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When is a space Menger at infinity?

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Abstract

We try to characterize those Tychonoff spaces X such that $\beta X \setminus X$ has the Menger property.

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1. Introduction

A space X is Menger (or has the Menger property) if for any sequence of open coverings $\{\mathcal{U}_n : n < \omega\}$ one may pick finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n$ in such a way that $\bigcup\{\mathcal{V}_n : n < \omega\}$ is a covering. This equivals to say that X satisfies the selection principle $\mathsf{S}_{\rm fin}(\mathcal{O}, \mathcal{O})$. It is easy to see the following chain of implications:

$$\sigma\text{-compact} \longrightarrow \text{Menger} \longrightarrow \text{Lindelöf}$$

An important result of Hurewicz [4] states that a space X is Menger if and only if player 1 does not have a winning strategy in the associated game $\mathsf{G}_{\rm fin}(\mathcal{O}, \mathcal{O})$ played on X. This highlights the game-theoretic nature of the Menger property, see [7] for more.

Henriksen and Isbell ([3]) proposed the following:

Definition 1.1. A Tychonoff space X is Lindelöf at infinity if $\beta X \setminus X$ is Lindelöf.

They discovered a very elegant duality in the following:

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Proposition 1.2 ([3]). *A Tychonoff space is Lindelöf at infinity if and only if it is of countable type.*

A space X is of countable type provided that every compact set can be included in a compact set of countable character in X.

A much easier and well-known fact is:

Proposition 1.3. *A Tychonoff space is Čech-complete if and only if it is σ-compact at infinity.*

These two propositions suggest the following:

Question 1.4. *When is a Tychonoff space Menger at infinity?*

Before beginning our discussion here, it is useful to note these well known facts:

Proposition 1.5. *The Menger property is invariant by perfect maps.*

Corollary 1.6. *X is Menger at infinity if, and only if, for any Y compactification of X, $Y \setminus X$ is Menger.*

Fremlin and Miller [6] proved the existence of a Menger subspace X of the unit interval $[0, 1]$ which is not σ-compact. The space X can be taken nowhere locally compact and so $Y = [0, 1] \setminus X$ is dense in $[0, 1]$. Since the Menger property is invariant under perfect mappings, we see that $\beta Y \setminus Y$ is still Menger. Therefore, a space can be Menger at infinity and not σ-compact at infinity. Another example of this kind, stronger but not second countable, is Example 3.1 in the last section.

On the other hand, the irrational line shows that a space can be Lindelöf at infinity and not Menger at infinity.

Consequently, the property $\mathcal{M}$ characterizing a space to be Menger at infinity strictly lies between countable type and Čech-complete.

Of course, taking into account the formal definition of the Menger property, we cannot expect to have an answer to Question 1.4 as elegant as Henriksen-Isbell's result.

2. A characterization

Definition 2.1. Let $K \subset X$. We say that a family $\mathcal{F}$ is a **closed net at** K if each $F \in \mathcal{F}$ is a closed set such that $K \subset F$ and for every open A such that $K \subset A$, there is an $F \in \mathcal{F}$ such that $F \subset A$.

Lemma 2.2. *Let X be a T_1 space. If $(F_n)_{n\in\omega}$ is a closed net at K, for $K \subset X$ compact, then $K = \bigcap_{n\in\omega} F_n$.*

Proof. Simply note that for each $x \notin K$, there is an open set V such that $K \subset V$ and $x \notin V$. □

Lemma 2.3. *Let Y be a regular space and let X be a dense subspace of Y. Let $K \subset X$ be a compact subset. If $(F_n)_{n\in\omega}$ is a closed net at K in X, then $(\overline{F_n}^Y)_{n\in\omega}$ is a closed net at K in Y.*

Proof. In the following, all the closures are taken in Y. Let A be an open set in Y such that $K \subset A$. By the compactness of K and the regularity of Y, there is an open set B such that $K \subset B \subset \overline{B} \subset A$. Thus, there is an $n \in \omega$ such that $K \subset F_n \subset B \cap X$. Note that $K \subset \overline{F}_n \subset \overline{B} \subset A$. □

Lemma 2.4. *Let X be a compact Hausdorff space. If $K = \bigcap_{n\in\omega} F_n$, where $(F_n)_{n\in\omega}$ is a decreasing sequence of closed sets, then $(F_n)_{n\in\omega}$ is a closed net at K.*

Proof. If not, then there is an open set V such that $K \subset V$ and, for every $n \in \omega$, $F_n \setminus V \neq \varnothing$. By compactness, there is an $x \in \bigcap_{n\in\omega} F_n \setminus V = K \setminus V$. Contradiction with the fact that $K \subset V$. □

Theorem 2.5. *Let X be a Tychonoff space. X is Menger at infinity if, and only if, X is of countable type and for every sequence $(K_n)_{n\in\omega}$ of compact subsets of X, if $(F^n_p)_{p\in\omega}$ is a decreasing closed net at K_n for each n, then there is an $f : \omega \longrightarrow \omega$ such that $K = \bigcap_{n\in\omega} F^n_{f(n)}$ is compact and $(\bigcap_{k\leq n} F^k_{f(k)})_{n\in\omega}$ is a closed net for K.*

Proof. In the following, every closure is taken in βX.

Suppose that X is Menger at infinity. By Lemma 1.2 X is of countable type. Let $(F^n_p)_{p,n\in\omega}$ be as in the statement. Note that, by Lemma 2.3 and Lemma 2.2, $\bigcap_{p\in\omega} F^n_p = \bigcap_{p\in\omega} \overline{F^n_p}$ for each $n \in \omega$. Thus, for each $n \in \omega$, $(V^n_p)_{p\in\omega}$, where $V^n_p = \beta X \setminus \overline{F^n_p}$, is an increasing covering for $\beta X \setminus X$. Since $\beta X \setminus X$ is Menger, there is an $f : \omega \longrightarrow \omega$ such that $\beta X \setminus X \subset \bigcup_{n\in\omega} V^n_{f(n)}$. Note that $K = \bigcap_{n\in\omega} \overline{F^n_{f(n)}}$ is compact and it is a subset of X. By Lemma 2.4, $(\bigcap_{k\leq n} \overline{F^k_{f(k)}})_{n\in\omega}$ is a closed net at K in βX, therefore, $(\bigcap_{k\leq n} F^k_{f(k)})_{n\in\omega}$ is a closed net at K in X. Conversely, for each $n \in \omega$, let $\mathcal{W}_n$ be an open covering for $\beta X \setminus X$. We may suppose that each $W \in \mathcal{W}_n$ is open in βX. By regularity, we can take a refinement $\mathcal{V}_n$ of $\mathcal{W}_n$ such that, for every $x \in \beta X \setminus X$, there is a $V \in \mathcal{V}_n$ such that $x \in V \subset \overline{V} \subset W_V$ for some $W_V \in \mathcal{W}_n$. Since X is of countable type, By Lemma 1.2 we may suppose that each $\mathcal{V}_n$ is countable. Fix an enumeration for each $\mathcal{V}_n = (V^n_k)_{k\in\omega}$. Define $A^n_k = \beta X \setminus (\bigcup_{j\leq k} \overline{V^n_j})$. Note that each $K_n = \bigcap_{k\in\omega} \overline{A^n_k}$ is compact and a subset of X. By Lemma 2.4, $(\overline{A^n_k})_{k\in\omega}$ is a closed net at K_n. Thus, $(\overline{A^n_k} \cap X)_{k\in\omega}$ is a closed net at K_n in X. Therefore, there is $f : \omega \longrightarrow \omega$ such that $K = \bigcap_{n\in\omega}(\overline{A^n_{f(n)}} \cap X)$ is compact and $(\bigcap_{k\leq f(n)} \overline{A^k_{f(k)}} \cap X)_{n\in\omega}$ is a closed net at K. So, by Lemma 2.3, $K = \bigcap_{n\in\omega} \overline{(\overline{A^n_{f(n)}} \cap X)}$. Since $\bigcap_{n\in\omega} \overline{(\overline{A^n_{f(n)}} \cap X)} = \bigcap_{n\in\omega} \overline{A^n_{f(n)}}$ and by the fact that $K \subset X$, it follows that $\beta X \setminus X \subset \bigcup_{n\in\omega} \beta X \setminus \overline{A^n_{f(n)}} \subset \bigcup_{n\in\omega} Int(\bigcup_{j\leq f(n)} \overline{V^n_j}) \subset \bigcup_{n\in\omega} \bigcup_{j\leq f(n)} W_{V^n_j}$. Therefore, letting $\mathcal{U}_n = \{W_{V^n_j} : j \leq f(n)\} \subset \mathcal{W}_n$, we see that the collection $\bigcup_{n\in\omega} \mathcal{U}_n$ covers $\beta X \setminus X$, and we are done. □

Property $\mathcal{M}$ given in the above theorem does not look very nice and we wonder whether there is a simpler way to describe it, at least in some special cases.

Recall that a metrizable space is always of countable type. Moreover, a metrizable space is complete if and only if it is σ-compact at infinity. Therefore, we could hope for a "nicer" $\mathcal{M}$ in this case.

Question 2.6. *What kind of weak completeness characterizes those metrizable spaces which are Menger at infinity?*

Proposition 2.7. *Let X be a Tychonoff space. If X is Menger at infinity then for every sequence $(K_n)_{n\in\omega}$ of compact sets, there is a sequence $(Q_n)_{n\in\omega}$ of compact sets such that:*

(1) *each $K_n \subset Q_n$;*
(2) *each Q_n has a countable base at X;*
(3) *for every sequence $(B^n_k)_{n,k\in\omega}$ such that, for every $n \in \omega$, $(B^n_k)_{k\in\omega}$ is a decreasing base at K_n, then there is a function $f : \omega \longrightarrow \omega$ such that $K = \bigcap_{n\in\omega} \overline{B^n_{f(n)}}$ is compact and $(\bigcap_{k\leq n} \overline{B^k_{f(k)}})_{n\in\omega}$ is a closed net at K.*

Proof. Suppose X is Menger at infinity. Let $(K_n)_{n\in\omega}$ be a sequence of compact sets. Since X is Menger at infinity, X is Lindelöf at infinity. Thus, by Proposition 1.2, for each K_n, there is a compact $Q_n \supset K_n$ such that Q_n has a countable base. Now, let $(B^n_k)_{k,n}$ be as in 3. Since each Q_n is compact and X is regular, each $(\overline{B^n_k})_{k\in\omega}$ is a decreasing closed net at Q_n. Thus, by Theorem 2.5, there is an $f : \omega \longrightarrow \omega$ as we need. □

To some extent, the Menger property is closer to σ- compactness rather than to Lindelöfness. Since a Čech- complete space has the Baire property, we may ask:

Question 2.8. *Is it true that a space Menger at infinity has the Baire property?*

We thank M. Sakai for calling our attention to the above question. He also noticed a partial answer to it:

Theorem 2.9 (Sakai)**.** *Let X be a first countable Tychonoff space. If X is Menger at infinity, then X is hereditarily Baire.*

Proof. According to a result of Debs [2], a regular first countable space is hereditarily Baire if and only if it contains no closed copy of the space of rationals $\mathbb{Q}$. To finish, it suffices to observe that $\mathbb{Q}$ is not Menger at infinity. □

We end this section presenting a selection principle that at first glance could be related with the Menger at infinity property.

Definition 2.10. We say that a family $\mathcal{U}$ of open sets of X is an **almost covering** for X if $X \setminus \bigcup\mathcal{U}$ is compact. We call $\mathcal{A}$ the family of all almost coverings for X.

Note that the property "being Menger at infinity" looks like something as $\mathsf{S}_{\rm fin}(\mathcal{A},\mathcal{A})$, but for a narrow class of $\mathcal{A}$. We will see that the "narrow" part is important.

Proposition 2.11. *If X satisfies $\mathsf{S}_{\rm fin}(\mathcal{A},\mathcal{A})$, then X is Menger.*

Proof. Let $(\mathcal{U}_n)_{n\in\omega}$ be a sequence of coverings of X. By definition, for each $n \in \omega$, there is a finite $U_n \subset \mathcal{U}_n$, such that $K = X \setminus \bigcup_{n\in\omega} \bigcup U_n$ is compact. Therefore, there is a finite $W \subset U_n$ such that $K \subset \bigcup W$. Thus, $X = W \cup \bigcup_{n\in\omega} \bigcup U_n$. □

Example 2.12. The space of the irrationals is an example of a space that is Menger at infinity but does not satisfy $\mathsf{S}_{\text{fin}}(\mathcal{A}, \mathcal{A})$ (by the Proposition 2.11).

Example 2.13. The one-point Lindelöfication of a discrete space of cardinality $\aleph_1$ is an example of a Menger space which does not satisfy $\mathsf{S}_{\text{fin}}(\mathcal{A}, \mathcal{A})$.

Example 2.14. ω is an example of a space that satisfies $\mathsf{S}_{\text{fin}}(\mathcal{A}, \mathcal{A})$, but it is not compact.

Proof. Let $(\mathcal{V}_n)_{n\in\omega}$ be a sequence of almost coverings for ω. Therefore, for each n, $F_n = \omega \setminus \bigcup \mathcal{V}_n$ is finite. For each n, let $V_n \subset \mathcal{V}_n$ be a finite subset such that $F_{n+1} \setminus F_n \subset \bigcup V_n$ and $\min(\omega \setminus \bigcup_{k<n} V_k) \in V_n$. Note that $\omega \setminus \bigcup_{n\in\omega} \bigcup V_n = F_0$. □

3. More than Menger at infinity

One may wonder whether the hypothesis "player 2 has a winning strategy in the Menger game $\mathsf{G}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ played on $\beta X \setminus X$" is strong enough to guarantee that X is Čech-complete. It turns out this is not the case, as the following example shows.

Example 3.1. Take the usual space of rational numbers $\mathbb{Q}$ and an uncountable discrete space D. Let $Y = \mathbb{Q} \times D \cup \{p\}$ be the one-point Lindelöfication of the space $\mathbb{Q} \times D$ and then let $X = \beta Y \setminus Y$. Since Y is nowhere locally compact, we have $Y = \beta X \setminus X$. X is not Čech-complete, since Y is not σ-compact, but player 2 has a winning strategy in $\mathsf{G}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ played on $\beta X \setminus X$. The latter assertion easily follows by observing that any open set containing p leaves out countably many points.

Therefore, to ensure the Čech-completeness of X, we need to assume something more on the space (see for instance Corollary 3.3 below). Moreover, the first example presented in the introduction shows that a metrizable space (actually a subspace of the real line) can be Menger at infinity, but not favorable for player 2 in the Menger game at infinity (see again Corollary 3.3).

Recall that a space X is sieve complete [5] if there is an indexed collection of open coverings $\langle \{U_i : i \in I_n\} : n < \omega \rangle$ together with mapps $\pi_n : I_{n+1} \to I_n$ such that $U_i = X$ for each $i \in I_0$ and $U_i = \bigcup \{U_j : j \in \pi_n^{-1}(i)\}$ for all $i \in I_n$. Moreover, we require that for any sequence of indexes $\langle i_n : n < \omega \rangle$ satisfying $\pi_n(i_{n+1}) = i_n$ if $\mathcal{F}$ is a filterbase in X and U_{i_n} contains an element of $\mathcal{F}$ for each $n < \omega$, then $\mathcal{F}$ has a cluster point.

Every Čech-complete space is sieve complete and every sieve complete space contains a dense Čech-complete subspace. In addition, a paracompact sieve complete space is Čech-complete and a sieve complete space is of countable type [9].

Telgársky presented a characterization of sieve completeness in terms of the Menger game played on $\beta X \setminus X$ (note that in [8] the Menger game is called the Hurewicz game and is denoted by $H(X)$):

Theorem 3.2 (Telgársky [8]). *Let X be a Tychonoff space. $\beta X \setminus X$ is favorable for player 2 in the Menger game if and only if X is sieve complete.*

Since a sieve-complete space has the Baire property, Question 2.8 has a positive answer for spaces which are Menger favorable at infinity.

Taking into account that a paracompact sieve complete space is Čech-complete, we immediately get:

Corollary 3.3. *Let X be a paracompact Tychonoff space. X is Čech-complete if and only if player 2 has a winning strategy in the game $\mathsf{G}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ played on $\beta X \setminus X$.*

In particular:

Corollary 3.4. *A metrizable space X is complete if and only if player 2 has a winning strategy in $\mathsf{G}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ played on $\beta X \setminus X$.*

Corollary 3.5. *A topological group G is Čech-complete if and only if player 2 has a winning strategy in $\mathsf{G}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ played on $\beta G \setminus G$.*

Proof. Every topological group of countable type is paracompact. □

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