*} \subset A^{a^*}$,
(6) $(A \cup B)^{a^*} = A^{a^*} \cup B^{a^*}$,
(7) $A^{a^*} - B^{a^*} = (A - B)^{a^*} - B^{a^*} \subset (A - B)^{a^*}$,
(8) *If $U \in \tau^a$, then $U \cap A^{a^*} = U \cap (U \cap A)^{a^*} \subset (U \cap A)^{a^*}$,*
(9) *If $U \in \tau^a$, then $(A - U)^{a^*} = A^{a^*} = (A \cup U)^{a^*}$,*
(10) *If $A \subseteq A^{a^*}$, then $A^{a^*}(\mathcal{I}) = aCl(A^{a^*}) = aCl(A)$.*

Theorem 2.3 ([1]). *Let $(X, \tau, \mathcal{I})$ an ideal in topological space and A, B subsets of X.Then for a-local functions the following properties hold:*

(1) $\tau^a \cap \mathcal{I} = \phi$;
(2) *If* $I \in \mathcal{I}$, *then* $aInt(I) = \phi$;
(3) *For every* $G \in \tau^a$, *then* $G \subseteq G^{a^*}$;
(4) $X = X^{a^*}$.

Theorem 2.4 ([1]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A subset of X. Then the following are equivalent:*

(1) $\mathcal{I} \sim^a \tau$,
(2) If a subset A of X has a cover a-open of sets whose intersection with A is in $\mathcal{I}$, then A is in $\mathcal{I}$, in other words $A^{a^*} = \phi$, then $A \in \mathcal{I}$,
(3) For every $A \subset X$, if $A \cap A^{a^*} = \phi$, $A \in \mathcal{I}$,
(4) For every $A \subset X$, $A - A^{a^*} \in \mathcal{I}$,
(5) For every $A \subset X$, if A contains no nonempty subset B with $B \subset B^{a^*}$, then $A \in \mathcal{I}$.

Theorem 2.5 ([1]). *Let $(X, \mathcal{I}, \tau)$ be an ideal topological space. Then $\beta(\mathcal{I}, \tau)$ is a basis for τ^{a^*}. $\beta(\mathcal{I}, \tau) = \{V - I_i : V \in \tau^a(x), I_i \in \mathcal{I}\}$ and β is not, in general, a topology.*

Theorem 2.6 ([2]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties hold:*

(1) *If* $A \subset X$, *then* $\Re_a(A)$ *is a-open.*
(2) *If* $A \subset B$, *then* $\Re_a(A) \subseteq \Re_a(B)$.
(3) *If* $A, B \in \wp(X)$, *then* $\Re_a(A \cup B) \subset \Re_a(A) \cup \Re_a(B)$.
(4) *If* $A, B \in \wp(X)$, *then* $\Re_a(A \cap B) = \Re_a(A) \cap \Re_a(B)$.
(5) *If* $U \in \tau^{a^*}$, *then* $U \subseteq \Re_a(U)$.
(6) *If* $A \subset X$, *then* $\Re_a(A) \subseteq \Re_a(\Re_a(A))$.
(7) *If* $A \subset X$, *then* $\Re_a(A) = \Re_a(\Re_a(A))$ *if and only if* $(X - A)^{a^*} = ((X - A)^{a^*})^{a^*}$.
(8) *If* $A \in \mathcal{I}$, *then* $\Re_a(A) = X - X^{a^*}$.
(9) *If* $A \subset X$, *then* $A \cap \Re_a(A) = Int^{a^*}(A)$, *where* Int^{a^*} *is the interior of* τ^{a^*}.
(10) *If* $A \subset X$, $I \in \mathcal{I}$, *then* $\Re_a(A - I) = \Re_a(A)$.
(11) *If* $A \subset X$, $I \in \mathcal{I}$, *then* $\Re_a(A \cup I) = \Re_a(A)$.
(12) *If* $(A - B) \cup (B - A) \in \mathcal{I}$, *then* $\Re_a(A) = \Re_a(B)$.

Theorem 2.7 ([1]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A subset of X. If τ is a-compatible with $\mathcal{I}$. Then the following are equivalent:*

(1) *For every* $A \subset X$, *if* $A \cap A^{a^*} = \phi$ *implies* $A^{a^*} = \phi$,
(2) *For every* $A \subset X$, $(A - A^{a^*})^{a^*} = \phi$,
(3) *For every* $A \subset X$,$(A \cap A^{a^*})^{a^*} = A^{a^*}$.

Theorem 2.8 ([2]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim^a \mathcal{I}$. Then $\Re_a(A) = \cup\{\Re_a(U) : U \in \tau^a, \Re_a(U) - A \in \mathcal{I}\}$.*

Proposition 2.9 ([2]). *Let $(X,\tau,\mathcal{I})$ be an ideal topological space with $\tau^a \cap \mathcal{I} = \phi$.Then the following are equivalent:*

(1) $A \in \mathcal{U}(X,\tau,\mathcal{I})$,
(2) $\Re_a(A) \cap aInt(A^{a^*}) \neq \phi$,
(3) $\Re_a(A) \cap A^{a^*} \neq \phi$,
(4) $\Re_a(A) \neq \phi$,
(5) $Int^{a^*}(A) \neq \phi$,
(6) *There exists* $N \in \tau^a - \{\varnothing\}$ *such that* $N - A \in \mathcal{I}$ *and* $N \cap A \notin \mathcal{I}$.

Proposition 2.10 ([2]). *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then $\tau \sim^a \mathcal{I}$, $A \subseteq X$. If N is a nonempty a-open subset of $A^{a^*} \cap \Re_a(A)$, then $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.*

Theorem 2.11 ([2]). *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then $\tau \sim^a \mathcal{I}$ if and only if $\Re_a(A) - A \in \mathcal{I}$ for every $A \subseteq X$.*

3. $\mathcal{A}*$-HOMEOMORPHISM

Given an ideal topological space $(X,\tau,\mathcal{I})$ a topology τ^a finer than $\langle\Re_a(\tau)\rangle$ which $\langle\Re_a(\tau)\rangle$ is generated by the basis $\Re_a(\tau) = \{\Re_a(U) : U \in \tau^a\}$.

Definition 3.1 ([5]). A function $f : (X,\tau) \to (X,\sigma)$ is called

(1) a-continuous if the inverse image of a-open set is a-open.
(2) a-open if the image of a-open set is a-open.

Definition 3.2. Let $(X,\tau,\mathcal{I})$ and $(X,\sigma,\mathcal{J})$ be an ideal topological space. A bijection $f : (X,\tau,\mathcal{I}) \to (X,\sigma,\mathcal{J})$ is called

(1) $\mathcal{A}*$-homeomorphism if $f : (X,\tau^{a^*}) \to (Y,\sigma^{a^*})$ is a homeomorphism.
(2) $\Re_a$-homeomorphism if $f : (X,\Re_a(\tau)) \to (Y,\Re_a(\sigma))$ is a homeomorphism.

Theorem 3.3. *Let $(X,\tau,\mathcal{I})$ and $(X,\sigma,\mathcal{J})$ be an ideal topological space with $f : (X,\Re_a(\tau)) \to (X,\sigma,\mathcal{J})$ an a-open bijective, $\tau \sim^a \mathcal{I}$ and $f(\mathcal{I}) \subseteq \mathcal{J}$. Then $f(\Re_a(A)) \subseteq \Re_a(f(A))$ for every $A \subseteq X$.*

Proof. Let $A \subseteq X$ and let $y \in f(\Re_a(A))$. Then $f^{-1}(y) \in \Re_a(A)$ and there exists $U \in \tau^a$ such that $f^{-1}(y) \in \Re_a(U)$ and $\Re_a(U) - A \in \mathcal{I}$ by Theorem 2.8. Now $f(\Re_a(U)) \in \sigma^a(y)$ and $f(\Re_a(U)) - f(A) = f[\Re_a(U) - A] \in f(\mathcal{I}) \subseteq \mathcal{J}$. Thus $y \in \Re_a(f(A)$, and the proof is complete. □

Theorem 3.4. *Let $(X,\tau,\mathcal{I})$ and $(X,\sigma,\mathcal{J})$ be an ideal topological space with $f : (X,\tau) \to (X,\Re_a(\sigma))$ is a-continuous injection, $\sigma \sim^a \mathcal{J}$ and $f^{-1}(\mathcal{J}) \subseteq \mathcal{I}$. Then $\Re_a(f(A)) \subseteq f(\Re_a(A))$ for every $A \subseteq X$.*

Proof. Let $y \in \Re_a(f(A))$ where $A \subseteq X$. Then by Theorem 2.8, there exists $U \in \sigma^a$ such that $y \in \Re_a(U)$ and $\Re_a(U) - f(A) \in \mathcal{J}$. Now we have $f^{-1}(\Re_a(U)) \in \tau^a(f^{-1}(y))$ with $f^{-1}[\Re_a(U) - f(A)] \in \mathcal{I}$ then $f^{-1}[\Re_a(U)] - A \in \mathcal{I}$ and $f^{-1}(y) \in \Re_a(A)$ and hence $y \in f(\Re_a(A))$, and the proof is complete. □

Theorem 3.5. *Let $(X,\tau,\mathcal{I})$ and $(X,\sigma,\mathcal{J})$ be a bijective with $f(\mathcal{I})=\mathcal{J}$. Then the following properties are equivalent:*

(1) *f is $\mathcal{A}*$-homeomorphism;*
(2) *$f(A^{a^*})=[f(A)]^{a^*}$ for every $A\subseteq X$;*
(3) *$f(\Re_a(A))=\Re_a(f(A))$ for every $A\subseteq X$;*

Proof. (1) $\Rightarrow$ (2) Let $A\subseteq X$. Assume $y\notin f(A^{a^*})$. This implies that $f^{-1}(y)\notin A^{a^*}$, and hence there exists $U\in\tau^a(f^{-1}(y))$ such that $U\cap A\in\mathcal{I}$. Consequently $f(U)\in\sigma^{a^*}(y)$ and $f(U)\cap f(A)\in\mathcal{J}$, then $y\notin[f(A)]^{a^*}(\mathcal{J},\sigma^{a^*})=[f(A)]^{a^*}(\mathcal{J},\sigma)$. Thus $[f(A)]^{a^*}\subseteq f(A^{a^*})$. Now assume $y\notin[f(A)]^{a^*}$. This implies there exists a $V\in\sigma^{a^*}(y)$ such that $V\cap f(A)\in\mathcal{J}$, then $f^{-1}(V)\in\tau^{a^*}(f^{-1}(y))$ and $f^{-1}(V)\cap A\in\mathcal{I}$. Thus $f^{-1}(y)\notin A^{a^*}(\mathcal{I},\tau^{a^*})=A^{a^*}(\mathcal{I},\tau^a)$ and $y\notin f(A^{a^*})$. Hence $f(A^{a^*})\subseteq[f(A)]^{a^*}$ and $f(A^{a^*})=[f(A)]^{a^*}$.
(2) $\Rightarrow$ (3) Let $A\subseteq X$. Then $f(\Re_a(A))=f[X-(X-A)^{a^*}]=Y-f(X-A)^{a^*}=Y-[Y-f(A)]^{a^*}=\Re_a(f(A))$.
(3) $\Rightarrow$ (1) Let $U\in\tau^{a^*}$. Then $U\subseteq\Re_a(U)$ by Theorem 2.6 and $f(U)\subseteq f(\Re_a(U))=\Re_a(f(U))$. This shows that $f(U)\in\sigma^{a^*}$ and hence $f:(X,\tau^{a^*})\rightarrow(Y,\sigma^{a^*})$ is τ^{a^*}-open. Similarly, $f^{-1}:(Y,\sigma^{a^*})\rightarrow(X,\tau^{a^*})$ is σ^{a^*}-open and, f is $\mathcal{A}*$-homeomorphism. □

Theorem 3.6. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space, then $\langle\Re_a(\tau^{a^*})\rangle=\langle\Re_a(\tau^a)\rangle$.*

Proof. Note that for every $U\in\tau^a$ and for every $I\in\mathcal{I}$, we have $\Re_a(U-I)=\Re_a(U)$. Consequently, $\Re_a(\beta)=\Re_a(\tau^a)$ and $\langle\Re_a(\beta)\rangle=\langle\Re_a(\tau^a)\rangle$, where β is a basis for τ^a. It follows directly from Theorem 11 of [9] that $\langle\Re_a(\beta)\rangle=\langle\Re_a(\tau^{a^*})\rangle$, hence the theorem is proved. □

Theorem 3.7. *Let $f:(X,\tau,\mathcal{I})\rightarrow(Y,\sigma,\mathcal{J})$ be a bijection with $f(\mathcal{I})=\mathcal{J}$. Then the following are hold:*

(1) *If f is a $\mathcal{A}*$-homeomorphism, then f is a $\Re_a$-homeomorphism.*
(2) *If $\tau\sim^a\mathcal{I}$ and $\sigma\sim^a\mathcal{J}$ and f is a $\Re_a$-homeomorphism, then f is a $\mathcal{A}*$-homeomorphism.*

Proof. (1) Assume $f:(X,\tau^{a^*})\rightarrow(Y,\sigma^{a^*})$ is a $\mathcal{A}*$-homeomorphism, and let $\Re_a(U)$ be a basic open set in $\langle\Re_a(\tau^a)\rangle$ with $U\in\tau^a$. Then $f(\Re_a(U))=\Re_a(f(U))$ by Theorem 3.5. Then $f(\Re_a(U))\in\Re_a(\sigma^{a^*})$, but $\langle\Re_a(\tau^{a^*})\rangle=\langle\Re_a(\tau^a)\rangle$ by Theorem 3.6. Thus $f:(X,\Re_a(\tau))\rightarrow(Y,\Re_a(\sigma))$ is a-open. Similarly, $f^{-1}:(Y,\Re_a(\sigma))\rightarrow(X,\Re_a(\tau))$ is a-open and f is $\Re_a$-homeomorphism.
(2) Assume f is a $\Re_a$-homeomorphism, then $f(\Re_a(A))=\Re_a(f(A))$ for every $A\subseteq X$ by Theorems 3.4 and 3.3. Thus f is a $\mathcal{A}*$-homeomorphism by Theorem 3.5. □

4. Results related to topological groups

Given a topological group $(X,\tau,.)$ and an ideal $\mathcal{I}$ on X, denoted $(X,\tau,\mathcal{I},.)$ and $x\in X$, we denote by $x\mathcal{I}=\{xI:I\in\mathcal{I}\}$. We say that $\mathcal{I}$ is left translation invariant if for every $x \in X$ we have $x\mathcal{I} \subseteq \mathcal{I}$. Observe that if $\mathcal{I}$ is left translation invariant then $x\mathcal{I} = \mathcal{I}$ for every $x \in X$. We define $\mathcal{I}$ to be right translation invariant if and only if $\mathcal{I}x = \mathcal{I}$ for every $x \in X$ [8].

Given a topological group $(X, \tau, \mathcal{I})$, $\mathcal{I}$ is said to be τ^a-boundary [2], if $\tau^a \cap \mathcal{I} = \{\phi\}$.
Note that if $\mathcal{I}$ is left or right translation invariant, $X \notin \mathcal{I}$, and $I \sim^a \mathcal{I}$, then $\mathcal{I}$ is τ^a-boundary.

Definition 4.1 ([2])**.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is called a Baire set with respect to τ^a and $\mathcal{I}$, denoted $A \in \mathcal{B}_r(X, \tau, \mathcal{I})$, if there exists a a-open set U such that $A = U$ [mod $\mathcal{I}$]. Let $\mathcal{U}(X, \tau, \mathcal{I})$ be denoted $\{A \subseteq X :$ there exists $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A\}$.

Lemma 4.2. *Let (X, τ) and (X, σ) be two topological spaces and $\mathcal{F}$ be a collection of a-open mappings from X to Y. Let $U \in \tau^a - \{\phi\}$ and let A be a non empty subset of U. If $f(U) \in \mathcal{F}(A) = \{f(A) : f \in \mathcal{F}\}$ for every $f \in \mathcal{F}$, Then $\mathcal{F}(A) \in \sigma^a - \{\phi\}$.*

Proof. Let $y \in \mathcal{F}(A)$, then there exist $f \in \mathcal{F}$ such that $y \in f(A)$. Now, $A \subseteq U$, then $f(A) \subseteq f(U)$ and $y \in f(U)$. Then $f(U)$ is a-open in (Y, σ) (as f is a-open map). So there exists $V \in \sigma^a(y)$ such that $y \in V \subseteq f(U) \subseteq \mathcal{F}(A)$. So $\mathcal{F}(A) \in \sigma^a - \{\phi\}$. □

Theorem 4.3. *Let (X, τ) and (X, σ) be two topological spaces and $\mathcal{I}$ be an ideal (X, τ) with $\tau \sim^a \mathcal{I}$ and $\tau^a \cap \mathcal{I} = \{\phi\}$. Moreover, let $U \in \tau^a - \{\phi\}$, $A \subseteq X$, $U \subseteq A^{a*} \cap \Re_a(A)$ and $\mathcal{F}$ be a non-empty collection of a-open mappings from X to Y. Suppose $y \in \mathcal{F}(U) \Rightarrow U \cap \mathcal{F}^{-1}(y) \notin \mathcal{I}$, where $\mathcal{F}^{-1}(y) = \cup\{f^{-1}(y) : f \in \mathcal{F}\}$. Then $\mathcal{F}(U \cap A) \in \sigma^a - \{\phi\}$.*

Proof. Since U is a non-empty a-open set contained in $A^{a*} \cap \Re_a(A)$ and $\tau \sim^a \mathcal{I}$, by Proposition 2.10 it follows that $U - A \in \mathcal{I}$ and $U \cap A \notin \mathcal{I}$. For any $y \in \mathcal{F}(U)$, $U \cap \mathcal{F}^{-1}(y) \notin \mathcal{I}$ (by hypothesis) and we have $U \cap \mathcal{F}^{-1}(y) = U \cap \mathcal{F}^{-1}(y) \cap (A \cup A^c) = [U \cap \mathcal{F}^{-1}(y) \cap A] \cup [U \cap \mathcal{F}^{-1}(y) \cap A^c] \subseteq [U \cup \mathcal{F}^{-1}(y) \cap A] \cup (U - A)$ (where $A^c =$ complement of A). Since $U \cap \mathcal{F}^{-1}(y) \notin \mathcal{I}$ and $U - A \in \mathcal{I}$, we have $U \cap \mathcal{F}^{-1}(y) \cap A \notin \mathcal{I}$. Then for any $y \in \mathcal{F}(U)$, $U \cap \mathcal{F}^{-1}(y) \cap A \neq \{\phi\}$. Now for a given $f \in \mathcal{F}$, $k \in f(U) \Rightarrow k \in \mathcal{F}(U)$, then there exist $x \in U \cap A$ and $x \in g^{-1}(k)$ for some $g \in \mathcal{F}$, where $k = g(x) \Rightarrow k \in g(U \cap A)$, and $k \in \mathcal{F}(U \cap A)$. Hence $f(U) \subseteq \mathcal{F}(U \cap A)$, for all $f \in \mathcal{F}$. Then $\mathcal{F}(U \cap A) \in \sigma^a - \{\phi\}$ by Lemma 4.2. □

Lemma 4.4. *Let $\mathcal{I}$ be a left (right) translation invariant ideal on a topological group $(X, \tau, .)$ and $x \in X$. Then for any $A \subseteq X$ the following hold:*

(1) $x\Re_a(A) = \Re_a(xA)$, *and* $\Re_a(A)x = \Re_a(Ax)$,
(2) $xA^{a^*} = (xA)^{a^*}$ *(resp.* $A^{a^*}x = (Ax)^{a^*}$*).*

Proof. We assume that $\mathcal{I}$ is right translation invariant, the proof is similar for the case when $\mathcal{I}$ is left translation invariant would be .
(1) We first note that for any two subsets A and B of X, $(A-B)x = Ax - Bx$. In fact, $y \in (A-B)x$, then $y = tx$, for some $t \in A - B$. Now $t \in A$ then $tx \in Ax$. But $tx \in Bx \Rightarrow tx = bx$ for some $b \in B \Rightarrow t = b \in B$ a contradiction. So $y = tx \in Ax - Bx$. Again, $y \in Ax - Bx \Rightarrow y \in Ax$ and $y \notin Bx \Rightarrow y = ax$ for some $a \in A$ and $ax \notin Bx \Rightarrow a \notin B \Rightarrow y = ax$, where $a \in A - B \Rightarrow y \in (A-B)x$. Now, $y \in \Re_a(Ax) \Rightarrow y \in Ux$ for some $U \in \tau^a$ with $U - A \in \mathcal{I}$. Then $Ux = V \in \tau^a$ and $(U-A)x = Ux - Ax \in \mathcal{I}$ where $Ux \in \tau^a$. Then $y \in V$, where $V \in \tau^a$ and $V - Ax \in \mathcal{I} \Rightarrow y \in \cup\{V \in \tau^a : V - Ax \in \mathcal{I}\} = \Re_a(Ax)$. Thus $x\Re_a(A) \subseteq \Re_a(Ax)$.
Conversely, let $y \in \Re_a(Ax) = \cup\{U \in \tau^a : U - Ax \in \mathcal{I}\} \Rightarrow y \in U \in \tau^a$, where $U - Ax \in \mathcal{I}$. Put $V = Ux^{-1}$. Then $V \in \tau^a$. Now $yx^{-1} \in V$ and $V - A = Ux^{-1} - A = (U - Ax)x^{-1} \in \mathcal{I} \Rightarrow yx^{-1} \in \Re_a(A) \Rightarrow y \in \Re_a(A)x$. Thus $\Re_a(Ax) \subseteq \Re_a(A)x$ and hence $\Re_a(A)x = \Re_a(Ax)$
(2) In view of (1) $\Re_a(X-A)x = \Re_a((X-A)x)$, then $[X - A^{a^*}]x = X - (Ax)^{a^*}$ and $X - A^{a^*}x = X - (Ax)^{a^*}$ thus $A^{a^*}x = (Ax)^{a^*}$. □

Lemma 4.5. *Let $\mathcal{I}$ be an ideal space on a topological group $(X, \tau, .)$ such that $\mathcal{I}$ is left or right translation invariant and $\tau \sim^a \mathcal{I}$. Then $\mathcal{I} \cap \tau^a = \{\phi\}$.*

Proof. Since $X \notin \mathcal{I}$ and $\tau \sim^a \mathcal{I}$, by Theorem 2.4 there exist $x \in X$ such that for all $U \in \tau^a(x)$,

$$U = U \cap X \notin \mathcal{I} \tag{4.1}$$

Let $V \in \mathcal{I} \cap \tau^a$. If $V = \{\phi\}$ we have nothing to show. Suppose $V \neq \{\phi\}$. Without loosing of generality we may assume that $i \in V$ (i denoted the identity of X). For $y \in V$ then $y^{-1}V \in \tau^a$ and $y^{-1}V \in y^{-1}\mathcal{I}$ so that $y^{-1}V \in \mathcal{I}$ where $i \in y^{-1}V$. Thus $xV \in \tau^a$ and $xV \in x\mathcal{I}$ and hence $xV \in \mathcal{I}$. Thus $xV \in \tau^a \cap \mathcal{I}$, where xV is a neighborhood of x, which is contradicting (4.1) and hence $\mathcal{I} \cap \tau^a = \{\phi\}$. □

Theorem 4.6. *Let $(X, \tau, .)$ be a topological group and $\mathcal{I}$ be an ideal on X such that $\tau \sim^a \mathcal{I}$. Let $P \in \mathcal{U}(X, \tau, \mathcal{I})$ and $S \in \mathcal{P}(X) - \mathcal{I}$. Let $U, V \in \tau^a$ such that $U \cap S^{a^*} \neq \{\phi\}$, $V \cap aInt(P^{a^*}) \cap \Re_a(P) \neq \{\phi\}$. If $A = U \cap S \cap S^{a^*}$ and $B = V \cap aInt(P^{a^*}) \cap P \cap \Re_a(P)$ then the following hold:*

1. *If $\mathcal{I}$ is left translation invariant, then BA^{-1} is a non-empty a-open set contained in PS^{-1}.*
2. *If $\mathcal{I}$ is right translation invariant, then $A^{-1}B$ is a non-empty a-open set contained in $S^{-1}P$.*

Proof. (1) Since X is a topological group, $\tau \sim^a \mathcal{I}$ and $\mathcal{I}$ is right translation invariant, we have by Lemma 4.5, $\mathcal{I} \cap \tau^a = \{\phi\}$. Now by Theorem 2.2 $(U \cap S \cap S^{a^*})^{a^*} \subseteq (U \cap S)^{a^*}$ and by Theorem 2.7 we get $(U \cap S \cap (U \cap S)^{a^*})^{a^*} = (U \cap S)^{a^*}$. Hence

$$(U \cap S \cap S^{a^*})^{a^*} = (U \cap S)^{a^*} \tag{4.2}$$

Thus by Theorem 2.2 we have $U \cap S^{a^*} = U \cap (U \cap S)^{a^*} \subseteq (U \cap S)^{a^*} = (U \cap S \cap S^{a^*})^{a^*}$ by (*). Since $U \cap S^{a^*} \neq \{\phi\}$, we have $A \neq \{\phi\}$. Again, $A^{a^*} = (U \cap S \cap S^{a^*})^{a^*} \supseteq U \cap S^{a^*} \supseteq U \cap S^{a^*} \cap S = A$ i.e. $A \subseteq A^{a^*}$. For each $a \in A$, define $f_a : X \to X$ given by $f_a(x) = xa^{-1}$, and $\mathcal{F} = \{f_a : a \in A\}$. Since $A \neq \{\phi\}$, $\mathcal{F} \neq \{\phi\}$ and each f_a is a homeomorphism. Let $G = V \cap aInt((P)^{a^*}) \cap \Re_a(P)$. Now it is sufficient to show that $G \cap \mathcal{F}^{-1}(y) \notin \mathcal{I}$ for every $y \in \mathcal{F}(G)$. Because then by Theorem 4.3, $\mathcal{F}(G \cap P) = \mathcal{F}(B) = BA^{-1}$ is a non-empty a-open set in X contained in PS^{-1}. Let $y \in \mathcal{F}(G)$. Then $y = xa^{-1}$ for some $a \in A$ and $x \in G \Rightarrow \mathcal{F}^{-1}(y) = xa^{-1}A$. Thus $x \in xa^{-1}A \subseteq xa^{-1}A^{a^*}$ (as $A \subseteq A^{a^*}$) $\subseteq (xa^{-1}A)^{a^*}$ (by Lemma 4.4) $= (\mathcal{F}^{-1}(y))^{a^*} \Rightarrow N_x \cap \mathcal{F}^{-1}(y) \notin \mathcal{I}$ for some $N_x \in \tau^a(x)$. Thus BA^{-1} is a nonempty a-open subset of PS^{-1}. So in particular, as (2) is similar to (1). □

Corollary 4.7. *Let $(X, \tau, .)$ be a topological group and $\mathcal{I}$ be an ideal on X such that $\tau \sim^a \mathcal{I}$. Let $A \in \mathcal{U}(X, \tau, \mathcal{I})$ and $B \in P(X) - \mathcal{I}$.*

(1) *If $\mathcal{I}$ is right translation invariant, then $[B \cap B^{a^*}]^{-1}[A \cap aInt(A^{a^*}) \cap \Re_a(A)]$ is a non-empty a-open set contained in $B^{-1}A$.*
(2) *If $\mathcal{I}$ is left translation invariant, then $[A \cap aInt(A^{a^*}) \cap \Re_a(A)][B \cap B^{a^*}]^{-1}$ is a non-empty a-open set contained in AB^{-1}.*

Proof. We only show that $B^{a^*} \neq \{\phi\}$ and $A \cap aInt(A^{a^*}) \cap \Re_a(A) \neq \{\phi\}$, the rest follows from Theorem 4.6 by taking $U = V = X$. In fact, if $B^{a^*} = \{\phi\}$, then $B \cap B^{a^*} = \{\phi\}$ which gives in view of Theorem 2.4, $B \in \mathcal{I}$, a contradiction.
Again, $A \in \mathcal{U}(X, \tau, \mathcal{I}) \Rightarrow aInt(A^{a^*}) \cap \Re_a(A) \neq \{\phi\}$ (by Lemma 4.5 and Proposition 2.9) $\Rightarrow aInt(A^{a^*}) \cap \Re_a(A) \in \tau^a - \{\phi\}$. Now, $aInt(A^{a^*}) \cap \Re_a(A) = [A \cap aInt(A^{a^*}) \cap \Re_a(A)] \cup [A^c \cap aInt(A^{a^*}) \cap \Re_a(A)] \notin \mathcal{I}$ (by Lemma 4.5). Then $[A^c \cap aInt(A^{a^*}) \cap \Re_a(A)] \subseteq [A^c \cap \Re_a(A)] = \Re_a(A) - A \in \mathcal{I}$ by Theorem 2.11. Thus $A \cap aInt(A^{a^*}) \cap \Re_a(A) \notin \mathcal{I}$ and hence $A \cap aInt(A^{a^*}) \cap \Re_a(A) \neq \{\phi\}$. □

Corollary 4.8. *Let $(X, \tau, .)$ be a topological group and $\mathcal{I}$ be an ideal on X such that $\mathcal{I} \cap \tau^a = \{\phi\}$ and $A \in \mathcal{U}(X, \tau, \mathcal{I})$.*

(1) *If $\mathcal{I}$ is left translation invariant, then $e \in aInt(A^{-1}A)$.*
(2) *If $\mathcal{I}$ is right translation invariant, then $e \in aInt(AA^{-1})$.*
(3) *If $\mathcal{I}$ is left as well as right translation invariant, then $e \in aInt(AA^{-1} \cap A^{-1}A)$.*

Proof. It suffices to prove (1) only. We have, $A \in \mathcal{U}(X, \tau, \mathcal{I})$ then there exists $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A$. Now for any $x \in X$, $\Re_a(B)x \cap \Re_a(B) = \Re_a(Bx) \cap \Re_a(B) = \Re_a(Bx \cap B)$ (by Lemma 4.4 and Theorem 2.6). Thus if $\Re_a(B)x \cap \Re_a(B) \neq \{\phi\}$, then $Bx \cap B \neq \{\phi\}$. Now, if $x \in [\Re_a(B)]^{-1}[\Re_a(B)]$ then $x = y^{-1}z$ for some $y, z \in \Re_a(B)$, then $yx = z = t$ (say) $\Rightarrow t \in \Re_a(B)x$ and $t \in \Re_a(B) \Rightarrow \Re_a(B)x \cap \Re_a(B) \neq \{\phi\} \Rightarrow x \in \{x \in X : \Re_a(B)x \cap \Re_a(B) \neq \{\phi\}\}$ then $[\Re_a(B)]^{-1}[\Re_a(B)] \subseteq \{x \in X : \Re_a(B)x \cap \Re_a(B) \neq \{\phi\}\} \subseteq \{x \in X : Bx \cap B \neq \{\phi\}\} \subseteq B^{-1}B \subseteq A^{-1}A$. Since $\Re_a(B) \neq \{\phi\}$ by Proposition 2.9 as $B \in \mathcal{U}(X, \tau, \mathcal{I})$ and $\Re_a(B)$ is a-open for any $B \subseteq X$, we have $e \in [\Re_a(B)]^{-1}[\Re_a(B) \subseteq aInt(A^{-1}A)$. □

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