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Fuzzy metric spaces and applications to perceptual colour-differences

Memoria presentada por

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Certifican: que la presente memoria *"Fuzzy metric spaces and applications to perceptual colour-differences"* ha sido realizada bajo su dirección por D. Juan José Miñana Prats, en el Departamento de Matemática Aplicada de la Universitat Politècnica de València, y constituye su tesis para optar al grado de Doctor.

Y para que así conste, presentan la referida tesis, firmando el presente certificado.

Valencia, marzo de 2015

Fdo. Valentín Gregori Gregori

Fdo. Samuel Morillas Gómez

Agraïments

M'agradaria agrair el suport de totes aquelles persones que d'una o altra manera han contribuït a la realització d'aquesta tesi.

En primer lloc als directors d'aquesta tesi, Valentín Gregori i Samuel Morillas, per dedicar part del seu temps a la meva formació com a investigador i a orientar-me en la meva carrera investigadora.

En segon lloc, m'agradaria agrair la confiança dipositada en mi per part dels meus pares, que després d'alguns fracassos, em recolzaren a l'hora de començar la Llicenciatura en Matemàtiques, a la meva germana Aida, per formar part de la meva vida, i especialment a la meva companya de viatge Ana, per estar sempre al meu costat en tot, sigui on sigui i faça el fred que faça.

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RESUMEN ESPAÑOL

La matemática fuzzy ha constituido un amplio campo en la investigación, desde que en 1965 L. A. Zadeh introdujo el concepto de conjunto fuzzy. En particular, la construcción de una teoría satisfactoria de espacios métricos fuzzy ha sido un problema investigado por muchos autores. En 1994, George y Veeramani introdujeron y estudiaron una noción de espacio métrico fuzzy que constituía una modificación de la anteriormente dada por Kramosil y Michalek. Muchos autores han contribuido al estudio de este tipo de métricas fuzzy, desde el punto de vista matemático y de sus aplicaciones. En esta tesis hemos contribuido al desarrollo del estudio de estas métricas fuzzy, desde el punto de vista matemático, y hemos abordado el problema de la medida de la diferencia perceptual de color utilizando una de estas métricas.

Las contribuciones que aportamos en esta tesis a dicho estudio, se resumen a continuación:

(i) Hemos hecho un estudio detallado del espacio métrico fuzzy $(X, M, \cdot)$ donde M está dada sobre $[0, \infty[$ por la expresión $M(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ y de otros espacios métricos fuzzy relacionados con el. Como consecuencia de este estudio hemos introducido cinco cuestiones en la teoría de las métricas fuzzy relacionadas con continuidad, extensión, contractividad y completación.

(ii) Hemos respondido a una cuestión abierta construyendo un espacio métrico fuzzy $(X, M, *)$ en el cual la asignación $f(t) = \lim_n M(a_n, b_n, t)$, donde $\{a_n\}$ y $\{b_n\}$ son sucesiones M-Cauchy, no es una función continua sobre t. La respuesta a esta cuestión nos ha permitido caracterizar la clase de los espacios métricos fuzzy strong completables.

(iii) Hemos introducido y estudiado un concepto más fuerte que el de convergencia de sucesiones en espacios métricos fuzzy, al que hemos llamado s-convergencia. En nuestro estudio hemos conseguido una caracterización de aquellos espacios métricos fuzzy en los cuales toda sucesión convergente es s-convergente y hemos dado una clasificación de los espacios métricos fuzzy atendiendo a su comportamiento con respecto a los diferentes tipos de convergencia que se da en él.

(iv) Hemos estudiado, en el contexto de los espacios métricos fuzzy, cuando ciertas familias de bolas abiertas centradas en un punto son base local de este punto.

(v) Hemos respondido a dos cuestiones abiertas relacionadas con la convergencia standard, un concepto más fuerte que el de convergencia de sucesiones en espacios métricos fuzzy, introducido de forma natural a partir del concepto de sucesión de Cauchy standard (introducido en [74]). Estas respuestas nos han llevado a establecer unas condiciones bajo las cuales un concepto relacionado con el concepto de sucesión de Cauchy y un concepto relacionado con el de convergencia deberían satisfacer para ser consideradas *compatibles*.

(vi) Como aplicación práctica, hemos mostrado que una cierta métrica fuzzy es útil para medir diferencia perceptual de color entre muestras de color.

RESUMEN VALENCIANO

La matemàtica fuzzy ha constituït un ampli camp en la investigació, des que el 1965 L. A. Zadeh va introduir el concepte de conjunt fuzzy. En particular, la construcció d'una teoria satisfactòria d'espais mètrics fuzzy ha estat un problema investigat per molts autors. El 1994, George i Veeramani introduiren i estudiaren una noció d'espai mètric fuzzy que constituïa una modificació de la donada per Kramosil i Michalek anteriorment. Molts autors han contribuït a l'estudi d'aquest tipus de mètriques fuzzy, des del punt de vista matemàtic i de les seves aplicacions. En aquesta tesi hem contribuït al desenvolupament de l'estudi d'aquestes mètriques fuzzy, des del punt de vista matemàtic, i hem abordat el problema de la mesura de la diferència perceptiva de color utilitzant aquestes mètriques.

Les contribucions que aportem en aquesta tesi a tal estudi es resumeixen a continuació:

(i) Hem fet un estudi detallat de l'espai mètric fuzzy $(X, M, \cdot)$ on M està donada sobre $[0, \infty[$ per l'expressió $M(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ i d'altres espais mètrics fuzzy relacionats amb ell. Com a conseqüència d'aquest estudi hem introduït cinc qüestions en la teoria de les mètriques fuzzy relacionades amb continuïtat, extensió, contractividad i completació.

(ii) Hem respost a una qüestió oberta construint un espai mètric fuzzy $(X, M, *)$ en el qual l'assignació $f(t) = \lim_n M(a_n, b_n, t)$, on $\{a_n\}$ i $\{b_n\}$ són successions M-Cauchy, no és una funció contínua sobre t. La resposta a aquesta qüestió ens ha permès caracteritzar la classe dels espais mètrics fuzzy strong completables.

(iii) Hem introduït i estudiat un concepte més fort que el de convergència de successions en espais mètrics fuzzy, al qual hem anomenat s-Convergència. En el nostre estudi hem aconseguit una caracterització d'aquells espais mètrics fuzzy en els quals tota successió convergent és s-convergente i hem donat una classificació dels espais mètrics fuzzy atenent al seu comportament respecte als diferents tipus de convergència que es dóna en ell.

(iv) Hem estudiat, en el context dels espais mètrics fuzzy, quan certes famílies de boles obertes centrades en un punt són base local d'aquest punt.

(v) Hem respost a dues qüestions obertes relacionades amb la convergència estàndard, un concepte més fort que el de convergència de successions en espais mètrics fuzzy, introduït de forma natural a partir del concepte de successió de Cauchy estàndard (introduït en [74]). Aquestes respostes ens han portat a establir unes condicions sota les quals un concepte relacionat amb el concepte de successió de Cauchy i un concepte relacionat amb el de convergència haurien de satisfer per a ser considerats *compatibles*.

(vi) Com a aplicació pràctica, hem mostrat que una certa mètrica fuzzy és útil per mesurar la diferència perceptiva de color entre mostres de color.

RESUMEN INGLÉS

Fuzzy mathematics has constituted a wide field of research, since L. A. Zadeh introduced in 1965 the concept of fuzzy set. In particular, the problem of constructing a satisfactory theory of fuzzy metric spaces has been investigated by several authors. In 1994, George and Veeramani introduced and studied a notion of fuzzy metric space that constituted a modification of the one given by Kramosil and Michalek. Several authors have contributed to the study of this kind of fuzzy metrics, from the mathematical point of view and for their applications. In this thesis we have contributed to develop the study of these fuzzy metrics, from the mathematical point of view, and we approached the problem of measuring perceptual colour-difference between samples of colour using one of these fuzzy metrics.

The contributions of the study carried out in this thesis is summarized as follows:

(i) We have made a detailed study of the fuzzy metric space $(X, M, \cdot)$ where M is given on $X = [0, \infty[$ by $M(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ and others related to it. As a consequence we have introduced five questions in fuzzy metrics related to continuity, extension, contractivity and completion.

(ii) We have answered an open question constructing a fuzzy metric space $(X, M, *)$ in which the assignment $f(t) = \lim_n M(a_n, b_n, t)$, where $\{a_n\}$ and $\{b_n\}$ are M-Cauchy sequences in X, is not a continuous function on t. The response to this question has allowed us to characterize the class of completable strong fuzzy metric spaces.

(iii) We have introduced and studied a stronger concept than convergence of sequences in fuzzy metric spaces, which we call s-convergence. In our study, we have gotten a characterization of those spaces in which every convergent sequence is s-convergent and we have given a classification of fuzzy metrics attending to the behaviour of the fuzzy metric with respect to the different types of convergence.

(iv) We have studied, in the context of fuzzy metric spaces, when certain families of open balls centered at a point are local bases for this point.

(v) We have answered two open questions related to standard convergence, a stronger concept than convergence of sequences in fuzzy metric spaces, introduced in a natural way attending to the concept of standard Cauchy sequence (introduced in [74]). These responses have led us to establish conditions under which Cauchyness and convergence should be considered *compatible*.

(vi) As a practical application, we have shown that a certain fuzzy metric is useful for measuring perceptual colour-differences between colour samples.

Contents

Chapter 1

Introduction. Objectives

1.1 Background of study

The fuzzy theory was initiated by Lofti A. Zadeh [95] in 1965, who introduced the concept of fuzzy set as an assignment of a value in $[0,1]$ to each element of a classical set. This value represents the degree of membership of the element to the fuzzy set. Formally, given a non-empty set X, each application $A : X \to [0,1]$ is called a *fuzzy set* on X.

One of the first research topics that appeared in fuzzy mathematics was fuzzy topology. The first work on fuzzy topology was done by C. L. Chang [7] in 1968. According to Chang, a fuzzy topology $\mathcal{T}$ in X is a family of fuzzy sets on X that is closed for unions and for finite intersections. This family also contain the constant functions 0 and 1. There are other concepts of fuzzy topology. For instance, the concept of fuzzy topology introduced by R. Lowen [50, 51], the concept given, independently, by U. Höle [40] and

M. Ying [94] or the concept given by A. Šostak [85, 86] (rediscovered by Chattopadhyay, Hazra and Samanta [8]).

One of the most interesting and most studied problems in fuzzy topology is obtaining an appropriate notion of fuzzy metric space. The study of metric spaces is based on the notion of distance between points. However, in many real situations this distance cannot be exactly determined. This problem, that belongs to the fuzzy field, was previously approached from the point of view of the probability theory. Indeed, in 1942 K. Menger [55] introduced the so-called probabilistic metric spaces. These spaces have been widely studied, for instance [9, 10, 38, 80]. In the Menger's theory the concept of distance is considered to be statistical or probabilistic, i.e. he proposed to associate a distribution function F_{xy}, to every pair of elements x, y instead of associating a number, and for any positive number t, interpreted $F_{xy}(t)$ as the probability that the distance from x to y be less than t.

Kramosil and Michalek [48] gave a notion of fuzzy metric space which could be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric space due to Menger [55]. Later, George and Veeramani [19, 21] introduced and studied a notion of fuzzy metric space $(X, M, *)$, where $*$ is a continuous t-norm, which constitutes a modification of the one due to Kramosil and Michalek. From now on, by fuzzy metric we mean a fuzzy metric in the sense of George and Veeramani. We notice that many concepts and properties stated for fuzzy metrics can be given for KM-fuzzy metrics (fuzzy metrics in the sense of Kramosil and Michalek in the original version [48]) or in a modern version [19, 23]. For this reason, sometimes, the term fuzzy metric in a wide sense can make reference to any of them. Several authors have contributed to the development of this theory, for instance [56, 57, 75, 90, 91]. In particular, it has been proved that the class of topological spaces which are fuzzy metrizable agrees with the class of metrizable topological spaces [20, 30] and then, some classical theorems on metric completeness and metric (pre)compactness have been adapted to the realm of fuzzy metric spaces [30]. Nevertheless, the theory of fuzzy metric completion is, in this context, very different from the classical theories of metric completion and probabilistic metric completion. In fact, there are fuzzy metric spaces which are not completable ([31, Example 2], [32, Example 2] and [25, Example 14]). A characterization of completable fuzzy metric spaces was given in [32, Theorem 1].

This type of fuzzy metrics are interesting for Engineering problems mainly due to the following two advantages with respect to classical metrics: First, values given by fuzzy metrics are in the interval]0,1] regardless the nature of the distance concept being measured. This implies that it is easy to combine different distance criteria that may originally be in quite different ranges but fuzzy metrics take to a common range. Second, fuzzy metrics match perfectly with the employment of other fuzzy techniques since the value given by a fuzzy metric can be directly employed or interpreted as a fuzzy certainty degree. This allows to straightforwardly include fuzzy metrics as part of other complex fuzzy systems.

Recently, they have been applied to colour image filtering improving some filters when replacing classical metrics and allowing the design of new filtering methods [4, 62, 65, 66]. In fact, the use of this type of fuzzy metrics is interesting within image filtering due to three main reasons: (i) The t parameter in the fuzzy metric allows to include adaptivity to context and indeed image processing needs to be adaptive given the variability from one image to another which may be due not only to image content but also to acquisition process and device; (ii) fuzzy techniques provide an appropriate framework to develop soft-adaptive solutions to the problem of distinguishing between noise and image features and some fuzzy metrics have been found to be more appropriate in this context; and (iii) in the filtering problem usually different distance criteria need to be used simultaneously, for which fuzzy metrics are able to provide simple, efficient and effective solutions.

1.2 Objectives

The objective of this work is to continue the develop of the theory of fuzzy metric and to find for a certain fuzzy metric a practical application. From the mathematical point of view we have studied well known topics on this field as contractivity, convergence, completeness and completion.

The organization of the thesis is as follows. It is divided in eight chapters. Next, we explain, briefly, the content of each of them.

Chapter 1 describes the general background and the objectives of the thesis. Further, it contains all the necessary preliminaries about fuzzy metrics used in this work.

In Chapter 2 we study the fuzzy metrics M^* and M_0, where $M^*(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ is defined on $[0, \infty[$ and $M_0(x, y, t) = \frac{\min\{x,y\}}{\max\{x,y\}}$ is defined on $]0, \infty[$. Our study is detailed as follows. First, we show that $(]0, \infty[, M_0, \cdot)$ is complete (Theorem 4). Nevertheless we prove that $(]0, \infty[, M^*, \cdot)$ is not complete and completable, and so we have constructed its completion. Then, we study some aspects of the continuity of M_0 and the uniform continuity of M_0, fixed one component. Then, we construct an extension of M^* to $\mathbb{R}$. And finally, we see some aspects about contractivity with respect to M_0. These studies create an appropriate context to introduce five questions related to contractivity (Problem 5), continuity (Problem 3), extension (Problem 4) and completion of fuzzy metrics (Problems 1 and 2). Also, as a practical application we show that the fuzzy metric M^* can be used to approach the problem of measuring perceptual colour-difference between colour samples.

In Chapter 3 we construct a non-completable fuzzy metric space $(X, M, *)$ (Proposition 9). For it, we prove that the assignment $f(t) = \lim_n M(a_n, b_n, t)$ is a well-defined function on $]0, \infty[$, which is not continuous for two particular Cauchy sequences $\{a_n\}$ and $\{b_n\}$ of X. We also prove that the constructed fuzzy metric space is not strong (non-Archimedean).

In Chapter 4 we prove that the conditions, in our reformulation (Theorem 6), given by V. Gregori and S. Romaguera characterizing completable fuzzy metric spaces constitute an independent axiomatic system. For it we use the constructed non-completable fuzzy metric space of Chapter 3 which, at the same time, leads us to obtain a characterization of the class of completable strong fuzzy metric spaces (Theorem 8).

In Chapter 5 we introduce and study a stronger concept than convergence of sequences called s-convergence (Definition 14), and we characterize those fuzzy metric spaces in which convergent sequences are s-convergent (Corollary 9). In such a case M is called an s-fuzzy metric. On the other hand, given a fuzzy metric space $(X, M, *)$, if $(N_M, *)$ is a fuzzy metric on X where $N_M(x, y) = \bigwedge\{M(x, y, t) : t > 0\}$ then it is proved that the topologies deduced from M and N_M coincide if and only if M is an s-fuzzy metric (Theorem 10). A classification of fuzzy metrics attending to the behaviour of fuzzy metrics with respect to the different types of convergence and involving some well-known classes of fuzzy metrics is given at the end of this chapter (Diagram 5.1).

In Chapter 6 we study when certain families of open balls centered at a point are local bases at this point. This question is related to the concept of s-convergence and also to the concept of p-convergence introduced by Mihet [57]. The main results obtained in this chapter are Corollaries 12 and 13, and Theorem 11.

In Chapter 7 we answer two questions posed by S. Morillas and A. Sapena [On Cauchy sequences in fuzzy metric spaces, Proceedings of the Conference in Applied Topology WiAT'13 101-108] related to standard convergence (Definition 9). This last concept was introduced in a natural way by the authors after that L. Ricarte and S. Romaguera introduced in [74] the concept of standard Cauchy sequence in order to extend the classical theory of continuous domains to fuzzy setting. In particular, we prove the existence of a standard convergent sequence which is not standard Cauchy (Example 16). This result leads us to establish what conditions should satisfy a concept about sequential convergence to be considered *compatible* with a concept of Cauchyness (Definiton 17).

1.3 Preliminaries

Let us recall, [79], that a *t-norm* is a binary operation $* : [0,1]\times[0,1] \to [0,1]$ such that $([0,1], \leq, *)$ is an ordered Abelian topological monoid with unit 1.

Definition 1. (George and Veeramani [19]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;

(GV2) $M(x, y, t) = 1$ if and only if $x = y$;

(GV3) $M(x, y, t) = M(y, x, t)$;

(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(GV5) $M(x, y, \_) :]0, \infty[\rightarrow]0, 1]$ is continuous.

Some particular continuous t-norms used in this work are the minimum, denoted by $\wedge$, the usual product, denoted by $\cdot$, and the Lukasievicz t-norm, denoted by $\mathfrak{L}$ ($x \mathfrak{L} y = max\{0, x + y - 1\}$).

The axiom (GV1) is justified by the authors because in the same way that a classical metric does not take the value ∞ then M cannot take the value 0. The axiom (GV2) is equivalent to the following:

$$M(x, x, t) = 1 \text{ for all } x \in X, t > 0 \text{ and } M(x, y, t) < 1 \text{ for all } x \neq y, t > 0.$$

The axiom (GV2) gives the idea that only when $x = y$ the degree of nearness of x and y is *perfect*, or simply 1, and then $M(x, x, t) = 1$ for each $x \in X$ and for each $t > 0$. In this manner the values 0 and ∞ in the classical theory of metric spaces are identified with 1 and 0, respectively, in this fuzzy theory. Axioms (GV3) and (GV4) are a fuzzy version of the symmetry and the triangular inequality, respectively. Finally, in (GV5) the authors only assume that the variable t behave nicely, that is, they assume that for fixed x and y, the function $t \rightarrow M(x, y, t)$ is continuous without any imposition for M as $t \rightarrow \infty$.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ is a *fuzzy metric* on X. Also, if confusion is not possible, we will say that (X, M) is a fuzzy metric space or M is a fuzzy metric on X. This terminology will be also extended along this work in other concepts, as usual, without explicit mention.

Lemma 1. *(Grabiec [23]) The real function $M(x,y,\_)$ of Axiom (GV5) is non-decreasing for all $x,y \in X$.*

In the definition of Kramosil and Michalek, [48], M is a fuzzy set on $X \times X \times [0,\infty[$ that satisfies (GV3) and (GV4), and (GV1), (GV2), (GV5) are replaced by (KM1), (KM2), (KM5), respectively, below:

(KM1) $M(x,y,0) = 0$;

(KM2) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$;

(KM5) $M(x,y,\_) : [0,\infty[\to [0,1]$ is left continuous.

We will refer to these fuzzy metric spaces as KM-fuzzy metric spaces. It is worth nothing that, by defining the probabilistic metric $F_{xy}(t) = M(x,y,t)$, every KM-fuzzy metric space $(X,M,*)$ becomes a generalized Menger space, [73], under the continuous t-norm $*$. On the other hand a fuzzy metric space can be considered a KM-fuzzy metric space if we extend M defining $M(x,y,0) = 0$ for all $x,y \in X$.

George and Veeramani proved in [19] that every fuzzy metric M on X generates a topology $\mathcal{T}_M$ on X which has as a base the family of open sets of the form $\{B_M(x,\epsilon,t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x,\epsilon,t) = \{y \in X : M(x,y,t) > 1-\epsilon\}$ for all $x \in X$, $\epsilon \in]0,1[$ and $t > 0$.

Let (X,d) be a metric space and let M_d a function on $X \times X \times]0,\infty[$ defined by

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}$$

Then $(X,M_d,\cdot)$ is a fuzzy metric space [19] and M_d is called the *standard* *fuzzy metric* induced by d. The topology $\mathcal{T}_{M_d}$ coincides with the topology on X deduced from d.

Definition 2. A fuzzy metric M on X is said to be *stationary* [32] if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

Proposition 1. *(George and Veeramani [19]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ converges to x if and only if $\lim_n M(x_n, x, t) = 1$, for all $t > 0$.*

Definition 3. (George and Veeramani [19]), Schweizer and Sklar [80]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *M-Cauchy* if for each $\epsilon \in]0, 1[$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1-\epsilon$ for all $n, m \geq n_0$. Equivalently, $\{x_n\}$ is M-Cauchy if $\lim_{n,m} M(x_n, x_m, t) = 1$, where $\lim_{n,m}$ denotes the double limit as $n \to \infty$, and $m \to \infty$. If confusion is not possible we will say, simply, that $\{x_n\}$ is Cauchy. X is called *M-complete* if every Cauchy sequence in X is convergent with respect to $\mathcal{T}_M$. In such a case M is also said to be complete.

Definition 4. (Mihet [57]). Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *p-convergent* to x_0 if $\lim_n M(x_n, x_0, t_0) = 1$, for some $t_0 > 0$.

Definition 5. (Gregori et al. [25]). We say that the fuzzy metric space $(X, M, *)$ is *principal* (or simply, M is principal) if the family $\{B_M(x, r, t) : r \in]0, 1[\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$.

Theorem 1. *(Gregori et al. [25]). A fuzzy metric space is principal if and only if every p-convergent sequence is convergent.*

Definition 6. (Gregori and Romaguera [31]). Let $(X, M, *)$ and $(Y, N, \diamond)$ be two fuzzy metric spaces. A mapping f from X to Y is called an *isometry* if for each $x, y \in X$ and $t > 0$, $M(x,y,t) = N(f(x), f(y), t)$ and, in this case, if f is a bijection, X and Y are called *isometric*. A *fuzzy metric completion* of $(X, M, *)$ is a complete fuzzy metric space $(\tilde{X}, \tilde{M}, \tilde{*})$ such that $(X, M, *)$ is isometric to a dense subspace of $\tilde{X}$. X is called *completable* if it admits a fuzzy metric completion.

Proposition 2. *(Gregori and Romaguera [31]). If a fuzzy metric space has a fuzzy metric completion then it is unique up to isometry.*

In [32] is given the following characterization about completion of a fuzzy metric space.

Theorem 2. *Let $(X, M, *)$ be a fuzzy metric space, and let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in X. Then $(X, M, *)$ is completable if and only if it satisfies the following conditions:*

(C1) The function $t \to \lim_n M(a_n, b_n, t)$ is a continuous function on $]0, \infty[$ with values in $]0, 1]$.

(C2) If $\lim_n M(a_n, b_n, s) = 1$ for some $s > 0$ then $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$.

Remark 1. Suppose $(\tilde{X}, \tilde{M}, \tilde{*})$ is a fuzzy metric completion of $(X, M, *)$. Attending to the last proposition and the construction of the completion, [32], we can consider that $X \subset \tilde{X}$, $\tilde{*}$ is $*$, and that $\tilde{M}$ is defined on $\tilde{X}$ by

$$\tilde{M}(x,y,t) = \lim_n M(x_n, y_n, t)$$

for all $x, y \in \tilde{X}, t > 0$, where $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X that converge to x and y, respectively.

Remark 2. Cauchy sequences are defined in the same way in fuzzy metric spaces and KM-fuzzy metric spaces. Then it is easy to verify [76] that a fuzzy metric space $(X, M, *)$ is complete if and only if the corresponding KM-fuzzy metric space is also complete. Further if $(X, M, *)$ admits completion this completion agrees with the completion of the corresponding KM-fuzzy metric space. Recall that every KM-fuzzy metric space has a completion which is unique up to an isometry, [76, 83].

It is not the aim of this work to point out the analogies or differences between the results obtained for fuzzy metric spaces and the corresponding ones for KM-fuzzy metric spaces, in the next sections.

Definition 7. Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric M (or the fuzzy metric space $(X, M, *)$) is said to be *strong* if it satisfies for each $x, y, z \in X$ and each $t > 0$

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t) \quad (GV4')$$

Theorem 3. *(Gregori et al. [28, Theorem 35]) Let $(X, M, *)$ be a strong fuzzy metric space and suppose that $*$ is integral (i.e. $a * b > 0$ whenever $a, b \in]0, 1]$). If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and $t > 0$ then $\{M(x_n, y_n, t)\}_n$ converges in $]0, 1]$.*

Let $(X, M, *)$ be a non-stationary fuzzy metric. Define the family of functions $\{M_t : t > 0\}$ where, for each $t > 0$, $M_t : X^2 \to]0, 1]$ is given by $M_t(x, y) = M(x, y, t)$. Then $(X, M, *)$ is strong if and only if $(X, M_t, *)$ is a stationary fuzzy metric for each $t > 0$. In this case we will say that $\{M_t : t > 0\}$ is the *family of stationary fuzzy metrics associated* to M. Clearly, this family characterizes M in the sense that $M(x, y, t) = M_t(x, y)$ for all $x, y \in X, t > 0$. If $(X, M, *)$ is strong then $\mathcal{T}_M = \bigvee\{\mathcal{T}_{M_t} : t > 0\}$. Moreover, it is easy to verify that the sequence $\{x_n\}$ in X is M-Cauchy if and only if $\{x_n\}$ is M_t-Cauchy for each $t > 0$.

Proposition 3. *(Sapena and Morillas [68]) Let* $\{(M_t, *) : t > 0\}$ *be a family of stationary fuzzy metrics on* X*.*

(i). The function M *on* $X^2 \times]0, \infty[$ *defined by* $M(x, y, t) = M_t(x, y)$ *is a fuzzy metric on* X *when considering the* t*-norm* $*$*, if and only if* $\{M_t : t > 0\}$ *is an increasing family (i.e.* $M_t \leq M_{t'}$ *if* $t < t'$*) and the function* $M_{xy} :]0, \infty[\to]0, 1]$ *is a continuous function, for each* $x, y \in X$*. In such case:*

(ii). $(M, *)$ *is strong and* $\{(M_t, *) : t > 0\}$ *is the family of stationary fuzzy metrics deduced from* M*.*

Remark 3. (About terminology) If $(X, M, \wedge)$ is strong then (GV4') becomes

$$M(x, z, t) \geq min\{M(x, y, t), M(y, z, t)\} \quad (GV4'')$$

and in this case we say that M is a *fuzzy ultrametric* [28].

Let d be a metric on X. Now, we can consider the standard fuzzy metric M_d on X. Further, if $d(x, y) < 1$ for all $x, y \in X$ then we can also consider the stationary fuzzy metric $(N, \mathfrak{L})$ on X, where $N(x, y) = 1 - d(x, y)$. Then d is an ultrametric (a non-Archimedean metric) if and only if M_d is a fuzzy ultrametric, [77], if and only if N is a fuzzy ultrametric [28]. Further, condition (GV4") is stronger than (GV4) in the same way that $d(x, z) \leq max\{d(x, y), d(y, z)\}$ is stronger than the usual triangular inequality.

Following terminology of probabilistic metric spaces, [24, 43], some authors call non-Archimedean fuzzy metrics those that also satisfy equation (GV4'). Notice that in this case there is not any correspondence, in the above sense, between non-Archimedean metrics and non-Archimedean fuzzy metrics since M_d always satisfies $M_d(x,z,t) \geq M_d(x,y,t) \cdot M_d(y,z,t)$ and also because all stationary fuzzy metrics would be non-Archimedean. Further (GV4') is not stronger than (GV4) and it means that if we replace (GV4) by (GV4') then M could not be a fuzzy metric on X. (Indeed, $M(x,y,t) = \frac{1/t}{1/t+d(x,y)}$ satisfies (GV1)-(GV3), (GV4') and (GV5) and it does not satisfies (GV4).)

Definition 8. (Ricarte and Romaguera [74]). A sequence $\{x_n\}$ is called *std*-Cauchy if given $\epsilon \in]0,1[$ there exists $n_\epsilon \in \mathbb{N}$, depending on ϵ, such that $M(x_n, x_m, t) > \frac{t}{t+\epsilon}$, for all $n, m \geq n_\epsilon$ and for all $t > 0$. X is called *std-complete* if every *std*-Cauchy sequence in X is convergent.

Definition 9. (Morillas and Sapena [69]). A sequence $\{x_n\}$ in X is called *std*-convergent to $x_0 \in X$ if given $\epsilon \in]0,1[$ there exists $n_\epsilon \in \mathbb{N}$, depending on ϵ, such that $M(x_n, x_0, t) > \frac{t}{t+\epsilon}$, for all $n \geq n_\epsilon$ and for all $t > 0$.

Chapter 2

Some questions in fuzzy metric spaces

The material of this chapter is an adaptation to the thesis of the content of the paper by Valentín Gregori, Juan-José Miñana and Samuel Morillas, "Some questions in fuzzy metric spaces", published in the JCR-journal Fuzzy Sets and Systems **204** (2012) 71-85.

2.1 Introduction

The concept of fuzzy metric includes in its definition a parameter, t, that allows to introduce novel (fuzzy metric) concepts with respect to the classical metric concepts. For instance, the concepts of principal and strong fuzzy metric were motivated by the study of the p-convergence, [57], and the generalization of non-Archimedean fuzzy metrics, [77], respectively. Moreover, recently, fuzzy metrics have been applied to colour image filtering by replacing classical metrics and some improvements have been achieved [4, 5, 64, 61, 62, 63, 65, 66]. In this context, the presence of the t parameter is indeed a key issue because it allows the fuzzy metric to perform adaptively which is beneficial to improve performance. In particular, a fuzzy metric used frequently in the above cited papers has been the fuzzy metric M^* defined on $[0,\infty[$ (the set of non-negative real numbers) by $M^*(x,y,t) = \frac{min\{x,y\}+t}{max\{x,y\}+t}$.

In this chapter, we study some aspects of the fuzzy metric M^* and as well as the well-known fuzzy metric M_0 given by $M_0(x,y) = \frac{min\{x,y\}}{max\{x,y\}}$ on $]0,\infty[$ (the set of positive real numbers). This study is carried out in such a manner (see Remark 4) that it creates an appropriate context to introduce five questions in fuzzy metric spaces (relative to completion, uniform continuity, extension and contractivity) which is the second aim of this section. In spite of the risk of this proposal, [17] (Preface), we do hope that these problems will provide the basis of much future research. Finally, as practical application, we show that this fuzzy metric is useful for measuring perceptual colour-differences between colour samples.

So, the structure of the chapter is as follows. In Section 2.2 it is proved that $(]0,\infty[, M_0)$ is complete and we construct the completion of $(]0,\infty[, M^*)$ where M^* is given by the above expression. In Section 2.3 we study some aspects on the continuity of M_0. In Section 2.4 an extension of M^* (defined on $[0,\infty[$) to $\mathbb{R}$ is constructed. In Section 2.5 we study some aspects about contractivity with respect to M_0, and, finally, in Section 2.6 we show a new application of these fuzzy metrics.

2.2 Introducing the examples. On completeness and completion.

Throughout this chapter $(]0,\infty[, M_0, \cdot)$ will be the stationary fuzzy metric space where M_0 is defined by $M_0(x,y) = \frac{min\{x,y\}}{max\{x,y\}}$, [19]. It is easy to verify that $\mathcal{T}_{M_0}$ is the usual topology of $\mathbb{R}$ restricted to $]0,\infty[$.

Also, $([0,\infty[, M^*, \cdot)$ will be the fuzzy metric space where M^* is defined by $M^*(x,y,t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$, [91]. Its subspace $(]0,\infty[, M^*, \cdot)$ will take an interesting role in this section.

We omit the proof of the next proposition.

Proposition 4. *Consider the fuzzy metric M^* on $[0,\infty[$ (respectively, on $]0,\infty[$).*

(i) $\mathcal{T}_{M^*}$ is the usual topology of $\mathbb{R}$ restricted to $[0,\infty[$ (respectively, to $]0,\infty[$).

(ii) M^* is principal.

(iii) M^* is strong.

Since M^* is strong so we can consider its associated family of stationary fuzzy metrics $\{M_t^* : t > 0\}$ defined on $[0,\infty[$ (respectively, on $]0,\infty[$), i.e.

$M_t^*(x,y) = M^*(x,y,t)$, for each $t > 0$, and by (ii) we have:

(iv) $\mathcal{T}_{M_t^*}$ is the usual topology of $\mathbb{R}$ restricted to $[0,\infty[$ (respectively, to $]0,\infty[$), for each $t > 0$.

The infimum (denoted by $\wedge$) of a family of stationary fuzzy metrics associated to a strong fuzzy metric was studied in [28]. In the case of M^* we have the next proposition.

Proposition 5.

(i) Consider M^* on $[0,\infty[$. Then $\bigwedge_{t>0} M_t^*$ is not a fuzzy metric on $[0,\infty[$.

(ii) Consider M^* on $]0,\infty[$. Then $\bigwedge_{t>0} M_t^*$ is the fuzzy metric M_0.

Proof.
(i) If we take $y \neq 0$ then

$$\bigwedge_{t>0} M_t^*(0,y) = \inf\left\{\frac{t}{y+t} : t > 0\right\} = 0$$

and then $\bigwedge_{t>0} M_t^*$ is not a fuzzy metric on $[0,\infty[$.
(ii) For each $x, y, t \in]0,\infty[$ we have that

$$\bigwedge_{t>0} M_t^*(x,y) = \inf\left\{\frac{\min\{x,y\}+t}{\max\{x,y\}+t} : t > 0\right\} = \frac{\min\{x,y\}}{\max\{x,y\}} > 0$$

and so, $\bigwedge_{t>0} M_t^*$ is the fuzzy metric M_0. □

From now on, for simplicity, by a convergent sequence (in reference to $\mathcal{T}_{M^*}$ or $\mathcal{T}_{M_0}$) we mean that it is convergent with respect to the usual topology of $\mathbb{R}$ restricted to the corresponding domain.

Taking into account Remark 2 we could obtain the next theorem using results of KM-fuzzy metric spaces, [73], but we choose to prove it, since it is illustrative within the context of the chapter (see Remark 4).

Theorem 4. $(]0,\infty[, M_0, \cdot)$ *is complete.*

Proof.
Recall that $\mathcal{T}_{M_0}$ is the usual topology of $\mathbb{R}$ restricted to $]0,\infty[$. We will characterize the M_0-Cauchy sequences.
Firstly, we will see that M_0-Cauchy sequences in $]0,\infty[$ are bounded for the usual metric of $\mathbb{R}$. Indeed, if $\{a_n\}$ is a non-bounded sequence in $]0,\infty[$, then for a given $\epsilon \in]0,\infty[$ and for any $n \in \mathbb{N}$ we can find $m \in \mathbb{N}$ with $m > n$ such that $\epsilon \cdot a_m > a_n$ and so $M_0(a_n, a_m) = \frac{a_n}{a_m} < \epsilon$ and thus $\{a_n\}$ is not M_0-Cauchy.

Now we will see that if $\{a_n\}$ is a sequence in $]0,\infty[$ that converges to 0 then $\{a_n\}$ is not M_0-Cauchy. Indeed, if $\{a_n\}$ converges to 0 then for a fixed $\epsilon \in]0,1[$ and for any $n \in \mathbb{N}$ we can find $m \in \mathbb{N}$ with $m > n$ such that $a_m < \epsilon \cdot a_n$ and so $M_0(a_n, a_m) = \frac{a_m}{a_n} < \epsilon$ and then $\{a_n\}$ is not M_0-Cauchy.

Finally, we will see that if $\{a_n\}$ is an M_0-Cauchy sequence in $]0,\infty[$ then $\{a_n\}$ converges in $]0,\infty[$. Let $\{a_n\}$ an M_0-Cauchy sequence in $]0,\infty[$ and hence, as we have seen above, $\{a_n\}$ is bounded. Then there exist $a \in [0,\infty[$ and a subsequence $\{a_{n_i}\}_i$ of $\{a_n\}$ such that $\lim_i a_{n_i} = a$. Now, $\{a_{n_i}\}_i$ is also an M_0-Cauchy sequence and hence, for the last paragraph, $a > 0$. We will show that $\{a_n\}$ converges to a.

If $\{a_n\}$ does not converges to a then there exist $\delta' > 0$ such that infinite terms of $\{a_n\}$ are in (the compact of $\mathbb{R}$) $I = [0, a-\delta'] \cup [a+\delta', K]$, where K is an upper bound of $\{a_n\}$. Then there exist a subsequence $\{a_{n'_j}\}_j$ of $\{a_n\}$ in I and $b \in I$ such that $\lim_j a_{n'_j} = b$, and, as above, $b > 0$. Suppose that $b < a$.

Let $\delta > 0$ with $\delta < \min\left\{b, \frac{a-b}{3}\right\}$ and let $\epsilon = \frac{b+\delta}{a-\delta} > 1$. Since $\lim_i a_{n_i} = a$ and $\lim_j a_{n'_j} = b$ then there exists $p \in \mathbb{N}$ such that $a_{n_i} \in]a-\delta, a+\delta[$ for each $i \geq p$ and $a_{n'_j} \in]b-\delta, b+\delta[$ for each $j \geq p$.

Given $n \in \mathbb{N}$ we choose $q_n = \max\{n, p\}$ and then for $i, j \geq q_n$ we have $M_0(a_{n_i}, a_{n'_j}) < \frac{b+\delta}{a-\delta} = \epsilon$ and so $\{a_n\}$ is not M_0-Cauchy, a contradiction.

A similar argument can be made if $b > a$.

In consequence $\{a_n\}$ is M_0-Cauchy iff $\{a_n\}$ converges in $]0, \infty[$. □

Since a compact fuzzy metric space is precompact and complete, [30], then we have the next corollary.

Corollary 1. $(]0, \infty[, M_0, \cdot)$ *is not precompact.*

Proposition 6. $(]0, \infty[, M_t^*, \cdot)$ *is not complete for each* $t > 0$.

Proof.
Recall that $\mathcal{T}_{M_t^*}$ is the usual topology of $\mathbb{R}$ restricted to $]0, \infty[$, for each $t > 0$.

Now, the sequence $\{\frac{1}{n}\}$ is not convergent in $]0, \infty[$ because $0 \notin]0, \infty[$, but it is M_t^*-Cauchy for each $t > 0$. Indeed,

$$\lim_{m,n} M_t^*\left(\frac{1}{n}, \frac{1}{m}\right) = \lim_{m,n} \frac{\min\{\frac{1}{n}, \frac{1}{m}\} + t}{\max\{\frac{1}{n}, \frac{1}{m}\} + t} = 1, \text{ for each } t > 0.$$

□

In the proof of the last proposition we have just obtained that $\{\frac{1}{n}\}$ is Cauchy in $(]0,\infty[,M^*,\cdot)$ and so the next corollary is immediate.

Corollary 2. $(]0,\infty[,M^*,\cdot)$ *is not complete.*

Lemma 2. *Take* $t>0$ *and consider the fuzzy metric space* $(]0,\infty[,M_t^*,\cdot)$*. Let* $\{x_n\}$ *be a sequence in* $]0,\infty[$*. Then* $\{x_n\}$ *is* M_t^**-Cauchy if and only if* $\{x_n\}$ *converges in* $[0,\infty[$*.*

Proof.
Fix $t>0$, and let $\{x_n\}$ be an M_t^*-Cauchy sequence in $]0,\infty[$.
Then $\lim_{m,n} M_t^*(x_n,x_m)=\lim_{m,n}\frac{\min\{x_n,x_m\}+t}{\max\{x_n,x_m\}+t}=1$, but this expression is equivalent to $\lim_{m,n}\frac{\min\{x_n+t,x_m+t\}}{\max\{x_n+t,x_m+t\}}=1$ and so $\{x_n+t\}$ is an M_0-Cauchy sequence in $]0,\infty[$, so by Theorem 4 $\{x_n+t\}$ converges in $]0,\infty[$, then $\{x_n\}$ is convergent and clearly $\{x_n\}$ converges in $[0,\infty[$.

Conversely, if $\{x_n\}$ converges in $]0,\infty[$, then clearly it is M_t^*-Cauchy for each $t>0$. Now, suppose $\{x_n\}$ is a sequence in $]0,\infty[$ that converges to 0. Then, $\lim_{m,n}\min\{x_n,x_m\}=\lim_{m,n}\max\{x_n,x_m\}=0$ and therefore, for a fixed $t>0$ we have that $\lim_{m,n} M_t^*(x_n,x_m)=\lim_{m,n}\frac{\min\{x_n,x_m\}+t}{\max\{x_n,x_m\}+t}=1$, and so $\{x_n\}$ is M_t^*-Cauchy. □

Since M^* is strong by the above lemma we have the next corollary.

Corollary 3. *Consider the fuzzy metric space* $(]0,\infty[,M^*,\cdot)$*. Then a sequence* $\{x_n\}$ *in* $]0,\infty[$ *is* M^**-Cauchy if and only if* $\{x_n\}$ *converges in* $[0,\infty[$*.*

Theorem 5. $(]0,\infty[,M^*,\cdot)$ *is completable.*

Proof.

Let $\{a_n\}$ and $\{b_n\}$ be two M^*-Cauchy sequences in $(]0,\infty[, M^*, \cdot)$. First we will prove that $(C1)$ of Theorem 2 is satisfied.

From [68, 28] $\{a_n\}$ and $\{b_n\}$ are M_t^*-Cauchy sequences in $]0,\infty[$ for all $t > 0$ and so, by the previous lemma, $\{a_n\}$ and $\{b_n\}$ converge to a and b, respectively, in $[0,\infty[$.

Suppose, without lost of generality, that $a \leq b$. Then, it is an easy exercise to prove that $\lim_n(min\{a_n, b_n\}) = a$ and $\lim_n(max\{a_n, b_n\}) = b$.

Thus, for $t > 0$ we have that

$$\lim_n M^*(a_n, b_n, t) = \lim_n \frac{\min\{a_n, b_n\} + t}{\max\{a_n, b_n\} + t} = \frac{a+t}{b+t} > 0.$$

We have just obtained that the function $t \longrightarrow \lim_n M^*(a_n, b_n, t)$ is a continuous function on $]0,\infty[$ with values in $]0,1]$, and $(C1)$ of Theorem 2 is satisfied.

Next we will prove that $(C2)$ of Theorem 2 is also satisfied.

Suppose that for some $t_0 > 0$ $\lim_n M^*(a_n, b_n, t_0) = \lim_n \frac{\min\{a_n,b_n\}+t_0}{\max\{a_n,b_n\}+t_0} = 1$. Then, as we have seen in the first part of the proof, we can assert that there exist $\lim_n(\min\{a_n, b_n\})$ and $\lim_n(\max\{a_n, b_n\})$ and obviously, in this case,
$\lim_n(\min\{a_n, b_n\}) = \lim_n(\max\{a_n, b_n\})$. Consequently

$$\lim_n M^*(a_n, b_n, t) = \lim_n \frac{\min\{a_n, b_n\} + t}{\max\{a_n, b_n\} + t} = 1, \text{ for all } t > 0$$

and $(C2)$ of Theorem 2 is satisfied. So $(]0,\infty[, M^*, \cdot)$ is completable. □

The completion of $(]0,\infty[, M^*, \cdot)$.

Denote by $(\tilde{X}, \tilde{M}, \cdot)$ the completion of $(]0,\infty[, M^*, \cdot)$. By Corollary 3 M^*-Cauchy sequences in $]0,\infty[$ are the convergent sequences in $[0,\infty[$, then attending to [32] we can identify the equivalent class of M^*-Cauchy sequences in $]0,\infty[$ that converge to $p \in [0,\infty[$ with p and so $\tilde{X}$ is identified with $[0,\infty[$.

Now, attending to Remark 1 the fuzzy completion $\tilde{M}$ of M^* is defined in a such manner that if $\{a_n\}$ is a convergent sequence to 0 and $b \in]0,\infty[$ then for $t > 0$, $\tilde{M}(0, b, t) = \tilde{M}(b, 0, t) = \lim_n \frac{\min\{a_n,b\}+t}{\max\{a_n,b\}+t} = \frac{t}{b+t}$. On the other hand $\tilde{M}(0, 0, t) = 1$ for all $t > 0$ and then $\tilde{M}$ is given by $\tilde{M}(a, b, t) = \frac{\min\{a,b\}+t}{\max\{a,b\}+t}$ for each $a, b \in [0,\infty[$, $t > 0$ and therefore $\tilde{M}$ is the fuzzy metric M^* on $[0,\infty[$ defined at the beginning of this section.

From [28] Theorem 40, the following corollary is immediate.

Corollary 4. $([0,\infty[, M_t^*, \cdot)$ *is the completion of* $(]0,\infty[, M_t^*, \cdot)$ *for each* $t >$ 0.

Remark 4. Using similar arguments to the above ones in Theorem 4 one can shows that $([0,\infty[, M^*, \cdot)$ is complete. Now, the mapping $i : (]0,\infty[, M^*, \cdot) \to ([0,\infty[, M^*, \cdot)$ given by $i(x) = x$ for each $x \in]0,\infty[$, is an isometry and by (i) of Proposition 4 $]0,\infty[$ is dense in $([0,\infty[, \mathcal{T}_{M^*})$, and since the completion of a fuzzy metric space is unique, up to isometry [31], then $([0,\infty[, M^*, \cdot)$ is the completion of $(]0,\infty[, M^*, \cdot)$.

For obtaining the completion of $(]0,\infty[, M^*, \cdot)$ we have preferred the above constructive method because it allows us to introduce in its appropriate context the following open question.

Problem 1. To find a fuzzy metric space $(X, M, *)$ where for two M-Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the assignment $f(t) = \lim_n M(a_n, b_n, t)$ for all $t > 0$, does not define a continuous function on t.

It is known that the completion of a strong fuzzy metric is strong, [28] Lemma 39. On the other hand we have just obtained above that the completion of the principal fuzzy metric space $(]0, \infty[, M^*, \cdot)$ is $([0, \infty[, M^*, \cdot)$, which is also principal. Now, the next is an open question.

Problem 2. If the principal fuzzy metric space $(X, M, *)$ admits completion $(\tilde{X}, \tilde{M}, *)$, is it also principal?

2.3 On continuity and uniform continuity

We have just seen above that the fuzzy metric M^* on $]0, \infty[$ can be extended to $[0, \infty[$ by means of the fuzzy metric $\tilde{M}$ in such a manner that $]0, \infty[$ is dense in $([0, \infty[, \mathcal{T}_{\tilde{M}})$. Now, this situation is not possible for $(]0, \infty[, M_0)$ as shows the next proposition.

Proposition 7. *Consider the fuzzy metric space* $(]0, \infty[, M_0, \cdot)$ *and let* $\tilde{M}_0$ *an extension of* M_0 *to* $[0, \infty[$. *Then* $\{0\}$ *is* $\mathcal{T}_{\tilde{M}_0}$*-open.*

Proof.
$]0, \infty[$ is $\tilde{M}_0$-complete and then it is $\mathcal{T}_{\tilde{M}_0}$-closed. □

Consequently, we cannot find an extension $\tilde{M}_0$ of M_0 such that $\mathcal{T}_{\tilde{M}_0}$ coincides with the usual topology of $\mathbb{R}$ restricted to $[0, \infty[$.

Example 1. The fuzzy metric $\tilde{M_0}$ on $[0, \infty[$ given by

$$\tilde{M_0}(y,x) = \tilde{M_0}(x,y) = \begin{cases} M_0(x,y), & x, y \in]0, \infty[\\ \frac{1}{2y}, & x = 0, y \geq 1 \\ \frac{y}{2}, & x = 0, y < 1 \\ 1, & x = y = 0 \end{cases}$$

is an extension of M_0 to $[0, \infty[$ and $\{0\}$ is clearly open of $\mathcal{T}_{\tilde{M_0}}$.

From [75] we know that $M_0(x, y)$ is continuous on $]0, \infty[^2$ (endowed with the producto topology). Now, the continuous function M_0 does not admit any continuous extension N to $[0, \infty[^2$ endowed with the usual topology of $\mathbb{R}$. Indeed, if N were so, then since $\{\frac{1}{n}\}$ and $\{\frac{1}{n^2}\}$ converge to 0 it should be $N(0,0) = \lim_n M_0(1/n, 1/n) = 1$ and also $N(0,0) = \lim_n M_0(1/n, 1/n^2) = \lim_n \frac{1/n^2}{1/n} = 0$, a contradiction.

Definition 10. We will say that the fuzzy metrics M_1 and M_2 on X are *uniformly equivalent* if the identity mappings $i : (X, M_1) \to (X, M_2)$ and $i : (X, M_2) \to (X, M_1)$ are uniformly continuous [20]. In that case, obviously $\{x_n\}$ is an M_1-Cauchy sequence if and only if $\{x_n\}$ is an M_2-Cauchy sequence.

Now the fuzzy metrics M^* and M_0 on $]0, \infty[$ are topologically equivalent on $]0, \infty[$, i.e. $\mathcal{T}_{M^*} = \mathcal{T}_{M_0}$ on $]0, \infty[$, but they are not uniformly equivalent on $]0, \infty[$ because $(]0, \infty[, M_0)$ is complete but $(]0, \infty[, M^*)$ is not complete (Notice that the identity mapping $i : (]0, \infty[, M_0) \to (]0, \infty[, M^*)$ is uniformly continuous since $M_0(x, y) \leq M^*(x, y, t)$ for each $x, y \in]0, \infty[$, $t > 0$, but $i : (]0, \infty[, M^*) \to (]0, \infty[, M_0)$ is not uniformly continuous since $\{\frac{1}{n}\}$ is a Cauchy sequence in $(]0, \infty[, M^*)$ but it is not M_0-Cauchy).

Definition 11. (Gregori, Romaguera and Sapena [34]) Let $(X, M, *)$ be a fuzzy metric space. A mapping $f : X \to \mathbb{R}$ is called $\mathbb{R}$-*uniformly continuous* if given $\epsilon > 0$ we can find $s > 0$, $\delta \in]0,1[$ such that $M(x,y,s) > 1-\delta$ implies $|f(x) - f(y)| < \epsilon$.

Proposition 8. *Consider the fuzzy metric space* $(]0,\infty[, M_0)$. *For a fixed* $y > 0$ *the mapping* $M_0^y :]0,\infty[\to]0,\infty[$ *given by* $M_0^y(x) = \frac{\min\{x,y\}}{\max\{x,y\}}$ *for all* $x \in]0,\infty[$ *is* $\mathbb{R}$*-uniformly continuous.*

Proof.
Let $\epsilon > 0$. We distinguish three cases: (a) $x, x' \leq y$, (b) $x, x' \geq y$, (c) $x \leq y, x' > y$ (or $x' \leq y, x > y$).

(a) Choose $\delta \in]0,1[$ with $\delta < \epsilon$. Suppose that $x, x' \in]0,\infty[$ satisfy $M_0(x,x') > 1-\delta$. Without lost of generality we can suppose $x \leq x'$. Then we have that $\frac{x}{x'} > 1-\delta$ and hence

$$|M_0^y(x') - M_0^y(x)| = \frac{x'}{y} - \frac{x}{y} = \frac{1}{y}(x'-x) < \frac{1}{y}(x' - x'(1-\delta)) = \frac{x'}{y}\delta \leq \delta < \epsilon$$

With similar arguments the other cases can be proved, and then M_0^y is $\mathbb{R}$-uniformly continuous. □

The next is an open question.

Problem 3. Let $(X, N, *)$ be a stationary fuzzy metric space. Is the real function $N_y(x) = N(x,y)$ for each $x \in X$, $\mathbb{R}$-uniformly continuous for all $y \in X$?

2.4 Extending fuzzy metrics

2.4.1 A related fuzzy pseudo-metric

Consider the fuzzy set N on $\mathbb{R}^2\times]0,\infty[$ given by

$$N(x,y,t) = \frac{\min\{|x|,|y|\}+t}{\max\{|x|,|y|\}+t} \tag{2.1}$$

It is easy to verify that N satisfies axioms (GV1),(GV3) and (GV5). Also, N satisfies the triangular inequality. Indeed, for $x,y,z \in \mathbb{R}$, $t>0$ we have

$$N(x,z,t+s) = \frac{\min\{|x|,|z|\}+t+s}{\max\{|x|,|z|\}+t+s} = M^*(|x|,|z|,t+s) \geq$$

$$\geq M^*(|x|,|y|,t)\cdot M^*(|y|,|z|,s) = \frac{\min\{|x|,|y|\}+t}{\max\{|x|,|y|\}+t}\cdot\frac{\min\{|y|,|z|\}+s}{\max\{|y|,|z|\}+s} =$$

$$= N(x,y,t)\cdot N(y,z,s)$$

Also, for $x=y$ we have that $N(x,y,t)=1$ for all $t>0$ but the converse is, in general, false since for $x\neq 0$ we have that $N(x,-x,t)=1$ but $x\neq -x$. Consequently $(\mathbb{R},N,\cdot)$ is a fuzzy pseudo-metric space, [33], but it is not a fuzzy metric space.

The mapping $j:]-\infty,0]\to[0,\infty[$ defined by $j(x)=-x$ is a bijection and then $(]-\infty,0],M',\cdot)$ and $([0,\infty[,M^*,\cdot)$ are two fuzzy isometric spaces [31], where M' is given by $M'(x,y,t)=M^*(j(x),j(y),t)=M^*(-x,-y,t)=M^*(|x|,|y|,t)$ for all $x,y\in]-\infty,0]$, $t>0$. So M' is, obviously, strong and principal.

Notice that M^* and M' can be defined both two in their corresponding domains by the expresion (2.1), i.e. $N|_{[0,\infty[}=M^*$ and $N|_{]-\infty,0]}=M'$.

Remark 5. Section 5.1 admits the following easy generalization. Let $(M,*)$ be a fuzzy metric on a set of non-negative real numbers A. Put $-A=\{x\in$

$\mathbb{R} : -x \in A\}$. Define $N(x, y, t) = M(|x|, |y|, t)$ for all $x, y \in -A \cup A$, $t > 0$. Then, $(N, *)$ is a fuzzy pseudo-metric on $-A \cup A$.

2.4.2 A fuzzy metric extension of M^*

We have just seen that the fuzzy pseudometric N on $\mathbb{R}$ satisfies

$$N|_{[0,\infty[} = M^* \text{ and } N|_{]-\infty,0]} = M' \tag{2.2}$$

Now we will construct a fuzzy metric $\bar{M}$ on $\mathbb{R}$ such that $\bar{M}|_{[0,\infty[} = M^*$ and $\bar{M}|_{]-\infty,0]} = M'$. For it we consider the family $\{M_t^* : t > 0\}$ of stationary fuzzy metrics on $[0, \infty[$ associated to M^*, and the family $\{M_t' : t > 0\}$ of stationary fuzzy metrics on $]-\infty, 0]$ associated to M'.

Then, since $]-\infty, 0] \cap [0, \infty[= \{0\}$, from [29] Proposition 19 we have for each fixed $t > 0$ that the function

$$\bar{M}_t(x, y) = \begin{cases} M_t^*(x, y) & \text{if } x, y \in [0, \infty[\\ M_t'(x, y) & \text{if } x, y \in]-\infty, 0] \\ M_t^*(x, 0) \cdot M_t'(0, y) & \text{if } x \in]0, \infty[, y \in]-\infty, 0[\\ M_t'(x, 0) \cdot M_t^*(0, y) & \text{if } x \in]-\infty, 0[, y \in]0, \infty[\end{cases}$$

is a stationary fuzzy metric on $\mathbb{R}$, such that $\bar{M}_t|_{]-\infty,0]} = M_t'$ and $\bar{M}_t|_{[0,\infty[} = M_t^*$.

Attending (2.2), we can be written

$$\bar{M}_t(x, y) = \begin{cases} \frac{\min\{|x|,|y|\}+t}{\max\{|x|,|y|\}+t} & x, y \in [0, \infty[\text{ or } x, y \in]-\infty, 0] \\ \frac{t}{|x|+t} \cdot \frac{t}{|y|+t} & \text{elsewhere} \end{cases}$$

Obviously $\{\bar{M}_t : t > 0\}$ is an increasing family, i.e. $t < t'$ implies $\bar{M}_t \leq \bar{M}_{t'}$.

Now we define $\bar{M}(x,y,t) = \bar{M}_t(x,y)$ for all $x, y \in \mathbb{R}$, $t > 0$. Then, obviously $\bar{M}$ satisfies (GV1)-(GV3) and (GV5).

We prove that $\bar{M}$ satisfies the triangular inequality. Let $x, y, z \in \mathbb{R}$, $t, s > 0$. Then, since $\{\bar{M}_t : t > 0\}$ is an increasing family we have $\bar{M}(x,z,t+s) = \bar{M}_{t+s}(x,z) \geq \bar{M}_{t+s}(x,y) \cdot \bar{M}_{t+s}(y,z) \geq \bar{M}_t(x,y) \cdot \bar{M}_s(y,z) = \bar{M}(x,y,t) \cdot \bar{M}(y,z,s)$ and so $(\bar{M}, \cdot)$ is a fuzzy metric on $\mathbb{R}$ which obviously satisfy $\bar{M}|_{[0,\infty[} = M^*$ and $\bar{M}|_{]-\infty,0]} = M'$.

The following is an open question.

Problem 4. Let H and K be two distinct sets with $H \cap K \neq \emptyset$. Let $(M_H, *)$ and $(M_K, *)$ be two non-stationary fuzzy metrics on H and K, respectively, that agree in $H \cap K$. Does it exist a fuzzy metric M on $H \cup K$ such that $M|_H = M_H$ and $M|_K = M_K$?

2.5 Contractivity in $(]0,\infty[, M_0, \cdot)$

2.5.1 On contractivity

Let (X, M) be a fuzzy metric space.

In order to obtain satisfactory results in the fuzzy setting, related to the classical Banach contraction theorem, several concepts of M-contractivity on a mapping $f : (X, M) \to (X, M)$ have been given, for instance [23, 36, 39, 56, 57, 58, 59, 73, 81, 84, 84, 89, 90] among others.

The weaker contractivity condition on f which makes sense when M is stationary is given by the formula

$$M(f(x), f(y)) \geq M(x, y) \text{ for } x, y \in X$$

and in fact, it is obtained from the concept of B-contraction, [23, 81], given by the expression $M(f(x), f(y), kt) \geq M(x, y, t)$ for all $x, y \in X$, $t > 0$ and some fixed $k \in]0,1[$. Now, for stationary fuzzy metrics this concept is not really appropriate (in the same way that the contractivity condition $d(f(x), f(y)) \leq d(x, y)$ is not appropriate for a metric space (X, d)). Indeed, the identity mapping $i : X \to X$ satisfies $M(f(x), f(y)) = M(x, y)$ for all $x, y \in X$ and all points of X are fixed of i. Further, in the case of the fuzzy metric space $(]0, \infty[, M_0, \cdot)$ the mapping $f :]0, \infty[\to]0, \infty[$ given by $f(x) = ax$, where $a \in \mathbb{R}^+ \sim \{1\}$, also satisfies $M(f(x), f(y)) = M(x, y)$ for all $x, y \in]0, \infty[$ but f has not any fixed point. Then, a stronger contractivity condition than the above one is needed. So, we adopt the next definition.

Definition 12. Let M be a stationary fuzzy metric on X. A mapping $f : X \to X$ is *fuzzy M-contractive* (a fuzzy contraction) if

$$M(f(x), f(y)) > M(x, y) \text{ for } x, y \in X, x \neq y \tag{2.3}$$

This concept comes from the fuzzy Edelstein contractives notion stated by Grabiec [23] as $M(f(x), f(y), t) > M(x, y, t)$ for $x, y \in X$, $x \neq y$, $t > 0$, where M is a fuzzy metric on X. The author proved that a fuzzy Edelstein contractive mapping on a compact KM-fuzzy metric space has a unique fixed point.

Notice that (2.3) is satisfied by almost all fuzzy M-contractive concepts in the literature when M is stationary.

We can get a class of fuzzy M_0-contractive mappings with a unique fixed point in $]0, \infty[$ as follows. Consider the continuous increasing functions f :

$[0,\infty[\to [0,\infty[$ with $f(0)=0$ such that $f''(x)<0$ for all $x\in]0,\infty[$ (f'' denotes the second derivative of f). Using arguments from Analysis one can verify that for $0<x<y$ it is satisfied that $\frac{f(x)}{x}>\frac{f(y)}{y}$, i.e. $\frac{f(x)}{f(y)}>\frac{x}{y}$ and hence f is fuzzy M_0-contractive. It is easy to verify that such functions have at most a unique fixed point in $]0,\infty[$. Further, f has a (unique) fixed point if and only if $f'(x)=1$ for some $x\in]0,\infty[$. Notice that $\ln(1+x)$ satisfies $f''(x)<0$ for $x\in]0,\infty[$ but $f'(x)\neq 1$ for $x\in]0,\infty[$, and clearly $\ln(1+x)$ has not any fixed point in $]0,\infty[$. The mappings $f_\lambda(x)=\sqrt{x+\lambda}$ for $x\in]0,\infty[$, with a fixed $\lambda>0$, fulfill all conditions of this paragraph and they play an interesting role in the following.

Mihet [56] pointed out that the mapping $f(x)=x+a$ for $x\in]0,\infty[$, with a fixed $a>0$, is fuzzy M_0-contractive but it has not any fixed point in $]0,\infty[$. Then, in order to guarantee the existence of fixed points for such a mappings Mihet introduced and studied the next concept for KM-fuzzy metric spaces that we rewrite in our context.

Definition 13. Let $(X,M,*)$ be a fuzzy metric space and let φ be a decreasing continuous mapping $\varphi:[0,1]\to[0,1]$ such that $\varphi(t)>t$ for all $t\in]0,1[$. A mapping $f:X\to X$ is called *φ-contractive* if $M(f(x),f(y),t)\geq \varphi(M(x,y,t))$ for all $x,y\in X$, $t>0$. Obviously in this case f satisfies (2.3).

The author proved, [58], that a fuzzy φ-contractive mapping in a strong complete fuzzy metric space has a unique fixed point.

As a consequence, since the above commented mappings $f(x)=x+a$ and $\ln(1+x)$ satisfy (2.3) and they have not any fixed point in $]0,\infty[$, these mappings are fuzzy M_0-contractive but they are not φ-contractive in $(]0,\infty[,M_0)$.

We see that the mappings $f_\lambda:]0,\infty[\to]0,\infty[$, with $\lambda>0$, defined by

$f_\lambda(x) = \sqrt{x+\lambda}$ are φ-contractive. Indeed, if $x < y$ we have $M(f_\lambda(x), f_\lambda(y)) = \frac{\sqrt{x+\lambda}}{\sqrt{y+\lambda}} \geq \sqrt{\frac{x}{y}} = \varphi(M(x,y))$ where $\varphi(t) = \sqrt{t}$, independently of $\lambda > 0$.

Then each mapping f_λ has a unique fixed point $a_\lambda \in]0,\infty[$.

Now we can define the mapping $g :]0,\infty[\to]0,\infty[$ by $g(\lambda) = a_\lambda$. So, $g(\lambda) = \frac{1+\sqrt{1+4\lambda}}{2}$ and thus g is a continuous function on $]0,\infty[$. Then it arises the following question.

Problem 5. Let $(X, M, *)$ be a strong complete fuzzy metric space and let $f_\lambda : X \to X$ be a family of φ-contractive mappings for the same function φ, for all $\lambda > 0$. Suppose that for each $x \in X$ the mapping $f_x :]0,\infty[\to X$, where $f_x(\lambda) = f_\lambda(x)$, is continuous on $\lambda > 0$. Write a_λ the unique fixed point of f_λ for each $\lambda > 0$. Is the mapping $g :]0,\infty[\to X$ defined by $g(\lambda) = a_\lambda$ continuous?

Remark 6. This problem has been formulated according to the previous results but obviously it admits other versions. We notice that the analogous problem formulated in metric spaces has positive answered [78]. If the condition of continuity of f_x on $]0,\infty[$ is removed the answer to this question is negative as it has been proved in [92].

2.6 Application of the fuzzy metric M_0 to measure perceptual colour differences

Apart from the interesting theoretical properties of the fuzzy metrics studied in previous sections, it is interesting as well to note that they have application in a variety of practical problems. Indeed, they have been previously used to filter colour images and to measure the degree of consistency of elements in a dataset [5, 64, 61, 67].

Here we focus on a different application of the fuzzy metric M_0 that takes advantage of the homotetique invariant property that this fuzzy metric satisfies. Indeed, M_0 fulfills that, for any $\lambda \in \mathbb{R}$:

$$(I)\ M_0(\lambda x, \lambda y) = M_0(x, y)$$

Also, if $z > 0$,

$$(II)\ M_0(x + z, y + z) > M_0(x, y) \text{ if } x \neq y$$

As we will see later on, there exist practical problems where these properties are pretty interesting. However, in practical applications it is more appropriate to use the M^* fuzzy metric (which also satisfies (II)), instead of the M_0, because the presence of the t parameter makes this fuzzy metric more adaptive to the particular problem. On the other hand, M_0 is in fact M^* when $t = 0$. Notice that both M_0 and M^* are suitable only for scalar values and that for vector values the combination of several fuzzy metrics needs to be considered.

In particular, one application that matches the behaviour of these two fuzzy metrics regards the modeling of the perception of physical magnitudes such as colours, sounds or weights. It is known that the perception threshold of changes in these magnitudes increases as the magnitudes themselves increase [16, 18, 87]. That is to say, the perceived difference between two magnitude values x, y is different that for the values $x + k, y + k$, whenever $k > 0$. In particular, the perceived difference will be larger in the former case than in the latter, which agrees with (II). This situation can be observed in the case of perceptual colour differences and, since the M^* fuzzy metric behaves accordingly to this situation, M^* can be used to appropriately devise colour difference formulas as explained in the following.

A colour sample is usually represented as a tern in a particular colour space. Among the different colour spaces, a well-known one, specially in computer graphics, is the Hue-Chroma-Lightness (HCL) colour space [45], where a colour sample $\mathbf{s}$ is represented as a tern $\mathbf{s} = (H_s, C_s, L_s)$. In such a tern: Hue, H_s, is usually represented as an angle in $[0°, 360°]$ where $0°$, $90°$, $180°$, and $270°$ correspond to approximately pure red, yellow, green and blue, respectively. $C_s \in [0, 100]$ represents the Chroma of the colour, where 0 is associated with neutral gray, black or white; and $L_s \in [0, 100]$ represents the Lightness of the sample, where 0 represents no lightness (absolute black colour) and 100 represents the maximum lightness (absolute white colour).

A series of experimental datasets: BFD-P, Leeds, RIT-Dupont, and Witt, which are combined to form the COM dataset, have been obtained in order to characterize the perceptual difference between pairs of colour samples [2, 47, 52, 53, 93, 96]. In these datasets each pair of colour samples is associated with a value ΔV which represents the experimental perceptual difference between them. On the other hand, colour difference formulas are used to obtain, from two terns representing a pair of colour samples, the computed perceptual difference between them, usually denoted by ΔE. Since the objective of colour difference formulas is to model human perception, all formulas try to obtain ΔE values as close (or correlated) as possible to the ΔV values. One well-known colour difference formula is the CIELAB formula [97], that corresponds with the Euclidean distance in the CIELAB colour space.

The performance of a colour difference formula is assessed by measuring

Figure 2.1: Values of STRESS obtained by different colour difference formulas for the COM dataset.

colour difference formula	STRESS
CIELAB	0.428
CIE94	0.335
CIEDE2000	0.292
$\Delta E_{M_1^*}$	0.347
$\Delta E_{M_1^*}$	0.348

how close the ΔE values computed for the experimental datasets are to the ΔV values. A well established figure of merit for this closeness is the STRESS coefficient [54], which provides values in the interval $[0, 1]$, where lower values indicate a higher closeness. In Table 2.1, we can see that the value of STRESS for the CIELAB formula over the COM dataset is 0.428.

By analysing the experimental datasets, it has been observed that the sensitivity to differences in Chroma decreases as the value of Chroma increases. Notice that this fact is related to the Weber-Frechner and Stevens observations [16, 18, 87]. According to this, we propose to use the M^* fuzzy metric to model the similarity between two Chroma values C_s, C_r as

$$M^*(C_s, C_r) = \frac{min\{(C_s, C_r)\} + k_C}{max\{(C_s, C_r)\} + k_C},$$

where k_C is a parameter to adjust the behaviour as desired.

An analogous observation can be made with respect to Lightness. So we propose to measure the similarity between two Lightness values L_s, L_r as

$$M^*(L_s, L_r) = \frac{min\{(L_s, L_r)\} + k_L}{max\{(L_s, L_r)\} + k_L},$$

where k_L is another adjusting parameter.

Using these two expressions we build a more complex expression to obtain a new colour difference formula. We want also to take into account the CIELAB colour difference, ΔE^*_{ab}, so, we employ the standard fuzzy metric deduced from ΔE^*_{ab}. Given that the product of these fuzzy metrics is as well a fuzzy metric, [77], we can use a productory to join these three criteria. Finally, to obtain a difference formula we use the involutive negation as follows:

$$\Delta E_{M_1^*}(\mathbf{s},\mathbf{r}) = 1 - \left(M^*(L_s, L_r) M^*(C_s, C_r) \frac{t}{t + \Delta E^*_{ab}} \right), \tag{2.4}$$

where k_L, k_C and t are parameters able to tune the importance of each criterion. However, since ΔE^*_{ab} also includes Lightness and Chroma differences, alternatively we propose to replace ΔE^*_{ab} in Eq. (4) with ΔH, which represents only Hue differences in ΔE^*_{ab} and is given by
$\Delta H = \sqrt{\Delta E^{*2}_{ab} - |L_s - L_r|^2 - |C_s - C_r|^2}$, and so obtaining

$$\Delta E_{M_2^*}(\mathbf{s},\mathbf{r}) = 1 - \left(M^*(L_s, L_r) M^*(C_s, C_r) \frac{t}{t + \Delta H} \right), \tag{2.5}$$

where we have three adjusting parameters, as above.

It is interesting to point out that $\Delta E_{M_1^*}$ can be seen as a modification of the ΔE^*_{ab} using a correction term inspired in the Weber-Fechner and Stevens laws which are represented by an appropriate fuzzy metric. On the other hand, $\Delta E_{M_2^*}$ is a colour difference formula that corresponds with the representation of the Weber-Frechner and Stevens laws by means of fuzzy metrics.

We have performed extensive experimental assessments varying the values of the adjusting parameters k_L, k_C and t in the range $[0, 100]$ to obtain the optimal parameter setting for the formulas proposed in Eq. (4)-(5). With optimal parameter setting, $\Delta E_{M_1^*}$ is able to obtain a STRESS value for the COM dataset of 0.347 (with $k_L = 2, k_C = 4, t = 11$), whereas $\Delta E_{M_2^*}$ obtained STRESS of 0.348 (with $k_L = 4, k_C = 12, t = 40$). Notice that, in both cases, a significative improvement with respect to ΔE^*_{ab} is obtained. This means that M^* has been successfully used to take into account the facts related to the Weber-Fechner and Stevens laws. It should be also noted that whereas ΔE^*_{ab} does not incorporate these laws, they are considered in more recent colour difference formulas such as the CIE94 [98] and CIEDE2000 [99] formulas. We also compare the performance of the proposed formulas with these recent ones in Table 2.1, where we can see that the performance of our formulas are pretty close to the one of the CIE94.

Chapter 3

On completable fuzzy metric spaces

The material of this chapter is an adaptation to the thesis of the content of the paper by Valentín Gregori, Juan-José Miñana and Samuel Morillas, "On completable fuzzy metric spaces", published in the JCR-journal Fuzzy Sets and Systems **267** (2015) 133-139.

3.1 Introduction

In this chapter we continue the study of fuzzy metric completion initiated by Gregori and Romaguera [31]. The theory of fuzzy metric completion is, in this context, very different from the classical theory of metric completion. Indeed, as it is well-known metric and Menger spaces are completable. Further, imitating the Sherwod's proof [83] one can prove that fuzzy metric spaces defined by Kramosil and Michalek are completable (other different proof can be found in [6]). In this sense non-completability is a specific feature of fuzzy metric spaces, since there are fuzzy metric spaces which are not completable [31, 32, 25]. The following characterization of completable fuzzy metric spaces was given (in a slightly different way) in [32]:

Theorem 6. *A fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following three conditions are fulfilled:*

(c1) $\lim_n M(a_n, b_n, s) = 1$ *for some* $s > 0$ *implies* $\lim_n M(a_n, b_n, t) = 1$ *for all* $t > 0$.

(c2) $\lim_n M(a_n, b_n, t) > 0$ *for all* $t > 0$.

(c3) The assignment $t \to \lim_n M(a_n, b_n, t)$ *for each* $t > 0$ *is a continuous function on* $]0, \infty[$*, provided with the usual topology of* $\mathbb{R}$.

In [31] and [32] two non-completable fuzzy metric spaces were given in which conditions (c2) and (c1), respectively, are not satisfied. Since then the following is an open question (which was posed formally in Problem 1 of Chapter 2): Does it exist a fuzzy metric space in which condition (c3) is not satisfied? In this chapter we answer in a positive way this question, constructing a fuzzy metric space (Proposition 9) in which (c3) is not satisfied (Example 2). In addition, we also show that this space is an example of a non-strong fuzzy metric space.

3.2 A non-completable fuzzy metric space

Next, we attend to the requirement of [27] Problem 25, constructing a fuzzy metric space $(X, M, *)$ in which for two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the assignment $f : \mathbb{R}^+ \to]0,1]$ given by $f(t) = \lim_n M(a_n, b_n, t)$ for all $t > 0$ is a non-continuous function on $\mathbb{R}^+$, endowed with the usual topology of $\mathbb{R}$ restricted to $\mathbb{R}^+$.

We start with the following lemma.

Lemma 3. *Let $A, B, C, a, b, c \in \mathbb{R}^+$ and $u, v, w \in]0,1[$ such that $A \geq a$, $B \geq b$, $C \geq c$, and $A \geq B \cdot C$, $a \geq b \cdot c$ and $u \geq m = \max\{v, w\}$. Then*

$$Au + a(1-u) \geq (Bv + b(1-v)) \cdot (Cw + c(1-w)). \tag{3.1}$$

Proof.
The following expressions are satisfied:

$$Au + a(1-u) \geq Am + a(1-m). \tag{3.2}$$

(Indeed, $Au + a(1-u) - Am - a(1-m) = (A-a)(u-m) \geq 0$).

$$Am + a(1-m) \geq (Bm + b(1-m)) \cdot (Cm + c(1-m)). \tag{3.3}$$

(Indeed, $Am + a(1-m) \geq BCm + bc(1-m) - (B-b)(C-c)m(1-m) = (Bm + b(1-m)) \cdot (Cm + c(1-m))$).

$$Bv + b(1-v) \leq Bm + b(1-m). \tag{3.4}$$

$$Cw + c(1-w) \leq Cm + c(1-m). \tag{3.5}$$

(Indeed, $Bm + b(1-m) - Bv - b(1-v) = (B-b)(m-v) \geq 0$. The proof of (3.5) is similar).

Now, using expressions (3.4), (3.5), (3.3) and (3.2), successively, we have

$$(Bv + b(1-v)) \cdot (Cw + c(1-w)) \leq (Bm + b(1-m)) \cdot (Cm + c(1-m)) \leq$$

$$Am + a(1-m) \leq Au + a(1-u).$$

□

Lemma 4. *Let d be the usual metric on $\mathbb{R}$ and consider on $]0,1]$ the standard fuzzy metric M_d induced by d. Then*

$$M_d(x,z,t+s) \geq M_d(x,y,t) \cdot M_d(y,z,2s)$$

for all $x,y,z \in]0,1]$, $d(y,z) < s \leq 1$ and $0 < t \leq d(x,y)$.

Proof.
Let $x,y,z \in]0,1]$, $d(y,z) < s \leq 1$ and $0 < t \leq d(x,y)$. We have

$$(t+s)(t+d(x,y))(2s+d(y,z)) =$$

$$= (t+s)(2st + td(y,z) + 2sd(x,y) + d(x,y)d(y,z)) \geq$$

$$(t+s)(2st + 2sd(x,y) + 2td(y,z)) \geq$$

$$\geq 2ts(t+s+d(x,y)+d(y,z)) \geq 2ts(t+s+d(x,z)).$$

So,

$$M_d(x,z,t+s) = \frac{t+s}{t+s+d(x,z)} \geq \frac{t}{t+d(x,y)} \cdot \frac{2s}{2s+d(y,z)} = M_d(x,y,t) \cdot M_d(y,z,2s).$$

□

Proposition 9. *Let d be the usual metric on $\mathbb{R}$ restricted to $]0,1]$ and consider the standard fuzzy metric M_d induced by d.*

We define on $]0,1] \times]0,1] \times]0,\infty[$ the function

$$M(x,y,t) = \begin{cases} M_d(x,y,t), & 0 < t \leq d(x,y) \\ M_d(x,y,2t) \cdot \frac{t-d(x,y)}{1-d(x,y)} + M_d(x,y,t) \cdot \frac{1-t}{1-d(x,y)}, & d(x,y) < t \leq 1 \\ M_d(x,y,2t), & t > 1 \end{cases}$$

Then $(]0,1], M, \cdot)$ is a fuzzy metric space.

Proof.
Before starting the proof and tacking into account that

$$\frac{t-d(x,y)}{1-d(x,y)} + \frac{1-t}{1-d(x,y)} = 1$$

for all $t > 0$, we notice that the following inequalities are satisfied:

$$M_d(x,y,2t) \geq M_d(x,y,2t) \cdot \frac{t-d(x,y)}{1-d(x,y)} + M_d(x,y,t) \cdot \frac{1-t}{1-d(x,y)} \geq M_d(x,y,t) \tag{3.6}$$

for all $x, y \in]0, 1]$ and for all $d(x, y) < t \leq 1$.

Clearly, M satisfies $(GV1)$ and $(GV3)$.

It is left to the reader to verify that M satisfies $(GV2)$ and $(GV5)$.

Now, we will see that M satisfies the triangle inequality

$$M(x, z, t + s) \geq M(x, y, t) \cdot M(y, z, s)$$

for all $x, y, z \in]0, 1]$ and $s, t > 0$.

We distinguish three possibilities.

(a) Suppose $0 < t + s \leq d(x, z)$.

In this case $M(x, z, t + s) = M_d(x, z, t + s)$.

Under this possibility we can consider the following cases.

(a.1) Suppose $0 < t \leq d(x, y)$ and $0 < s \leq d(y, z)$.

In this case $M(x, y, t) = M_d(x, y, t)$ and
$M(y, z, s) = M_d(y, z, s)$.

Since

$$M_d(x, z, t + s) \geq M_d(x, y, t) \cdot M_d(y, z, s)$$

we have

$$M(x, z, t + s) \geq M(x, y, t) \cdot M(y, z, s).$$

(a.2) Suppose $0 < t \leq d(x,y)$ and $d(y,z) < s \leq 1$.

In this case $M(x,y,t) = M_d(x,y,t)$ and
$M(y,z,s) = M_d(y,z,2s) \cdot \frac{s-d(y,z)}{1-d(y,z)} + M_d(y,z,s) \cdot \frac{1-s}{1-d(y,z)}$.

By Lemma 4 we have that $M_d(x,z,t+s) \geq M_d(x,y,t) \cdot M_d(y,z,2s)$. Thus, by (3.6) we have that

$$M(x,z,t+s) \geq M(x,y,t) \cdot M(y,z,s).$$

(The case $d(x,y) < t \leq 1$ and $0 < s \leq d(y,z)$ is proved in a similar way.)

(b) Suppose now that $d(x,z) < t+s \leq 1$.

In this case
$M(x,z,t+s) = M_d(x,z,2(t+s)) \cdot \frac{t+s-d(x,z)}{1-d(x,z)} + M_d(x,z,t+s) \cdot \frac{1-(t+s)}{1-d(x,z)}$.

Under this possibility we can consider the following cases.

(b.1) Suppose $0 < t \leq d(x,y)$ and $0 < s \leq d(y,z)$. In this case $M(x,y,t) = M_d(x,y,t)$ and $M(y,z,s) = M_d(y,z,s)$. By (3.6) we have that

$$M(x,z,t+s) \geq M_d(x,z,t+s) \geq M_d(x,y,t) \cdot M_d(y,z,s)$$

and so

$$M(x,z,t+s) \geq M(x,y,t) \cdot M(y,z,s).$$

(b.2) Suppose $0 < t \leq d(x,y)$ and $d(y,z) < s \leq 1$.

In this case $M(x,y,t) = M_d(x,y,t)$ and
$M(y,z,s) = M_d(y,z,2s) \cdot \frac{s-d(y,z)}{1-d(y,z)} + M_d(y,z,s) \cdot \frac{1-s}{1-d(y,z)}$.

By (3.6) and Lemma 4 we have

$$M(x,z,t+s) \geq M_d(x,z,t+s) \geq M_d(x,y,t) \cdot M_d(y,z,2s)$$

and so

$$M(x,z,t+s) \geq M(x,y,t) \cdot M(y,z,s).$$

(The case $d(x,y) < t \leq 1$ and $0 < s \leq d(y,z)$ is proved in a similar way.)

(b.3) Suppose $d(x,y) < t \leq 1$ and $d(y,z) < s \leq 1$.

In this case $M(x,y,t) = M_d(x,y,2t) \cdot \frac{t-d(x,y)}{1-d(x,y)} + M_d(x,y,t) \cdot \frac{1-t}{1-d(x,y)}$ and $M(y,z,s) = M_d(y,z,2s) \cdot \frac{s-d(y,z)}{1-d(y,z)} + M_d(y,z,s) \cdot \frac{1-s}{1-d(y,z)}$.

Now, it is easy to verify that

$$\frac{t+s-d(x,z)}{1-d(x,z)} \geq \max\left\{\frac{t-d(x,y)}{1-d(x,y)}, \frac{s-d(y,z)}{1-d(y,z)}\right\}. \tag{3.7}$$

Put $u = \frac{t+s-d(x,z)}{1-d(x,z)}$, $v = \frac{t-d(x,y)}{1-d(x,y)}$, $w = \frac{s-d(y,z)}{1-d(y,z)}$, $A = M_d(x,z,2(t+s))$, $a = M_d(x,z,t+s)$, $B = M_d(x,y,2t)$, $b = M_d(x,y,t)$, $C = M_d(y,z,2s)$ and $c = M_d(y,z,s)$.

Obviously $u,v,w \in]0,1[$ and $A,B,C,a,b,c \in \mathbb{R}^+$. Now, by (3.7) and since $(M_d,\cdot)$ is a fuzzy metric on $\mathbb{R}$ then u,v,w,A,B,C,a,b,c fulfil the conditions of Lemma 3.1. Then

$$M(x,z,t+s) = Au + a(1-u) \geq (Bv + b(1-v)) \cdot (Cw + c(1-w))$$

and so

$$M(x,z,t+s) \geq M(x,y,t) \cdot M(y,z,s).$$

(c) Suppose $t+s > 1$.

In this case $M(x,z,t+s) = M_d(x,z,2(t+s))$.

Clearly, for all $x,y,z \in]0,1]$ and for all $s,t > 0$ we have that

$$M(x,y,t) \leq M_d(x,y,2t) \text{ and } M(y,z,s) \leq M_d(y,z,2s).$$

Since $M_d(x,z,2(t+s)) \geq M_d(x,y,2t) \cdot M_d(y,z,2s)$ for each $t,s > 0$ then

$$M(x,z,t+s) \geq M(x,y,t) \cdot M(y,z,s) \text{ for all } t,s > 0$$

Therefore, M satisfies the triangle inequality and hence $(]0,1], M, \cdot)$ is a fuzzy metric space. □

Proposition 10. *The sequence $\{a_n\}$, where $a_n = \frac{1}{n}$ for all $n = 1,2,\ldots,$ is a Cauchy sequence in $(]0,1], M, \cdot)$.*

Proof.
Fix $t > 0$. We can find $n_0 \in \mathbb{N}$ such that $\left|\frac{1}{n} - \frac{1}{m}\right| < t$ for each $m,n \geq n_0$. Then for $m,n \geq n_0$ we have

$$M(a_n, a_m, t) = \begin{cases} \frac{2t}{2t+\left|\frac{1}{n}-\frac{1}{m}\right|} \cdot \frac{t-\left|\frac{1}{n}-\frac{1}{m}\right|}{1-\left|\frac{1}{n}-\frac{1}{m}\right|} + \frac{t}{t+\left|\frac{1}{n}-\frac{1}{m}\right|} \cdot \frac{1-t}{1-\left|\frac{1}{n}-\frac{1}{m}\right|}, & 0 < t \leq 1 \\ \frac{2t}{2t+\left|\frac{1}{n}-\frac{1}{m}\right|}, & t > 1 \end{cases}$$

Hence, if $0 < t \leq 1$ we have $\lim_{n,m} M(a_n, a_m, t) = \frac{2t}{2t} \cdot t + \frac{t}{t} \cdot (1-t) = 1$, and if $t > 1$ we have $\lim_{n,m} M(a_n, a_m, t) = \frac{2t}{2t} = 1$

Then $\lim_{n,m} M(a_n, a_m, t) = 1$ for all $t > 0$. So $\{a_n\}$ is a Cauchy sequence in $(]0,1], M, \cdot)$. □

Remark 7. It is easy to see that $\mathcal{T}_M \succ \mathcal{T}_{M_d}$ and then $\mathcal{T}_M$ is finer than the usual topology of $\mathbb{R}$. Then, the sequence $\{\frac{1}{n}\}$ only could converges to 0 in $\mathcal{T}_M$, but $0 \notin]0,1]$ and, in consequence, $]0,1]$ is not complete.

Example 2. Let $(]0,1], M, \cdot)$ the above fuzzy metric space. Consider the Cauchy sequences $\{a_n\}$ and $\{b_n\}$ where $a_n = \frac{1}{n}$ and $b_n = 1$, for $n = 1, 2, \ldots$. We will see that the assignment $t \to \lim_n M(a_n, b_n, t)$ is a well-defined non-continuous function on $]0, \infty[$, endowed with the usual topology of $\mathbb{R}$.

Take $t \in]0,1[$. Then there exists $n_0 \in \mathbb{N}$ such that $\left|1 - \frac{1}{n}\right| > t$ for each $n \geq n_0$. Hence for each $n \geq n_0$ we have that $M(a_n, b_n, t) = \frac{t}{t + \left|1-\frac{1}{n}\right|}$ and so

$$\lim_n M(a_n, b_n, t) = \frac{t}{t+1}.$$

If $t = 1$, then $t > \left|1 - \frac{1}{n}\right|$ for all $n \in \mathbb{N}$, and so $M(a_n, b_n, t) = \frac{2}{2+\left|1-\frac{1}{n}\right|} \cdot \frac{1-\left|1-\frac{1}{n}\right|}{1-\left|1-\frac{1}{n}\right|} + \frac{1}{1+\left|1-\frac{1}{n}\right|} \cdot \frac{1-1}{1-\left|1-\frac{1}{n}\right|}$. Therefore

$$\lim_n M(a_n, b_n, 1) = \frac{2}{3}$$

And finally, take $t > 1$. Then we have that $M(a_n, b_n, t) = \frac{2t}{2t+\left|1-\frac{1}{n}\right|}$ and so

$$\lim_n M(a_n, b_n, t) = \frac{2t}{2t+1}.$$

Therefore, we can consider the function $f : \mathbb{R}^+ \to]0,1]$ defined by

$$f(t) = \lim_n M(a_n, b_n, t)$$

for each $t > 0$. Hence this function is given by

$$f(t) = \begin{cases} \frac{t}{t+1}, & 0 < t < 1 \\ \frac{2t}{2t+1}, & t \geq 1 \end{cases}$$

As one can see f is not continuous at $t = 1$.

Remark 8. Since M does not satisfy (c3), then by Theorem 6 the fuzzy metric space $(]0,1], M, \cdot)$ is not completable.

Remark 9. The fuzzy metric space of Example 2 is not strong. Indeed, if we take $x = 1, y = \frac{1}{2}, z = \frac{9}{20} \in]0,1]$ and $t = \frac{11}{20} > 0$, after a tedious computation one can verify that $M(x,z,t) < M(x,y,t) \cdot M(y,z,t)$.

Chapter 4

Characterizing a class of completable fuzzy metric spaces

The material of this chapter is an adaptation to the thesis of the content of the paper by Valentín Gregori, Juan-José Miñana, Samuel Morillas and Almanzor Sapena "Characterizing a class of completable fuzzy metric spaces", which is accepted for publication in the JCR-journal Topology and its Applications.

4.1 Introduction

In this chapter we study the characterization of completable fuzzy metric spaces, in the sense of Geroge and Veeramani, given by Gregori and Romaguera 2, which we reformulate, for our convenience, in Theorem 6 of Chapter 3.

There were in the literature examples of non-completable strong fuzzy metrics that do not satisfy $(c1)$ or $(c2)$ [31, 32], and in the last chapter we have constructed a non-completable fuzzy metric space which does not satisfy $(c3)$.

In this chapter we first observe that $(c1) - (c3)$ constitute an independent axiomatic system and then we will proof, after several lemmas, that strong fuzzy metrics satisfy $(c3)$, or in other words (Theorem 8): A strong fuzzy metric space $(X, M, *)$ is completable if and only if M satisfies $(c1)$ and $(c2)$. Several corollaries can be obtained from this theorem, for instance a characterization of completable fuzzy ultrametrics (Corollary 6) and also we could obtain that metric spaces admit a unique completion, but we do not insist on it because it is well-known from the properties of the standard fuzzy metric. Several examples illustrate our results.

The structure of the chapter is as follows. In Section 4.2 we prove that $(c1) - (c3)$ constitute an independent axiomatic system. In Section 4.3 we give a characterization for the class of completable strong fuzzy metrics.

4.2 Non-completable fuzzy metric spaces

In this section we will show that the axioms $(c1) - (c3)$ constitute an independent axiomatic system. To that end, we show three examples of non-completable fuzzy metric space, which do not satisfy anyone of these three axioms but they satisfy the other two.

Example 3. (Gregori and Romaguera [32, Example 2].) Let $\{x_n\}$ and $\{y_n\}$ be two strictly increasing sequences of positive real numbers, which converge to 1 with respect to the usual topology of $\mathbb{R}$, with $A \cap B = \emptyset$, where $A = \{x_n : n \in \mathbb{N}\}$ and $B = \{y_n : n \in \mathbb{N}\}$. Put $X = A \cup B$ and define a fuzzy set M on $X \times X \times]0, \infty[$ by:

$$
\begin{aligned}
&M(x_n, x_n, t) = M(y_n, y_n, t) = 1 \text{ for all } n \in \mathbb{N}, t > 0,\\
&M(x_n, x_m, t) = x_n \wedge x_m \text{ for all } n, m \in \mathbb{N} \text{ with } n \neq m, t > 0,\\
&M(y_n, y_m, t) = y_n \wedge y_m \text{ for all } n, m \in \mathbb{N} \text{ with } n \neq m, t > 0,\\
&M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m \text{ for all } n, m \in \mathbb{N}, t \geq 1,\\
&M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m \wedge t \text{ for all } n, m \in \mathbb{N}, t \in]0, 1[.
\end{aligned}
$$

As pointed out in [32], an easy computation shows that $(X, M, *)$ is a fuzzy metric space, where $*$ is the minimum t-norm, and it satisfies conditions $(c2)$ and $(c3)$ of Theorem 6. But M does not satisfy condition $(c1)$ of Theorem 6. Indeed, in [32] it was observed that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X such that $\lim_n M(x_n, y_n, t) = 1$ for all $t \geq 1$, but $\lim_n M(x_n, y_n, t) = t$ for all $t \in]0, 1[$.

Example 4. (Gregori and Romaguera [31, Example 2].) Let $\{x_n\}$ and $\{y_n\}$ be two sequences of distinct points such that $A \cap B = \emptyset$, where $A = \{x_n : n \geq 3\}$ and $B = \{y_n : n \geq 3\}$. Put $X = A \cup B$ and define a fuzzy set M on $X \times X \times]0, \infty[$ by:

$$M(x_n, x_m, t) = M(y_n, y_m, t) = 1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right],$$

$$M(x_n, y_m, t) = M(y_m, x_n, t) = \frac{1}{n} + \frac{1}{m},$$

for all $n, m \geq 3$. In [31], it was proved that $(X, M, *)$ is a fuzzy metric space, where $*$ is the Luckasievicz t-norm ($a * b = \max\{0, a + b - 1\}$), for which both $\{x_n\}_{n\geq 3}$ and $\{y_n\}_{n\geq 3}$ are Cauchy sequences. Clearly,

$$\lim_n M(x_n, y_n, t) = \lim_n \left(\frac{1}{n} + \frac{1}{n}\right) = 0.$$

Therefore, M does not satisfy condition $(c2)$.

On the other hand, M is a stationary fuzzy metric on X, and so it satisfies conditions $(c1)$ and $(c3)$, since, obviously, this two conditions are satisfied for stationary fuzzy metrics.

Example 5. Let d be the usual metric on $\mathbb{R}$ restricted to $]0, 1]$ and consider the standard fuzzy metric M_d induced by d. Put $X =]0, 1]$ and define a fuzzy set M on $X \times X] \times]0, \infty[$ by

$$M(x, y, t) = \begin{cases} M_d(x, y, t), & 0 < t \leq d(x, y) \\ M_d(x, y, 2t) \cdot \frac{t - d(x,y)}{1 - d(x,y)} + M_d(x, y, t) \cdot \frac{1 - t}{1 - d(x,y)}, & d(x, y) < t \leq 1 \\ M_d(x, y, 2t), & t > 1 \end{cases}$$

In the last chapter it is proved that $(X, M, *)$ is a fuzzy metric space, where $*$ is the usual product. Also, it is obtained that for the Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X, given by $a_n = \frac{1}{n}$ and $b_n = 1$ for all $n \in \mathbb{N}$, the assignment

$$\lim_n M(a_n, b_n, t) = \begin{cases} \frac{t}{t+1}, & 0 < t < 1 \\ \\ \frac{2t}{2t+1}, & t \geq 1 \end{cases}$$

is a well-defined function on $]0,\infty[$ which is not continuous at $t = 1$. Therefore, M does not satisfy condition $(c3)$.

Next, we will see that M satisfies conditions $(c1)$ and $(c2)$.

For proving that M satisfies $(c1)$, we suppose that $\{a_n\}$ and be $\{b_n\}$ are two Cauchy sequences in $]0,1]$ such that $\lim_n M(a_n, b_n, s) = 1$ for some $s > 0$. By Lemma 1, we can find $t_0 > 1$, with $t_0 > s$, such that $\lim_n M(a_n, b_n, t_0) = 1$. Then,

$$\lim_n M(a_n, b_n, t_0) = \lim_n M_d(a_n, b_n, 2t_0) = \lim_n \frac{2t_0}{2t_0 + |a_n - b_n|} = 1$$

and thus $\lim_n |a_n - b_n| = 0$.

Let $t > 0$. We distinguish two cases:

(1) If $t \in]0,1]$, then there exists $n_0 \in \mathbb{N}$ such that $|a_n - b_n| < t$ for all $n \geq n_0$, since $\lim_n |a_n - b_n| = 0$. Then

$$\lim_n M(a_n, b_n, t) =$$

$$= \lim_n \left(\frac{2t}{2t + |a_n - b_n|} \cdot \frac{t - |a_n - b_n|}{1 - |a_n - b_n|} + \frac{t}{t + |a_n - b_n|} \cdot \frac{1 - t}{1 - |a_n - b_n|} \right) =$$

$$= t + 1 - t = 1$$

(2) If $t > 1$, then

$$\lim_n M(a_n, b_n, t) = \lim_n \frac{2t}{2t + |a_n - b_n|} = 1$$

Therefore, $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$, and so M satisfies $(c1)$.

Now, we will prove that M satisfies $(c2)$. Suppose the contrary, i.e., there exist two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ such that $\lim_n M(a_n, b_n, s) = 0$ for some $s > 0$. First, we claim that M-Cauchy sequences are Cauchy for the usual metric d of $\mathbb{R}$ restricted to $]0, 1]$. Indeed, if $\{a_n\}$ is a Cauchy sequence in $(X, M, *)$, then $\lim_{n,m} M(a_n, a_m, t) = 1$ for all $t > 0$. In particular, for $t > 1$ we have that $\lim_{n,m} M(a_n, a_m, t) = \lim_{n,m} \frac{2t}{2t+|a_n-a_m|} = 1$, and so $\lim_{n,m} |a_n - a_m| = 0$, i.e., $\{a_n\}$ is Cauchy in $(\mathbb{R}, d)$.

Then, there exist $a, b \in [0, 1]$ such that $\{a_n\}$ and $\{b_n\}$ converge to a and b, respectively, for the usual topology of $\mathbb{R}$ restricted to $[0, 1]$. Therefore, $\lim_n |a_n - b_n| = |a - b|$.

We distinguish two cases:

(1) Suppose that $|a - b| = 0$. Then for $t_0 > 1$ we have that

$$\lim_n M(a_n, b_n, t_0) = \lim_n \frac{2t_0}{2t_0 + |a_n - b_n|} = \frac{2t_0}{2t_0 + |a - b|} = 1.$$

So $M(a_n, b_n, t) = 1$ for all $t > 0$, since M satisfies condition $(c1)$, a contradiction.

(2) Suppose that $|a - b| \in]0, 1]$. Taking into account our assumption and Lemma 1, we can find $0 < t_0 < |a - b|$, with $t_0 < s$, such that $\lim_n M(a_n, b_n, t_0) = 0$. Then there exists $n_0 \in \mathbb{N}$ such that $|a_n - b_n| > t_0$ for all $n \geq n_0$, and so

$$\lim_n M(a_n, b_n, t_0) = \lim_n \frac{t_0}{t_0 + |a_n - b_n|} = \frac{t_0}{t_0 + |a - b|} > 0,$$

a contradiction.

Therefore, M satisfies $(c2)$.

Consequently, $(c1) - (c3)$ constitute an independent axiomatic system.

4.3 Completable strong fuzzy metrics

In this section we will show that condition $(c3)$ in Theorem 6 can be omitted when $(X, M, *)$ is a strong fuzzy metric space.

We begin this section giving five lemmas.

Lemma 5. *Let $(X, M, *)$ be a strong fuzzy metric space and let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X. For each $t > 0$, the sequence $\{M(a_n, b_n, t)\}_n$ converges in $[0, 1]$ with the usual topology of $\mathbb{R}$ restricted to $[0, 1]$.*

Proof.
Fix $t > 0$. Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in X. Since $[0, 1]$ is compact the sequence $M(a_n, b_n, t) \in [0, 1]$ has a subsequence $\{M(a_{n_k}, b_{n_k}, t)\}_k$ that converges to some $c \in [0, 1]$. We will see that $\{M(a_n, b_n, t)\}_n$ converges to c.

Contrary, suppose that $\{M(a_n, b_n, t)\}_n$ does not converge to c. Then, we can find a subsequence $\{M(a_{m_i}, b_{m_i}, t)\}_i$ of $\{M(a_n, b_n, t)\}_n$ converging to $a \in [0, 1]$, with $a \neq c$.

Now, since M is strong, for each $i, k \in \mathbb{N}$ we have that

$$M(a_{n_k}, b_{n_k}, t) \geq M(a_{n_k}, a_{m_i}, t) * M(a_{m_i}, b_{m_i}, t) * M(b_{m_i}, b_{n_k}, t)$$

and taking limit as $i, k \to \infty$, we have that

$$\lim_k M(a_{n_k}, b_{n_k}, t) \geq \lim_i M(a_{m_i}, b_{m_i}, t).$$

With a similar argument, we can also obtain

$$\lim_i M(a_{m_i}, b_{m_i}, t) \geq \lim_k M(a_{n_k}, b_{n_k}, t).$$

So, $c = \lim_k M(a_{n_k}, b_{n_k}, t) = \lim_i M(a_{m_i}, b_{m_i}, t) = a$, a contradiction.

Therefore, $\lim_n M(a_n, b_n, t) = c$. □

Lemma 6. *Let* $(X, M, *)$ *be a fuzzy metric space, let* $\{a_n\}$ *be a Cauchy sequence in* X *and let* $\{t_n\}$ *be a strictly increasing (decreasing) sequence of positive real numbers converging to* $t_0 > 0$ *(for the usual topology of* $\mathbb{R}$*). Then* $\lim_{n,m} M(a_n, a_m, t_n) = 1$.

Proof.
It is immediate. □

Lemma 7. *Let* $(X, M, *)$ *be a strong fuzzy metric space. Let* $\{a_n\}$, $\{b_n\}$ *be two Cauchy sequences in* X *and let* $\{t_n\}$ *be a strictly increasing (decreasing) sequence of positive real numbers converging to* $t_0 > 0$ *(for the usual topology of* $\mathbb{R}$*). Then, the sequence* $\{M(a_n, b_n, t_n)\}_n$ *converges in* $[0, 1]$*, with the usual topology of* $\mathbb{R}$ *restricted to* $[0, 1]$.

Proof.
Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$ be a strictly increasing sequence of positive real numbers converging to $t_0 > 0$. Consider the sequence $\{M(a_n, b_n, t_n)\}_n \subset [0, 1]$. Since $[0, 1]$ is compact then, there exists a subsequence $\{M(a_{n_k}, b_{n_k}, t_{n_k})\}_k$ of $\{M(a_n, b_n, t_n)\}_n$ converging to $c \in [0, 1]$.

Suppose that $\{M(a_n, b_n, t_n)\}_n$ does not converge to c. Then, we can find a subsequence $\{M(a_{m_i}, b_{m_i}, t_{m_i})\}_i$ of $\{M(a_n, b_n, t_n)\}_n$ converging to $a \in [0, 1]$, with $a \neq c$.

Suppose, without loss of generality, that $a > c$. We will construct, by induction, two subsequences $\{M(a_{n_{k_l}}, b_{n_{k_l}}, t_{n_{k_l}})\}_l$ and $\{M(a_{m_{i_j}}, b_{m_{i_j}}, t_{m_{i_j}})\}_j$ of $\{M(a_{n_k}, b_{n_k}, t_{n_k})\}_k$ and $\{M(a_{m_i}, b_{m_i}, t_{m_i})\}_i$, respectively, as follows.

Take $m_{i_1} = m_1 \in \mathbb{N}$. We can choose $n_{k_1} \in \mathbb{N}$ such that $n_{k_1} > m_{i_1}$ and $t_{n_{k_1}} > t_{m_{i_1}}$ (since $\{t_{n_k}\}$ is strictly increasing). By Lemma 1 and using that M is strong, we have that

$$M(a_{n_{k_1}}, b_{n_{k_1}}, t_{n_{k_1}}) \geq M(a_{n_{k_1}}, b_{n_{k_1}}, t_{m_{i_1}}) \geq$$

$$M(a_{n_{k_1}}, a_{m_{i_1}}, t_{m_{i_1}}) * M(a_{m_{i_1}}, b_{m_{i_1}}, t_{m_{i_1}}) * M(b_{m_{i_1}}, b_{n_{k_1}}, t_{m_{i_1}}).$$

Now, we choose $m_{i_2} \in \mathbb{N}$ such that $m_{i_2} > n_{k_1}$. Given m_{i_2}, we can choose $n_{k_2} \in \mathbb{N}$ such that $n_{k_2} > m_{i_2}$ and $t_{n_{k_2}} > t_{m_{i_2}}$. By Lemma 1 and using that M is strong, we have that

$$M(a_{n_{k_2}}, b_{n_{k_2}}, t_{n_{k_2}}) \geq M(a_{n_{k_2}}, b_{n_{k_2}}, t_{m_{i_2}}) \geq$$

$$M(a_{n_{k_2}}, a_{m_{i_2}}, t_{m_{i_2}}) * M(a_{m_{i_2}}, b_{m_{i_2}}, t_{m_{i_2}}) * M(b_{m_{i_2}}, b_{n_{k_2}}, t_{m_{i_2}}).$$

Therefore, by induction on j we have that

$$M(a_{n_{k_j}}, b_{n_{k_j}}, t_{n_{k_j}}) \geq$$

$$M(a_{n_{k_j}}, a_{m_{i_j}}, t_{m_{i_j}}) * M(a_{m_{i_j}}, b_{m_{i_j}}, t_{m_{i_j}}) * M(b_{m_{i_j}}, b_{n_{k_j}}, t_{m_{i_j}}).$$

Taking limit as $j \to \infty$, by Lemma 6 we have that.

$$c = \lim_j M(a_{n_{k_j}}, b_{n_{k_j}}, t_{n_{k_j}}) \geq \lim_j M(a_{m_{i_j}}, b_{m_{i_j}}, t_{m_{i_j}}) = a,$$

a contradiction.

Therefore, $\lim_n M(a_n, b_n, t_n) = c$.

If $\{t_n\}$ is strictly decreasing, it is proved in a similar way. □

Lemma 8. *Let $(X, M, *)$ be a strong fuzzy metric space. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$, $\{s_n\}$ be two strictly increasing (decreasing) sequences of positive real numbers converging to $t_0 > 0$ (for the usual topology of $\mathbb{R}$). Then, $\lim_n M(a_n, b_n, t_n) = \lim_n M(a_n, b_n, s_n)$.*

Proof.
Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$, $\{s_n\}$ be two strictly increasing sequence of positive real numbers converging to $t_0 > 0$. By Lemma 7, there exist $a, c \in [0,1]$ such that $\lim_n M(a_n, b_n, t_n) = a$ and $\lim_n M(a_n, b_n, s_n) = c$. Contrary, suppose that $\lim_n M(a_n, b_n, t_n) \neq \lim_n M(a_n, b_n, s_n)$. Suppose, without loss of generality, that $a < c$.

In a similar way that in the proof of the above lemma, we will construct two subsequences $\{M(a_{n_k}, b_{n_k}, t_{n_k})\}_k$ and $\{M(a_{m_i}, b_{m_i}, s_{m_i})\}_i$ of the sequences $\{M(a_n, b_n, t_n)\}_n$ and $\{M(a_n, b_n, s_n)\}_n$, respectively, where $t_{n_k} > s_{m_k}$ for all $k \in \mathbb{N}$ and we have that

$$M(a_{n_k}, b_{n_k}, t_{n_k}) \geq$$

$$M(a_{n_k}, a_{m_k}, s_{m_k}) * M(a_{m_k}, b_{m_k}, s_{m_k}) * M(b_{m_k}, b_{n_k}, s_{m_k})$$

for each $k \in \mathbb{N}$.

Taking limit as $k \to \infty$, by Lemma 6 we have that

$$a = \lim_k M(a_{n_k}, b_{n_k}, t_{n_k}) \geq \lim_k M(a_{m_k}, b_{m_k}, s_{m_k}) = c,$$

a contradiction.

Therefore, $\lim_n M(a_n, b_n, t_n) = \lim_n M(a_n, b_n, s_n)$.

The case in which $\{t_n\}$ and $\{s_n\}$ are strictly decreasing is proved in a similar way. □

Lemma 9. *Let $(X, M, *)$ be a strong fuzzy metric space. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$ be a strictly increasing (decreasing) sequence of positive real numbers converging to $t_0 > 0$ (for the usual topology of $\mathbb{R}$). Then, $\lim_n M(a_n, b_n, t_n) = \lim_n M(a_n, b_n, t_0)$.*

Proof.
Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$ be a strictly increasing sequence of positive real numbers converging to $t_0 > 0$.

By Lemma 7, there exists $a \in [0, 1]$ such that $\lim_n M(a_n, b_n, t_n) = a$ and by Lemma 5, there exists $c \in [0, 1]$ such that $\lim_n M(a_n, b_n, t_0) = c$. Note that, by Lemma 1, since $\{t_n\}$ is strictly increasing converging to t_0, we have that for each $n \in \mathbb{N}$ we have that $M(a_n, b_n, t_n) \leq M(a_n, b_n, t_0)$ and so $a \leq c$.

Since $\lim_n M(a_n, b_n, t_0) = c$, for each $\epsilon \in]0, 1[$, with $\epsilon < c$, we can find $n_\epsilon \in \mathbb{N}$ such that $M(a_{n_\epsilon}, b_{n_\epsilon}, t_0) \in]c - \epsilon/2, c + \epsilon/2[$. By axiom $(GV5)$ we can find $\delta_{n_\epsilon} > 0$ such that $M(a_{n_\epsilon}, b_{n_\epsilon}, t) \in]c - \epsilon, c + \epsilon[$ for each $t \in]t_0 - \delta_{n_\epsilon}, t_0[$.

Suppose that $c > a$. Taking into account the last paragraph, we will construct a sequence $\{M(a_{n_k}, b_{n_k}, s_k)\}_k$, where $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are subsequences of $\{a_n\}$ and $\{b_n\}$, respectively, converging to c, as follows.

Let $i_1 \in \mathbb{N}$, with $\frac{1}{i_1} < \min\{c, t_0\}$, then there exist $n_1 \in \mathbb{N}$ and $s_1 \in]t_0 - \frac{1}{i_1}, t_0[$ such that $M(a_{n_1}, b_{n_1}, s_1) > c - \frac{1}{i_1}$. Choose $i_2 \in \mathbb{N}$, with $\frac{1}{i_2} < t_0 - s_1$, then we can find $n_2 \in \mathbb{N}$, with $n_2 > n_1$ and $s_2 \in]t_0 - \frac{1}{i_2}, t_0[$, such that $M(a_{n_2}, b_{n_2}, s_2) > c - \frac{1}{i_2}$. Thus, in this way by induction on k, we construct the sequence $\{M(a_{n_k}, b_{n_k}, s_k)\}_k$, which obviously satisfies $\lim_k M(a_{n_k}, b_{n_k}, s_k) = c$. On the other hand, $\{s_k\}$ is a strictly increasing sequence of positive real numbers converging to t_0. Therefore, by Lemma 8 $\lim_k M(a_{n_k}, b_{n_k}, r_k) = c$ for each strictly increasing sequence $\{r_k\}$ of positive real numbers converging to t_0. In particular, if we consider the subsequence $\{t_{n_k}\}$ of $\{t_n\}$, then $\lim_k M(a_{n_k}, b_{n_k}, t_{n_k}) = c$, a contradiction, since $\lim_n M(a_n, b_n, t_n) = a < c$.

Therefore, $\lim_n M(a_n, b_n, t_n) = c$.

The case of $\{t_n\}$ strictly decreasing is proved in a similar way. □

Theorem 7. *Let $(X, M, *)$ be a strong fuzzy metric space, and let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X. Then the assignment*

$$t \to \lim_n M(a_n, b_n, t), \text{ for each } t > 0$$

is a continuous function on $]0, \infty[$ provided with the usual topology of $\mathbb{R}$.

Proof.
Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in X. By Lemma 5, the assignment $t \to \lim_n M(a_n, b_n, t)$ for each $t > 0$, is a well-defined function on $]0, \infty[$ to $[0, 1]$.

Next, we will see that this function is continuous. First we see that for each $t > 0$ the mentioned function is left-continuous.

Fix $t_0 > 0$. By Lemma 5, we have that there exists $c \in [0,1]$ such that $\lim_n M(a_n, b_n, t_0) = c$. We distinguish two cases:

(1) Suppose that $c = 0$. By Lemma 1 and Lemma 5 we have that $\lim_n M(a_n, b_n, s) = 0$ for all $s \in]0, t_0[$.

So, the function $t \to \lim_n M(a_n, b_n, t)$ is left-continuous at t_0.

(2) Suppose that $c \in]0,1]$ and suppose contrary that the function $t \to \lim_n M(a_n, b_n, t)$ is not left-continuous at t_0.

Then, there exists $\epsilon_0 \in]0,1[$ such that for each $\delta \in]0, t_0[$ we can find $t_\delta \in]t_0 - \delta, t_0[$ such that $b_\delta = \lim_n M(a_n, b_n, t_\delta) \notin]c - \epsilon_0, c + \epsilon_0[$. Note that, by Lemma 1, $b_\delta \leq c$ and so $b_\delta < c - \epsilon_0$.

On the other hand, given $t_\delta \in]t_0 - \delta, t_0[$, since $\lim_n M(a_n, b_n, t_\delta) = b_\delta < c - \epsilon_0$, for $\epsilon_0/2$ we can find $n(\delta) \in \mathbb{N}$ such that $M(a_n, b_n, t_\delta) \in]b_\delta - \epsilon_0/2, b_\delta + \epsilon_0/2[$ for each $n \geq n(\delta)$. Therefore, $M(a_n, b_n, t_\delta) < c - \epsilon_0/2$ for each $n \geq n(\delta)$.

Now, we will construct a sequence $\{M(a_{n_k}, b_{n_k}, t_k)\}_k$, where $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are subsequences of $\{a_n\}$ and $\{b_n\}$, respectively, as follows.

Consider $i_1 \in \mathbb{N}$, with $\frac{1}{i_1} < t_0$. We can find $t_1 \in]t_0 - \frac{1}{i_1}, t_0[$ such that $\lim_n M(a_n, b_n, t_1) < c - \epsilon_0$. Then, we can find $n(i_1) \in \mathbb{N}$ such that $M(a_n, b_n, t_1) < c - \epsilon_0/2$ for each $n \geq n(i_1)$. We choose $n_1 = n(i_1)$.

Consider now, $i_2 \in \mathbb{N}$, with $\frac{1}{i_2} \in]t_1, t_0[$. We can find $t_2 \in]t_0 - \frac{1}{i_2}, t_0[$ such that $\lim_n M(a_n, b_n, t_2) < c - \epsilon_0$. Then, we can find $n(i_2) \in \mathbb{N}$ such that $M(a_n, b_n, t_2) < c - \epsilon_0/2$ for each $n \geq n(i_2)$. We choose $n_2 \geq n(i_2)$, with $n_2 > n_1$.

So, by induction on k we construct the sequence $\{M(a_{n_k}, b_{n_k}, t_k)\}_k$, where $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are subsequences of $\{a_n\}$ and $\{b_n\}$, respectively, such that $M(a_{n_k}, b_{n_k}, t_k) < c - \epsilon_0/2$ for each $k \in \mathbb{N}$. Also, $\{t_k\}$

is a strictly increasing sequence of positive real numbers converging to t_0. Therefore, by Lemma 9, we have that $\lim_k M(a_{n_k}, b_{n_k}, t_k) = \lim_k M(a_{n_k}, b_{n_k}, t_0) = \lim_n M(a_n, b_n, t_0) = c$, a contradiction.

So, the above assignment is a left-continuous function at t_0.

In a similar way it is proved that $t \to \lim_n M(a_n, b_n, t)$ is right-continuous at t_0 using a strictly decreasing sequence $\{t_n\}$ converging to t_0 and thus it is continuous at t_0.

Hence, the assignment $t \to \lim_n M(a_n, b_n, t)$ is a continuous function on $]0, \infty[$. □

Theorem 8. *A strong fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following conditions are fulfilled:*

(c1) $\lim_n M(a_n, b_n, s) = 1$ *for some $s > 0$ implies* $\lim_n M(a_n, b_n, t) = 1$ *for all $t > 0$.*

(c2) $\lim_n M(a_n, b_n, t) > 0$ *for all $t > 0$.*

Proof.
The proof is immediate using Theorem 7 and Theorem 6. □

By Theorem 3 and the fact that the minimum t-norm is integral, the following corollaries are immediate.

Corollary 5. *Let* $(X, M, *)$ *be a strong fuzzy metric space and suppose that* $*$ *is integral. Then* $(X, M, *)$ *is completable if and only if for each pair of Cauchy sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* X *the condition* $(c1)$ *is satisfied.*

Corollary 6. *Let* $(X, M, *)$ *be a fuzzy ultrametric space. Then* $(X, M, *)$ *is completable if and only if for each pair of Cauchy sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* X *the condition* $(c1)$ *is satisfied.*

Remark 10. We cannot remove the condition that $*$ is integral in Corollary 5 as shows Example 4. In addition, the fuzzy metric of Example 3 is a non-completable fuzzy ultrametric which does not satisfy $(c1)$.

Chapter 5

A note on convergence in fuzzy metric spaces

The material of this chapter is an adaptation to the thesis of the content of the paper by Valentín Gregori, Juan-José Miñana and Samuel Morillas, "A note on convergence in fuzzy metric spaces", published in the journal Iranian Journal of Fuzzy Systems **11 (4)** (2014) 75-85.

5.1 Introduction

In this chapter we continue the work started in [25, 57], but in the opposite way, that is, we strengthen the condition of convergence on t. So, we introduce the following concept: A sequence $\{x_n\}$ in $(X, M, *)$ is called s-convergent if $\lim_n M(x_n, x_0, \frac{1}{n}) = 1$, for some $x_0 \in X$. This concept is close to convergence, and indeed, s-convergence implies convergence but the converse is not true, in general. A fuzzy metric space in which every convergent sequence is s-convergent will be called s-fuzzy metric space. Our first goal is to obtain a characterization of s-fuzzy metric spaces by means of local bases similar to the case of principal fuzzy metric spaces. Indeed, $(X, M, *)$ is an s-fuzzy metric space if and only if $\{\bigcap_{t>0} B(x, r, t) : r \in]0, 1[\}$ is a local base at x, for each $x \in X$ (Corollary 9).

The second goal is to characterize a certain class of fuzzy metrics by means of our concept. Indeed, for those fuzzy metrics M on X such that $N_M(x, y) = \bigwedge_{t>0} M(x, y, t)$ is a (stationary) fuzzy metric on X, we prove that the topologies on X deduced from M and N_M agree if and only if M is s-fuzzy metric (Theorem 10). Appropriate examples illustrate that the implications

$$s - \text{convergence} \Rightarrow \text{convergence} \Rightarrow p - \text{convergence},$$

have only one sense, in general.

Finally, to provide an overview, a classification of fuzzy metrics is drawn. This classification attends, specially, to the behaviour of fuzzy metrics with respect to the different types of convergence studied and it also involves some well-known families of fuzzy metrics used in this chapter.

The structure of the chapter is as follows. In Section 5.2 we introduce and study the concept of s-convergence, in Section 5.3 we study a certain class of s-fuzzy metrics and in Section 5.4 we classify fuzzy metric spaces in accordance with the concepts of p and s-convergence.

5.2 s-convergence

Definition 14. Let $(X, M, *)$ be a fuzzy metric space. We will say that a sequence $\{x_n\}$ in X is *s-convergent* to $x_0 \in X$ if $\lim_n M(x_n, x_0, \frac{1}{n}) = 1$.

Equivalently, $\{x_n\}$ is s-convergent to x_0 if for each $r \in]0,1[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_0, \frac{1}{n}) > 1 - r$ for all $n \geq n_0$, i.e. $x_n \in B(x_0, r, \frac{1}{n})$ for all $n \geq n_0$.

Under this terminology the following consequences are immediate:

Consequences 1.

(i) If M is stationary then convergent sequences are s-convergent.

(ii) Constant sequences are s-convergent.

In consequence:

(iii) If $\mathcal{T}_M$ is the discrete topology then convergent sequences are s-convergent.

Proposition 11. *Let $(X, M, *)$ be a fuzzy metric space. Each s-convergent sequence in X is convergent.*

Proof.
Suppose that $\{x_n\}$ is s-convergent to x_0. Let $t > 0$. We choose $n_0 \in \mathbb{N}$ such

that $\frac{1}{n_0} < t$. We have that $M(x_n, x_0, t) \geq M(x_n, x_0, \frac{1}{n})$ for all $n \geq n_0$, and so $\lim_n M(x_n, x_0, t) = 1$, for all $t > 0$ and so $\{x_n\}$ converges to x_0. $\square$

Now we will see that the converse of the last proposition is not true, in general.

Example 6. ([25, 91]) On $[0, \infty[$ we consider the principal fuzzy metric $(M, \cdot)$ where M is defined by

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}, \quad x, y \in [0, \infty[, t > 0.$$

Since $\lim_n M(\frac{1}{n}, 0, t) = \lim_n \frac{0+t}{\frac{1}{n}+t} = 1$ for all $t > 0$, then $\{\frac{1}{n}\}$ converges to 0, but it is not s-convergent to 0, since $\lim_n M(\frac{1}{n}, 0, \frac{1}{n}) = \frac{0+\frac{1}{n}}{\frac{1}{n}+\frac{1}{n}} = \frac{1}{2}$.

Further, if $\{x_n\}$ is a sequence that converges to x_0 in a fuzzy metric space $(X, M, *)$ we cannot ensure, in general, that $\lim_n M(x_n, x_0, \frac{1}{n})$ exists. Indeed, in the current example, if we consider the sequence $\{x_n\}$ given by $x_n = \frac{1}{n}$ if n is odd and $x_n = \frac{1}{n^2}$ if n is even, then $\{x_n\}$ converges to 0 and it is easy to see $\lim_n M(x_n, 0, \frac{1}{n})$ does not exist.

Proposition 12. *Let $(X, M, *)$ be a fuzzy metric space.*

(i) Each subsequence of an s-convergent sequence in X is s-convergent.

(ii) Each convergent sequence in X admits an s-convergent subsequence.

Proof.

(i) Suppose that $\{x_n\}$ is an s-convergent sequence to x_0 in X, and consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. If we take a fix $r \in]0,1[$, by our assumption there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_0, \frac{1}{n}) > 1-r$ for each $n \geq n_0$. Now, for all $k \in \mathbb{N}$ we have that $M(x_{n_k}, x_0, \frac{1}{k}) \geq M(x_{n_k}, x_0, \frac{1}{n_k})$, since $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Thus if we take k_0 such that $n_{k_0} \geq n_0$, then $M(x_{n_k}, x_0, \frac{1}{k}) \geq M(x_{n_k}, x_0, \frac{1}{n_k}) > 1 - r$ for each $k \geq k_0$.

(ii) Let $\{x_n\}$ be a convergent sequence to x_0 in X. We will construct the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ as follows:

Since $\{B(x_0, \frac{1}{m}, \frac{1}{m}) : m \geq 2\}$ is a local base at x_0 and $\{x_n\}$ converges to x_0, then for $k = 2$ we can find $n_2 \in \mathbb{N}$ with $n_2 \geq 2$ such that $x_{n_2} \in B(x_0, \frac{1}{2}, \frac{1}{2})$. By induction on k ($k \geq 3$) we choose $x_{n_k} \in B(x_0, \frac{1}{k}, \frac{1}{k})$, with $n_k \geq \max\{n_{k-1}, k\}$ and so we construct the sequence $\{x_{n_k}\}$. By construction $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Finally, we will see that $\{x_{n_k}\}$ is s-convergent. Let $r \in]0,1[$. We can find $k_0 \in \mathbb{N}$ such that $0 < \frac{1}{k_0} < r$ and then for all $k \geq k_0$ we have that $0 < \frac{1}{k} < \frac{1}{k_0} < r$. Thus $x_{n_k} \in B(x_0, \frac{1}{k}, \frac{1}{k}) \subset B(x_0, r, \frac{1}{k})$ for all $k \geq k_0$ and then $\{x_{n_k}\}$ is s-convergent.

□

Definition 15. We will say that $(X, M, *)$ is an *s-fuzzy metric space* or simply M is an s-fuzzy metric if every convergent sequence is s-convergent.

By Consequence 1 and the last definition we have the next corollary.

Corollary 7. *Let $(X, M, *)$ be a fuzzy metric space.*

(i) If $\mathcal{T}_M$ is the discrete topology then M is an s-fuzzy metric.

(ii) If M is stationary then M is an s-fuzzy metric.

Theorem 9. *Let $(X, M, *)$ be a fuzzy metric space. Take $x_0 \in X$ and let $\{t_n\}$ be a sequence of positive real numbers that converges to 0 in the usual topology of $\mathbb{R}$ restricted to $[0, \infty[$. Then each convergent sequence $\{x_n\}$ to x_0 satisfies that $\lim_n M(x_n, x_0, t_n) = 1$ if and only if $\bigcap_{t>0} B(x_0, r, t)$ is a neighborhood of x_0 for each $r \in]0, 1[$.*

Proof.
Suppose that $\bigcap_{t>0} B(x_0, r, t)$ is a neighborhood of x_0 for each $r \in]0, 1[$ and consider a convergent sequence $\{x_n\}$ to x_0 in X. Let $\epsilon \in]0, 1[$. Since $\bigcap_{t>0} B(x_0, \epsilon, t)$ is a neighborhood of x_0 there exists $n_\epsilon \in \mathbb{N}$ such that $x_n \in \bigcap_{t>0} B(x_0, \epsilon, t)$ for all $n \geq n_\epsilon$, i.e. $M(x_0, x_n, t) > 1 - \epsilon$ for all $t > 0$, and for all $n \geq n_\epsilon$. In particular $M(x_0, x_n, t_n) > 1 - \epsilon$ for all $n \geq n_\epsilon$. Thus $\lim_n M(x_0, x_n, t_n) = 1$.

Conversely, suppose that there exists $r_0 \in]0, 1[$ such that $\bigcap_{t>0} B(x_0, r_0, t)$ is not a neighborhood of x_0. Equivalently, $\bigcap_n B(x_0, r_0, t_n)$ is not a neighborhood of x_0. Recall that $\{B(x_0, \frac{1}{n}, \frac{1}{n}) : n \geq 2\}$ is a decreasing local base at x_0. So, for each $n \geq 2$ we have that $B(x_0, \frac{1}{n}, \frac{1}{n}) \nsubseteq \bigcap_n B(x_0, r_0, t_n)$. We construct a sequence $\{x_n\}$ taking $x_n \in B(x_0, \frac{1}{n}, \frac{1}{n}) \setminus (\bigcap_n B(x_0, r_0, t_n))$ for all $n \geq 2$. This sequence $\{x_n\}$ is convergent to x_0. (Indeed, let $\delta \in]0, 1[$ and $t > 0$, and consider $B(x_0, \delta, t)$, then there exists $n_0 \in \mathbb{N}$ such that $B(x_0, \frac{1}{n_0}, \frac{1}{n_0}) \subset B(x_0, \delta, t)$ and so for all $n \geq n_0$ we have that $B(x_0, \frac{1}{n}, \frac{1}{n}) \subset B(x_0, \delta, t)$ and then $x_n \in B(x_0, \delta, t)$ for all $n \geq n_0$). Now, we will see that $\lim_n M(x_n, x_0, t_n) \neq 1$, by contradiction. Suppose that $\lim_n M(x_n, x_0, t_n) = 1$. Then for $r_0 \in]0, 1[$ there exists $n_{r_0} \in \mathbb{N}$ such that $M(x_n, x_0, t_n) > 1 - r_0$ for all $n \geq n_{r_0}$ and in consequence $x_n \in B(x_0, r_0, t_n)$ for all $n \geq n_{r_0}$, a contradiction. $\square$

Using the sequence $\{\frac{1}{n}\}$ as $\{t_n\}$ in the above theorem and taking into account that for each $r \in]0,1[$ and $t > 0$ we have that $\bigcap_{s>0} B(x_0, r, s) \subset B(x_0, r, t)$ for each $x_0 \in X$, we obtain the next corollary.

Corollary 8. *Let $(X, M, *)$ be a fuzzy metric space and let $x_0 \in X$. Then the following are equivalent:*

(i) Each sequence converging to x_0 is s-convergent.

(ii) $\bigcap_{t>0} B(x_0, r, t)$ is a neighborhood of x_0 for each $r \in]0,1[$.

(iii) $\{\bigcap_{t>0} B(x_0, r, t) : r \in]0,1[\}$ is a local base at x_0.

From this corollary it is immediate to obtain the following corollary.

Corollary 9. *Let $(X, M, *)$ be a fuzzy metric space. Then the following are equivalent:*

(i) M is an s-fuzzy metric.

(ii) $\bigcap_{t>0} B(x, r, t)$ is a neighborhood of x for all $x \in X$, and for all $r \in]0,1[$.

(iii) $\{\bigcap_{t>0} B(x, r, t) : r \in]0,1[\}$ is a local base at x, for each $x \in X$.

Taking into account Theorem 1 we have the next corollary.

Corollary 10. *Each p-convergent sequence $\{x_n\}$ in X is s-convergent if and only if X is a principal s-fuzzy metric space.*

Proposition 13. *Let (X, d) be a metric space. Then $(X, M_d, \cdot)$ is an s-fuzzy metric space if and only if $\mathcal{T}(d)$ is the discrete topology.*

Proof.
Fix $r \in]0,1[$ and $x_0 \in X$. We will see that $\bigcap_{t>0} B(x_0,r,t) = \{x_0\}$. Indeed, $B(x_0,r,t) = \{y \in X : d(x,y) < \frac{t \cdot r}{1-r}\}$ for all $t > 0$ and so $\bigcap_{t>0} B(x_0,r,t) = \bigcap_{t>0}\{y \in X : d(x,y) < \frac{t \cdot r}{1-r}\} = \{y \in X : d(x,y) \leq 0\} = \{x_0\}$. Then by Corollary 9 $(X, M_d, \cdot)$ is an s-fuzzy metric space if and only if x_0 is isolated, that is $\mathcal{T}(d)$ is the discrete topology. □

5.3 On a class of s-fuzzy metrics

If $(X, M, *)$ is a fuzzy metric space we define the mapping N_M on X^2 given by $N_M(x,y) = \bigwedge_{t>0} M(x,y,t)$ for all $x, y \in X$. In this section we are interested in studying those non-stationary fuzzy metric spaces $(X, M, *)$ such that $(N_M, *)$ is a (stationary) fuzzy metric on X and we establish a relationship between those fuzzy metrics and s-fuzzy metrics. Notice that if X is a set with at least two elements and d is a metric on X it is obvious that $\bigwedge_{t>0} M_d(x,y,t) = 0$ for $x \neq y$, and so N_{M_d} is not a fuzzy metric on X.

We start with the following lemma (which proof we omit).

Lemma 10. *Let $(M, *)$ be a fuzzy metric on X. Then*

(i) $(N_M, *)$ is a stationary fuzzy metric on X if and only if $N_M(x,y) > 0$ for all $x, y \in X$. In such a case:

(ii) $\mathcal{T}_{N_M} \succ \mathcal{T}_M$.

Theorem 10. *Let $(M, *)$ be a fuzzy metric on X such that $N_M(x,y) > 0$*

for each $x, y \in X$. *Then*

$$\mathcal{T}_{N_M} = \mathcal{T}_M \text{ if and only if } M \text{ is an } s\text{-fuzzy metric.}$$

Proof.
Suppose that $\mathcal{T}_{N_M} = \mathcal{T}_M$.

Fix $x_0 \in X$, $r \in]0,1[$. We will see that $\bigcap_{t>0} B_M(x_0, r, t)$ is a $\mathcal{T}_M$-neighborhood of x_0.

Consider the open ball $B_{N_M}(x_0, r)$ relative to N_M. Since $\mathcal{T}_M = \mathcal{T}_{N_M}$ we can find $r_1 \in]0,1[$, $t_1 > 0$ such that $B_M(x_0, r_1, t_1) \subset B_{N_M}(x_0, r)$. We will see that $B_{N_M}(x_0, r) \subset \bigcap_{t>0} B_M(x_0, r, t)$. Indeed, if $y \in B_{N_M}(x_0, r)$ then $N_M(x_0, y) > 1 - r$, i.e. $\bigwedge_{t>0} M(x_0, y, t) > 1 - r$ and so $M(x_0, y, t) > 1 - r$ for all $t > 0$, i.e. $y \in B_M(x_0, r, t)$ for all $t > 0$. Then $y \in \bigcap_{t>0} B_M(x_0, r, t)$. Now, since $B_M(x_0, r_1, t_1) \subset B_{N_M}(x_0, r) \subset \bigcap_{t>0} B_M(x_0, r, t)$ then $\bigcap_{t>0} B_M(x_0, r, t)$ is a $\mathcal{T}_M$-neighborhood of x_0, and so by Corollary 9 M is an s-fuzzy metric.

Conversely, suppose that M is an s-fuzzy metric. By the last lemma we have that $\mathcal{T}_{N_M} \succ \mathcal{T}_M$. Now, we will see that $\mathcal{T}_M \succ \mathcal{T}_{N_M}$. Let $x_0 \in X$, $r \in]0,1[$ and consider $B_{N_M}(x_0, r)$. We will see that (the $\mathcal{T}_M$-neighborhood of x_0) $\bigcap_{t>0} B_M(x_0, \frac{r}{2}, t)$ is contained in $B_{N_M}(x_0, r)$. Indeed, if we consider $y \in \bigcap_{t>0} B_M(x_0, \frac{r}{2}, t)$ then $y \in B_M(x_0, \frac{r}{2}, t)$ for all $t > 0$, i.e $M(x_0, y, t) > 1 - \frac{r}{2}$ for all $t > 0$, so $\bigwedge_{t>0} M(x_0, y, t) \geq 1 - \frac{r}{2}$, thus $N_M(x_0, y) > 1 - r$ and so $y \in B_{N_M}(x_0, r)$. □

An example of s-fuzzy metric fulfilling all conditions of Theorem 10 is given later in Example 8. On the other hand the next example shows that the class of fuzzy metrics M such that N_M is a fuzzy metric is not contained in the class of s-fuzzy metrics and *vice-versa*.

Example 7. (a) (N_M is a fuzzy metric and M is not an s-fuzzy metric)

Let $X =]0,1]$ be endowed with the usual metric d of $\mathbb{R}$. We define

$$M(x,y,t) = \begin{cases} 1 - \frac{1}{2}d(x,y)^t, & \text{if } 0 < t \leq 1 \\ \\ 1 - \frac{1}{2}d(x,y), & \text{if } t > 1 \end{cases}$$

It is easy to verify that $\{(M_t, \mathfrak{L}) : t > 0\}$ is an increasing family of stationary fuzzy metrics on $]0,1]$, where $M_t(x,y) = M(x,y,t)$ for each $t > 0$. Also that $\mathcal{T}_{M_t}$ is $\mathcal{T}(d)$ (the usual topology of $\mathbb{R}$ restricted to $]0,1]$), for all $t > 0$. Then from [28, 68] one can conclude that $(X, M, \mathfrak{L})$ is a fuzzy metric space and $\mathcal{T}_M$ is $\mathcal{T}(d)$.

On the other hand, $N_M(x,y) = \bigwedge_{t>0} M(x,y,t) > 0$ for all $x, y \in]0,1]$, since

$$N_M(x,y) = \bigwedge_{t>0} M(x,y,t) = \begin{cases} 1, & \text{if } x = y \\ \\ \frac{1}{2}, & \text{if } x \neq y \end{cases}$$

By the last lemma we have that $(N_M, \mathfrak{L})$ is a fuzzy metric on X, and it is obvious that $\mathcal{T}_{N_M}$ is the discrete topology. Therefore $\mathcal{T}_{N_M} \neq \mathcal{T}_M$.

(b) (M is an s-fuzzy metric but N_M is not a fuzzy metric)

The fuzzy metric $(M, \cdot)$ of Example 9 is an s-fuzzy metric on X, but $N_M(\frac{1}{2}, \frac{1}{\pi}) = \bigwedge_{t>0} M(\frac{1}{2}, \frac{1}{\pi}, t) = \bigwedge_{t>0} \frac{2}{\pi} \cdot t = 0$ and so $(N_M, \cdot)$ is not a fuzzy metric on X.

5.4 A classification of fuzzy metric spaces

Let X be a non-empty set. Denote by $\mathcal{D}$ the family of fuzzy metrics that generate the discrete topology on X, and by $\mathcal{M}_s$ and $\mathcal{S}$ the families of s-fuzzy metrics and stationary fuzzy metrics on X, respectively. Attending to Consequences 1 we have that $\mathcal{D} \subset \mathcal{M}_s$ and $\mathcal{S} \subset \mathcal{M}_s$.

Also, denote the families of principal fuzzy metrics and standard fuzzy metrics on X by $\mathcal{P}$ and $\mathcal{M}_d$, respectively. From [25] we know that $\mathcal{S} \subset \mathcal{P}$ and $\mathcal{M}_d \subset \mathcal{P}$. Now, from our previous results and the implications

$$s - convergence \Rightarrow convergence \Rightarrow p - convergence$$

we can conclude the diagram of inclusions in Figure 5.1.

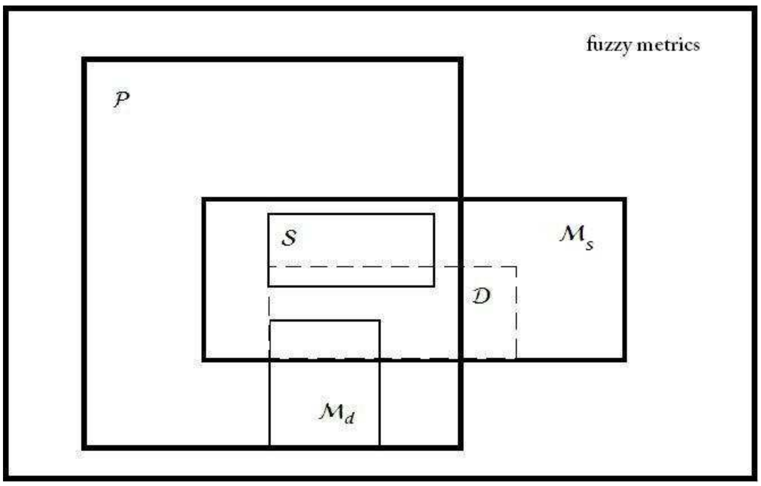

Figure 5.1: Diagram of inclusions

Next, we give examples which show that all (non-trivial) inclusions in the diagram are strict. In some cases, appropriate sequences with and without some type of convergence are also provided.

Notice that in Example 6 we have seen a principal non-s-fuzzy metric space.

Example 8. (A non-stationary principal s-fuzzy metric space).
Let $(]0,\infty[, M, \cdot)$ the fuzzy metric space, where M is the fuzzy metric of Example 6. It is known that M is principal [25]. Now,

$$N_M(x,y) = \bigwedge_{t>0} \frac{\min\{x,y\}+t}{\max\{x,y\}+t} = \frac{\min\{x,y\}}{\max\{x,y\}} > 0 \text{ for each } x,y \in]0,\infty[.$$

Then by Theorem 10 we have that M is an s-fuzzy metric, since $\mathcal{T}_{N_M} = \mathcal{T}_M$, [27].

Remark 11. Since the completion of the fuzzy metric space of Example 8 is the fuzzy metric space of Example 6, [27], then the completion of an s-fuzzy metric space is not necessarily an s-fuzzy metric space.

Example 9. (A non-stationary non-principal s-fuzzy metric space). Let $X =]0,1]$, $A = X \cap \mathbb{Q}$, $B = X \setminus A$. Define the function M on $X^2 \times \mathbb{R}^+$ by

$$M(x,y,t) = \begin{cases} \frac{\min\{x,y\}}{\max\{x,y\}} \cdot t, & (x \in A, y \in B) \text{ or } (x \in B, y \in A), t \in]0,1[, \\ \frac{\min\{x,y\}}{\max\{x,y\}}, & \text{elsewhere.} \end{cases}$$

In [25] it is proved that $(X, M, \cdot)$ is a fuzzy metric space which is not principal. (Notice that if we take $b \in B$ we have that the sequence $\{1 - \frac{b}{n}\}$ is p-convergent, since $\lim_n M(1-\frac{b}{n}, 1, 1) = \lim_n \frac{1-\frac{b}{n}}{1} = 1$, but it is not convergent, since $\lim_n M(1-\frac{b}{n}, 1, \frac{1}{2}) = \lim_n \frac{1-\frac{b}{n}}{1} \cdot \frac{1}{2} = \frac{1}{2}$.)

Now, we will see that M is an s-fuzzy metric on X. For it we will prove that $\bigcap_{t>0} B(x,r,t)$ is a neighborhood of x, for each $x \in X$ and each $r \in]0,1[$.

Fix $x \in X$ and $r \in]0,1[$. It is easy to verify that for $t \in]0, 1-r]$:

$$B(x,r,t) = \bigcap_{t>0} B(x,r,t) = \begin{cases} \left[x\cdot(1-r), \frac{x}{1-r}\right] \cap A, & x \in A, \\ \\ \left[x\cdot(1-r), \frac{x}{1-r}\right] \cap B, & x \in B. \end{cases}$$

On the other hand, if $n \geq 2$

$$B(x, \frac{1}{n}, \frac{1}{n}) = \begin{cases}]x\cdot(1-\frac{1}{n}), \frac{x}{1-\frac{1}{n}}[\cap A, & x \in A, \\ ]x\cdot(1-\frac{1}{n}), \frac{x}{1-\frac{1}{n}}[\cap B, & x \in B. \end{cases}$$

Therefore if we take $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < r$ we have that $B(x, \frac{1}{n}, \frac{1}{n}) \subset \bigcap_{t>0} B(x,r,t)$ and so $\bigcap_{t>0} B(x,r,t)$ is a neighborhood of x and then by Corollary 9 M is an s-fuzzy metric.

Example 10. (A non-principal non-s-fuzzy metric space). Let $A = \mathbb{R} \cap \mathbb{Q}$, $B = \mathbb{R} \setminus A$. Let d be the usual metric on $\mathbb{R}$. Define the function M on $\mathbb{R}^2 \times \mathbb{R}^+$ by

$$M(x,y,t) = \begin{cases} t\cdot M_d(x,y,t), & (x \in A, y \in B) \text{ or } (x \in B, y \in A), t \in]0,1[, \\ M_d(x,y,t), & \text{elsewhere.} \end{cases}$$

We will show that $(\mathbb{R}, M, \cdot)$ is a fuzzy metric space.

Obviously, M satisfies $(GV1)$, $(GV3)$ and $(GV5)$.

First, we will see that M satisfies $(GV2)$. Suppose that $M(x,y,t) = 1$ for $x \in A$, $y \in B$ and $t \in]0,1[$. Then $t \cdot M_d(x,y,t) = 1$, but since $t \in]0,1[$ we have that $M_d(x,y,t) > 1$, a contradiction. Therefore, $M(x,y,t) = M_d(x,y,t) = 1$ and so $x = y$. The converse is immediate.

Now, we will see that M satisfies $(GV4)$. Suppose that $x, y \in A$, $z \in B$ and let $t, s > 0$ such that $t + s \in]0,1[$. Then

$$M(x,z,t+s) = (t+s)\cdot M_d(x,z,t+s) > s\cdot M_d(x,y,t)\cdot M_d(y,z,s) =$$

$$M(x,y,t)\cdot M(y,z,s).$$

The other cases are proved in a similar way.

We will see that M is not principal and neither an s-fuzzy metric. For it, we will give a p-convergent sequence which is not convergent and a convergent sequence which is not s-convergent.

Consider the sequence $\{\frac{\pi}{n}\}$. Then $\lim_n M(\frac{\pi}{n},0,1) = \lim_n \frac{1}{1+\frac{\pi}{n}} = 1$ and so $\{\frac{\pi}{n}\}$ is p-convergent, but $\lim_n M(\frac{\pi}{n},0,\frac{1}{2}) = \lim_n \frac{(\frac{1}{2})^2}{\frac{1}{2}+\frac{\pi}{n}} = \frac{1}{2}$ and so $\{\frac{\pi}{n}\}$ is not convergent.

Now, consider the sequence $\{\frac{1}{n}\}$. For all $t>0$,

$$\lim_n M(\frac{1}{n},0,t) = \lim_n \frac{t}{t+\frac{1}{n}} = 1,$$

then $\{\frac{1}{n}\}$ is convergent, but $\lim_n M(\frac{1}{n},0,\frac{1}{n}) = \frac{\frac{1}{n}}{\frac{1}{n}+\frac{1}{n}} = \frac{1}{2}$ and so $\{\frac{1}{n}\}$ is not s-convergent.

Example 11. (A non-stationary non-principal s-fuzzy metric which generates the discrete topology). Let $X =]0,\infty[$ and let $\varphi : \mathbb{R}^+ \to]0,1]$ be a function given by

$$\varphi(t) = \begin{cases} t, & \text{if } t \in]0,1[\\ 1, & \text{elsewhere} \end{cases}$$

Define the function M on $X^2 \times \mathbb{R}^+$ by

$$M(x,y,t) = \begin{cases} 1, & x=y \\ \frac{\min\{x,y\}}{\max\{x,y\}}\cdot\varphi(t), & x\neq y \end{cases}$$

In [25] it is proved that $(M,\cdot)$ is a non-principal fuzzy metric on X and that $\mathcal{T}_M$ is the discrete topology, so M is an s-fuzzy metric. Clearly M is non-stationary.

Chapter 6

A note on local bases and convergence in fuzzy metric spaces

The material of this chapter is an adaptation to the thesis of the content of the paper by Valentín Gregori, Juan-José Miñana and Samuel Morillas, "A note on local bases and convergence in fuzzy metric spaces", published in the JCR-journal Topology and its Applications ***163*** *(2014) 142-148.*

6.1 Introduction

The convergence of a sequence to a point x_0 in a metric space (X, d) involves some local base constituted by balls centered at x_0. If ξ is any family of open balls centered at x_0 such that $\bigcap \xi = \{x_0\}$ and x_0 is not isolated in (X, d) then ξ is a local base at x_0. (In this paper $\bigcap \xi$ denotes the intersection of all members of ξ). The purpose of this chapter is to study this assertion in the fuzzy setting. We consider first, a general case, and later some families of open balls that, in a natural way, appear when studying p-convergence and s-convergence. Notice that a centered ball at x_0 in a fuzzy metric space $(X, M, *)$ is denoted by $B(x_0, r, t)$ where $r \in]0, 1[$, $t > 0$.

We show in this chapter that the above assertion is false, in general, for a fuzzy metric space (Example 12). Now, if ξ is constituted by open balls of the form $\{B(x_0, r, r) : r \in J\}$, where $J \subset]0, 1[$, or M is stationary (Definition 2) then the above assertion holds.

In [25] it is proved that any sequence p-convergent to x_0 in $(X, M, *)$ is convergent if and only if $\{B(x_0, r, t) : r \in]0, 1[\}$ is a local base at x_0, for each $t > 0$. Fuzzy metric spaces in which all p-convergent sequences are convergent were called principal. So it seems natural to study families of open balls, centered at x_0, for a fixed $t > 0$. We show that if $\mathcal{B}$ is any of these families the above assertion is true in principal fuzzy metric spaces, but in general it is false.

In Chapter 5 it is proved that any sequence convergent to x_0 is s-convergent in $(X, M, *)$ if and only if $\bigcap_{t>0} B(x_0, r, t)$ is a local neighbourhood of x_0 in $(X, \mathcal{T}_M)$, for each $r \in]0, 1[$. Fuzzy metric spaces in which all convergent sequences are s-convergent were called s-fuzzy metric spaces. So, it is natural to study families of open balls centered at x_0 with a fixed radius $r \in]0,1[$. If $\mathcal{D}$ is any of these families the above assertion is true in co-principal fuzzy metric spaces (Definition 16), and a similar result is obtained when $(X, \mathcal{T}_M)$ is compact (Theorem 11). The answer in a more general context is an open problem (Problem 6). Some examples are provided, along the chapter, that illustrate the theory.

The structure of the chapter is as follows. In Section 6.2 we study the question of when a family ξ of open balls centered at x_0 in a (principal) fuzzy metric space $(X, M, *)$, is a local base at x_0 provided that $\bigcap \xi = \{x_0\}$. The same question related to s-fuzzy metrics is studied in Section 6.3.

6.2 Local bases in (principal) fuzzy metric spaces

If ξ is a family of open sets in a metric space that constitutes a local base at x_0 then $\bigcap \xi = \{x_0\}$. Conversely, if we assume that x_0 is not isolated and ξ is constituted by a family of open balls centered at x_0 such that $\bigcap \xi = \{x_0\}$ then it can be asserted that ξ is a local base at x_0. We will see in the next example that this assertion is false, in general, in fuzzy metric spaces.

Example 12. Consider the fuzzy metric space, [25], $(X, M, \cdot)$ where $X =]0,1]$, $A = X \cap \mathbb{Q}$, $B = X \setminus A$ and M is given by

$$M(x,y,t) = \begin{cases} \frac{\min\{x,y\}}{\max\{x,y\}} \cdot t, & (x \in A, y \in B) \text{ or } (x \in B, y \in A), t \in]0,1[, \\ \frac{\min\{x,y\}}{\max\{x,y\}}, & \text{elsewhere.} \end{cases}$$

It is easy to see that $\{1\}$ is not open, and that $B(1,r,t) =]1-r,1]$ for all $r \in]0,1[$ and all $t > 1$. Consider for (some) $t > 1$ the family $\xi = \{B(1,r,t) : r \in]0,1[\}$. We have that $\bigcap \xi = \{1\}$ but ξ is not a local base at 1, since

$B(1,\frac{1}{2},\frac{1}{2}) =]\frac{1}{2},1] \cap \mathbb{Q}$ and obviously $B(1,r,t) \nsubseteq B(1,\frac{1}{2},\frac{1}{2})$ for all $r \in]0,1[$, and all $t > 1$.

The next proposition shows that the above assertion holds for stationary fuzzy metric spaces and at least for a particular case in fuzzy metric spaces.

Proposition 14. *Let $(X, M, *)$ be a (stationary) fuzzy metric space and suppose that x_0 is not isolated. Let $\mathcal{B} = \{B(x_0, r, r) : r \in J\}$ (or $\mathcal{B} = \{B(x_0, r) : r \in J\}$ if M is stationary). If $\bigcap \mathcal{B} = \{x_0\}$ then $\mathcal{B}$ is a local base at x_0.*

Proof.
Since $\{x_0\}$ is not open then $\inf J = 0$ and the conclusion is obvious. □

Denote by J_1 and J_2 two non-empty subsets of $]0,1[$ where $\inf J_1 = \inf J_2 = 0$. The following is an immediate corollary.

Corollary 11. *Let $(X, M, *)$ be a fuzzy metric space and suppose that x_0 is not isolated. Let $\mathcal{B} = \{B(x_0, r, t) : r \in J_1, t \in J_2\}$. If $\bigcap \mathcal{B} = \{x_0\}$ then $\mathcal{B}$ is a local base at x_0.*

This last proposition is false, in general, if we remove the condition that $\{x_0\}$ is not open, even if M is stationary, as illustrate the following examples.

Example 13. Consider the fuzzy metric space $(]0,1[, M, \cdot)$ where M is given by

$$M(x,y,t) = \begin{cases} 1, & x = y \\ xyt, & x \neq y, t \leq 1 \\ xy, & x \neq y, t > 1 \end{cases}$$

In [25], it is proved that $\mathcal{T}_M$ is the discrete topology.

Let $x_0 \in]0,1[$ and consider the family $\mathcal{B} = \{B(x_0, r, r) : r \in]\frac{1}{x_0+1}, 1[\}$.

It is easy to verify that $B(x_0, r, r) = \{x_0\} \cup]\frac{1-r}{rx_0}, 1[$. We have that $\bigcap \mathcal{B} = \{x_0\}$, but $\mathcal{B}$ is not a local base at x_0, since $\mathcal{B}$ does not contain $\{x_0\}$.

Example 14. Consider the stationary fuzzy metric space $([0,\infty[, M, \cdot)$, [27], where M is given by

$$M(y,x) = M(x,y) = \begin{cases} \frac{\min\{x,y\}}{\max\{x,y\}}, & x,y \in]0,\infty[\\ \frac{1}{2y}, & x = 0, y \geq 1 \\ \frac{y}{2}, & x = 0, y < 1 \\ 1, & x = y = 0 \end{cases}$$

It is easy to verify that $\{0\} \in \mathcal{T}_M$.

For $r \in]\frac{1}{2}, 1[$ we have that $B(0,r) = \{0\} \cup]2(1-r), \frac{1}{2(1-r)}[$. Consider the family $\mathcal{B} = \{B(0,r) : r \in J\}$, where $J =]\frac{1}{2}, 1[$. We have that $\bigcap \mathcal{B} = \{0\}$ but $\mathcal{B}$ is not a local base at 0.

Remark 12. (On principal fuzzy metric spaces) In any fuzzy metric space $(X, M, *)$ it is easy to verify that for a fixed $t_0 > 0$ it holds that $\bigcap\{B(x_0, r, t_0) : r \in]0,1[\} = \{x_0\}$. Then it makes sense to study families of open balls centered at x_0 with fixed t_0. Now, if M is not principal then we can find $x_0 \in X$ and $t_0 > 0$ such that $\xi = \{B(x_0, r, t_0) : r \in]0,1[\}$ is not a local base at x_0. So from $\bigcap \xi = \{x_0\}$ we cannot assert that ξ is a local base at x_0, even if x_0 is not isolated (indeed, this is the case of Example 12 since the family ξ is really $\{B(1, r, 1) : r \in]0,1[\}$). So, our aimed study only has sense in principal fuzzy metrics and the obtained results are the following.

Proposition 15. *Let $(X, M, *)$ be a fuzzy metric space and suppose that x_0 is not isolated. For a fixed $t_0 > 0$ consider a family $\zeta = \{B(x_0, r, t_0) : r \in J\}$ such that $\bigcap \zeta = \{x_0\}$. They are equivalent:*

(i) ζ *is a local base at* x_0.

(ii) $\{B(x_0, r, t_0) : r \in]0, 1[\}$ *is a local base at* x_0.

(iii) *Any sequence* $\{x_n\}$ *in* X *such that* $\lim_n M(x_n, x_0, t_0) = 1$ *is convergent (to* x_0*).*

Proof.

By [25] Theorem 11 we have that *(iii)* implies *(ii)*, and with similar arguments to the ones used in the proof of this theorem it is proved that *(ii)* implies *(iii)*. Then *(ii)* and *(iii)* are equivalents. Obviously, *(i)* implies *(ii)*. We see that *(ii)* implies *(i)*.

We claim that $\inf J = 0$ (in other case, $\{x_0\} = B(x_0, \alpha, t_0)$ for some $\alpha \in]0, 1[$, a contradiction). Now, consider an open ball $B(x_0, r, t)$. We can find $\delta \in]0, 1[$ such that $B(x_0, \delta, t_0) \subset B(x_0, r, t)$. Take $j \in J$ with $j < \delta$ and then $B(x_0, j, t_0) \subset B(x_0, \delta, t_0)$, so ζ is a local base at x_0. □

Corollary 12. *Let* $(X, M, *)$ *be a fuzzy metric space without isolated points. For each* $x \in X$ *and each* $t > 0$ *put* $\zeta_x^t = \{B(x, r, t) : r \in J\}$. *Then* $(X, M, *)$ *is principal if and only if* ζ_x^t *is a local base at* x, *for each* $x \in X$ *and each* $t > 0$, *whenever* $\bigcap \zeta_x^t = \{x\}$.

Remark 13. Notice that the converse of this corollary is true even if X has isolated points, since $\{B(x_0, r, t) : r \in]0, 1[\}$ is a local base at $x_0 \in X$, $t > 0$. Now, the fuzzy metric M of Example 13 is principal, and the family $\mathcal{B}$ satisfies $\bigcap \mathcal{B} = \{x_0\}$, where $\{x_0\}$ is open, and $\mathcal{B}$ is not a local base at x_0.

6.3 Local bases in s-fuzzy metric spaces

The study of families of balls centered at x_0 with fixed radius turns interesting when studying s-fuzzy metrics (see Theorem 8). Hence, we are interested in this type of families. Consider a family $\mathcal{D} = \{B(x_0, r_0, t) : t \in J\}$. In the next example we will see that from $\bigcap \mathcal{D} = \{x_0\}$ we cannot assert that $\mathcal{D}$ is a local base at x_0.

Example 15. Let $(X, M, \cdot)$ the fuzzy metric space of Example 13.

Consider the family of open balls $\mathcal{D} = \{B(\frac{2}{3}, \frac{2}{3}, t) : t \in]\frac{1}{2}, 1]\}$ centered at $x_0 = \frac{2}{3}$ with radius $r_0 = \frac{2}{3}$. We have that $B(\frac{2}{3}, \frac{2}{3}, t) = \{\frac{2}{3}\} \cup]\frac{1}{2t}, 1]$ for $t \in]\frac{1}{2}, 1]$ and then $\bigcap \mathcal{D} = \{\frac{2}{3}\}$. Now, $\mathcal{D}$ is not a local base at $\frac{2}{3}$ since $\mathcal{T}_M$ is the discrete topology.

The following is an open question.

Problem 6. Let $(X, M, *)$ be a fuzzy metric space, and suppose that x_0 is not isolated. Consider for a fixed $r_0 \in]0, 1[$ the family $\mathcal{D} = \{B(x_0, r_0, t) : t \in J\}$. If $\bigcap \mathcal{D} = \{x_0\}$, is $\mathcal{D}$ a local base at x_0?

Remark 14. (With respect to Problem 6). If x_0 is not isolated in $(X, M, *)$ and $\bigcap \{B(x_0, r_0, t) : t \in J\} = \{x_0\}$ then $\bigcap \{B(x_0, r_0, t) : t > 0\}$ is not a neighborhood of x_0 and thus there exists a convergent sequence to x_0 which is not s-convergent. So, if $(M, *)$ is an s-fuzzy metric without isolated points then $\bigcap \mathcal{D} \neq \{x_0\}$, for any $r_0 \in]0, 1[$.

For giving some partial answer to this problem we introduce a dual concept to principal fuzzy metrics, as follows.

Definition 16. We will say that the fuzzy metric space $(X, M, *)$ (or simply,

M) is *co-principal* if for each $x \in X$ and each $r \in]0,1[$, the family $\mathcal{D}_x^r = \{B(x,r,t) : t > 0\}$ is a local base at x.

Notice that if M is co-principal then M is an s-fuzzy metric space if and only if $\mathcal{T}_M$ is the discrete topology. Clearly, stationary fuzzy metrics (excepting trivial cases) are not co-principal.

Proposition 16. *The standard fuzzy metric is co-principal.*

Proof.
Let (X,d) be a metric space and consider the standard fuzzy metric space $(X, M_d, \cdot)$. As usual, $B_d(x;\delta)$ denotes the open ball in (X,d) with center x and radius δ.

Let $x \in X$ and $r \in]0,1[$. It is easy to see that $B_{M_d}(x,r,t) = B_d(x; \frac{rt}{1-r})$ for each $t > 0$. Since the family $\{B_d(x; \frac{rt}{1-r}) : t > 0\}$ is a local base at x for $\mathcal{T}(d)$ and $\mathcal{T}(d) = \mathcal{T}_{M_d}$, [19], we conclude that the family $\{B_{M_d}(x,r,t) : t > 0\}$ is a local base at x for $\mathcal{T}_{M_d}$. □

The proof of the next proposition is obvious.

Proposition 17. *Let $(X, M, *)$ be a fuzzy metric space and suppose that x_0 is not isolated. For a fixed $r_0 \in]0,1[$ consider a family $\mathcal{D} = \{B(x_0, r_0, t) : t \in J\}$ such that $\bigcap \mathcal{D} = \{x_0\}$. Then $\mathcal{D}$ is a local base at x_0 if and only if $\{B(x_0, r_0, t) : t > 0\}$ is a local base at x_0.*

Corollary 13. *Let $(X, M, *)$ be a co-principal fuzzy metric space without isolated points. Let $\mathcal{D} = \{B(x_0, r_0, t) : t \in J\}$. If $\bigcap \mathcal{D} = \{x_0\}$ then $\mathcal{D}$ is a local base at x_0.*

Notice that we cannot formulate last corollary as Corollary 12 because we cannot assert that $\bigcap\{B(x_0, r_0, t) : t > 0\}$ is $\{x_0\}$. The next theorem is a similar result to Corollary 13 replacing co-principal by compactness.

Theorem 11. *Let $(X, M, *)$ be a compact fuzzy metric space, let $\delta \in]0, 1[$ and suppose that x_0 is not isolated. Let $\mathcal{D} = \{B(x_0, \delta, t) : t \in J\}$. If $\bigcap \mathcal{D} = \{x_0\}$ then $\mathcal{D}_\epsilon$ is a local base at x_0, for each $\epsilon < \delta$ where $\mathcal{D}_\epsilon = \{B(x_0, \epsilon, t) : t \in J\}$.*

Proof.
We have that $\inf J = 0$, since we suppose that $\{x_0\}$ is not open. Take $\epsilon \in]0, \delta[$ and consider a sequence $\{t_n\} \subset J$ convergent to 0. Clearly $\bigcap_n B(x_0, \delta, t_n) = \bigcap_n B(x_0, \epsilon, t_n) = \{x_0\}$.

Take $\epsilon_1 \in]0, 1[$ such that $\epsilon < \epsilon_1 < \delta$. Since $B(x_0, \epsilon, t) \subset B[x_0, \epsilon_1, t] \subset B(x_0, \delta, t)$ for all $t > 0$, then $\bigcap_n B[x_0, \epsilon_1, t_n] = \{x_0\}$.

Put $V_n = B[x_0, \epsilon_1, t_n]$ for $n = 1, 2, ...$ We will see that $\{V_n : n \geq 1\}$ is a local base at x_0. Consider an open ball $B(x_0, r, t)$ with $r \in]0, 1[$, $t > 0$. Suppose, contrarily, that for all $n \geq 1$, $V_n \not\subset B(x_0, r, t)$. Then put $E_n = V_n \cap (B(x_0, r, t))^c \neq \emptyset$, for all $n = 1, 2, ...$

Since $\{V_n : n \geq 1\}$ is a decreasing family then $\{E_n : n \geq 1\}$ is also a decreasing family of closed sets with $E_n \neq \emptyset$ for each $n \geq 1$. Further, the intersection of finite elements of that family, $E_{n_1}, ..., E_{n_k}$, is non-empty (indeed, if $i = \max\{n_1, ..., n_k\}$, then $\bigcap_{j=1}^k E_{n_j} = E_i$). So, the family $\{E_n : n \geq 1\}$ has the finite intersection property. Since X is compact then $\bigcap E_n \neq \emptyset$, a contradiction (indeed, $y \in \bigcap_n E_n$ implies $y \in V_n$ for $n \geq 1$ with $y \neq x_0$).

So, there exists $m \in \mathbb{N}$ such that $V_m \subset B(x_0, r, t)$ and then $B(x_0, \epsilon, t_m) \subset B[x_0, \epsilon_1, t_m] \subset B(x_0, r, t)$. Hence $\{B(x_0, \epsilon, t) : t \in J\}$ is a local base at x_0. $\square$

Chapter 7

std-convergence in fuzzy metric spaces

The material of this chapter is an adaptation to the thesis of the content of the paper by Valentín Gregori and Juan-José Miñana, "std-Convergence in fuzzy metric spaces", published in the JCR-journal Fuzzy Sets and Systems **267** (2015) 140-143.

7.1 Introduction

For establishing relationships between the theory of complete fuzzy metric spaces and domain theory, Ricarte and Romaguera have introduced in [74] a stronger concept than Cauchy sequence, called standard Cauchy, briefly *std*-Cauchy. They have proved that the well-known theorem due to Edalat and Heckmann [13] that characterizes complete metric spaces by means of continuous domains can be obtained from their results in fuzzy metrics ([74], Corollary 1). Furthermore, the theory constructed in that chapter cannot be obtained from the metric case. Indeed, if M is a non-complete stationary fuzzy metric then it is *std*-complete but the uniformity $\mathcal{U}_M$ induced by M, see [30], is not complete and so all metrics compatible with $\mathcal{U}_M$ are not complete and then classical theory cannot be applied on M.

Inspired in the classical case the authors have introduced in [69], in a natural way, the concept of standard convergence, briefly *std*-convergence, and they have asked the following questions.

$Q1$: Is every *std*-convergent sequence a *std*-Cauchy sequence?

$Q2$: Let $\{x_n\}$ be a *std*-Cauchy and convergent sequence. Is $\{x_n\}$ *std*-convergent?

In this chapter we give negative response to $Q1$ in Example 16 and then we conclude that the concept of *std*-convergence is not *appropriate*. Then, for avoiding the proliferation of non-appropriate concepts related to convergence or Cauchyness, we create a framework in which the study of the relationship between both concepts to be more useful. So, we establish in Definition 17 when a concept of convergence is *compatible* with a concept of Cauchyness, and *vice-versa*. Later, we give a concept of convergence which is *compatible* with *std*-Cauchy. Finally, we give a positive answer to $Q2$.

7.2 Results

The next example gives a negative response to the first question Q1.

Example 16. (A *std*-convergent non-*std*-Cauchy sequence). Let d be the usual metric on $\mathbb{R}$ restricted to $[0, \infty[$ and consider the standard fuzzy metric induced by d. Let $X = [0, \infty[$. We define on $X \times X \times]0, \infty[$ the function

$$M(x, y, t) = \begin{cases} 1, & \text{if } x = y \\ M_d(x, 0, t) \cdot M_d(0, y, t), & \text{if } x \neq y \end{cases}$$

It is an easy exercise to prove that $(X, M, \cdot)$ is a fuzzy metric space.

Now, consider the sequence $\{x_n\}$ in X, where $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. We claim that $\{x_n\}$ is *std*-convergent to 0. Indeed, take $\epsilon \in]0, 1[$, then we can find $n_\epsilon \in \mathbb{N}$ such that $n_\epsilon > \frac{1}{\epsilon}$ and hence $M(x_n, 0, t) = \frac{t}{t+\frac{1}{n}} > \frac{t}{t+\epsilon}$, for all $n \geq n_\epsilon$ and for all $t > 0$. So $\{x_n\}$ is *std*-convergent to 0.

We claim that $\{x_n\}$ is not *std*-Cauchy. Indeed, if we suppose that $\{x_n\}$ is *std*-Cauchy, then for each $\epsilon \in]0, 1[$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$M(x_n, x_m, t) = \frac{t}{t + \frac{1}{n}} \cdot \frac{t}{t + \frac{1}{m}} > \frac{t}{t + \epsilon}$$

for all $n, m \geq n_\epsilon$ and $t > 0$. So, $\frac{t}{(t+\frac{1}{n_\epsilon})(t+\frac{1}{n_\epsilon})} > \frac{1}{t+\epsilon}$, for all $t > 0$.

Then, $\lim_{t \to 0} \frac{t}{(t+\frac{1}{n_\epsilon})(t+\frac{1}{n_\epsilon})} = 0 \geq \lim_{t \to 0} \frac{1}{t+\epsilon} = \frac{1}{\epsilon}$, a contradiction.

Remark 15. Attending to Definition 8 it is clear that a natural way of defining *std*-convergence is the one given by the authors in [69] (Definition

9). Unfortunately, as shows Example 16, this definition should be considered not appropriate.

Next we establish conditions under which a pair of concepts on convergence and Cauchyness, related to sequences, are considered *pairwise compatible*. These conditions have been chosen for preserving the natural structure among the concepts and also, for avoiding the unnecessary appearance of concepts or inner properties (which, finally, could distort the next diagrams).

Definition 17. Suppose it is given a sequential stronger (weaker, respectively) concept than Cauchy, say s-Cauchy (w-Cauchy, respectively). A concept on convergence, say s-convergence (w-convergence, respectively), is said to be compatible with s-Cauchy (w-Cauchy, respectively), and *vice-versa*, if the diagram of implications below on the left (on the right, respectively) is fulfilled

$$\begin{array}{ccc} s-convergence & \rightarrow & convergence \\ \downarrow & & \downarrow \\ s-Cauchy & \rightarrow & Cauchy \end{array} \qquad \begin{array}{ccc} convergence & \rightarrow & w-convergence \\ \downarrow & & \downarrow \\ Cauchy & \rightarrow & w-Cauchy \end{array}$$

and there is not any other implication, in general, among these concepts.

So, by Example 16 we can assert that the concept of *std*-convergence is not compatible with *std*-Cauchy. After the next remark we give a concept of convergence which is compatible with *std*-Cauchy.

Remark 16. (Existence of pairwise compatible s-concepts). Suppose that a concept of s-Cauchyness which is stronger than Cauchy, is given. Also, suppose that there is not any implication between convergence and s-Cauchyness. Then, there exists a concept of s-convergence compatible with s-Cauchy if and only if s-Cauchy and convergence are non-mutually exclusive concepts.

Indeed, in a such case we can give the next definition: A sequence $\{x_n\}$ is called s^*-convergent if it is convergent and s-Cauchy. Obviously, this concept of s^*-convergence is compatible with s-Cauchy. Further, any concept of s-convergence which is compatible with s-Cauchy is stronger than s^*-convergence.

Now, since every *std*-convergent sequence is convergent, [69], then Example 16 provides an example of a convergent sequence which is not *std*-Cauchy. On the other hand if $(X, M_d, \cdot)$ is a standard fuzzy metric then a sequence in X is *std*-Cauchy if and only if it is Cauchy. Hence, in a non-complete standard fuzzy metric space we can find *std*-Cauchy sequences which are not convergent. Further, every convergent sequence in $(X, M_d, \cdot)$ is *std*-Cauchy. Thus, by the last remark we can introduce the following definition of convergence which is compatible with *std*-Cauchy.

Definition 18. A sequence is called *std**-convergent if it is convergent and *std*-Cauchy.

Remark 17. (Existence of pairwise compatible w-concepts). Suppose that a concept of w-convergence which is weaker than convergence is given. Also, suppose that there is not any implication between w-convergence and Cauchy. Then, we can find concepts of Cauchyness compatible with w-convergence. Indeed, in a such case we can give the next definition: $\{x_n\}$ is called w^*-Cauchy if $\{x_n\}$ is Cauchy or w-convergent. Clearly, w^*-Cauchy is compatible with w-convergence. Further, any other concept of w-Cauchy which is compatible with w-convergence is weaker than w^*-Cauchy.

Finally, in the next proposition we response in a positive way to Question $Q2$.

Proposition 18. *Let $(X, M, *)$ be a fuzzy metric space and let $\{x_n\}$ be a* *std-Cauchy convergent sequence. Then* $\{x_n\}$ *is std-convergent.*

Proof.

Let $\{x_n\}$ be a *std*-Cauchy convergent sequence. Fix $\epsilon \in]0,1[$ and $t > 0$. Suppose that $\{x_n\}$ converges to x_0. Since $M(x, y, \_)$ is continuous for all $x, y \in X$, by Corollary 7.2 of [23] (or using Proposition 1 of [75]) we have that $\lim_m M(x_n, x_m, t) = M(x_n, x_0, t)$ for all $n \in \mathbb{N}$.

On the other hand, since $\{x_n\}$ is *std*-Cauchy we have that for $\delta \in]0, \epsilon[$ there exists $n_\delta \in \mathbb{N}$ such that

$$M(x_n, x_m, t) > \frac{t}{t+\delta} > \frac{t}{t+\epsilon}, \text{ for all } n, m \geq n_\delta \text{ and all } t > 0.$$

Then

$$M(x_n, x_0, t) = \lim_m M(x_n, x_m, t) \geq \frac{t}{t+\delta} > \frac{t}{t+\epsilon}, \text{ for all } n \geq n_\delta \text{ and all } t > 0$$

and so $\{x_n\}$ is *std*-convergent. □

Chapter 8

Discussion of the obtained results and conclusions

In Chapter 2 we have made a detailed study, from the mathematical point of view, of the fuzzy metrics M^* and M_0, where $M^*(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ is defined on $[0, \infty[$ and $M_0(x, y, t) = \frac{\min\{x,y\}}{\max\{x,y\}}$ is defined on $]0, \infty[$. As a consequence of our study, we have introduced five questions in fuzzy metric spaces (relative to completion, uniform continuity, extension and contractivity) that we think provide the basis of much future research. Further, in this chapter, from the practical application point of view, we have shown that the fuzzy metric M^* is useful to approach the problem of measuring perceptual colour differences between colour samples.

In Chapter 3, we have answered an open question by constructing a particular non-completable fuzzy metric space $(X, M, *)$. For it, we have seen that in this fuzzy metric space we can find two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ such that the assignment $f(t) = \lim_n M(a_n, b_n, t)$ is not continuous.

In addition, we have shown that the mentioned fuzzy metric space is not strong. This fact arises the question if there exists a strong fuzzy metric space fulfilling the requirements of the mentioned problem, which has been answered in a negative way in Chapter 4. As a consequence of this result, we have gotten a characterization of the class of completable strong fuzzy metrics. Further, in Chapter 4 we have showed that the conditions, in our reformulation (Theorem 6), of the theorem of characterization of completable fuzzy metric spaces constitute an independent system.

On the other hand, we have studied some different concepts related to convergence of sequences in fuzzy metric spaces. A significant difference between fuzzy metric and classical metrics is that the first one includes a t parameter in its definition. This fact allows us to introduce some (well-known) motivated concepts that in the classical theory have no sense. For instance, when working on contractivity, D. Mihet [57] introduced a weaker concept than convergence, called p-convergence. Then, the authors in [25] characterized those spaces in which p-convergent sequences are convergent. In Chapter 5 we continue the work started in [25, 57], but in the opposite way, that is, we introduce the concept of s-convergence, strengthen the condition of convergence on t. In that chapter we get a characterization of those fuzzy metric spaces in which convergent sequences are s-convergent, called s-fuzzy metric spaces, by means of local bases in a similar way to the case of principal fuzzy metric spaces. Further, we have obtained the following result. Given a fuzzy metric space $(X, M, *)$, if $(N_M, *)$ is a fuzzy metric on X where $N_M(x, y) = \bigwedge\{M(x, y, t) : t > 0\}$ then the topologies deduced from M and N_M coincide if and only if M is an s-fuzzy metric.

We have studied when certain families of open balls centered at a point are a local base. If (X, d) is a metric space it is well-known that if ξ is any family of open balls centered at a point x_0 such that $\bigcap \xi = \{x_0\}$ and x_0 is not isolated in (X, d) then ξ is a local base at x_0 ($\bigcap \xi$ denotes the intersection of all members of ξ). The results obtained in this chapter show that the above assertion is false, in general, for a fuzzy metric space (Example 12). Now, if ξ is constituted by balls of the form $\{B(x_0, r, r) : r \in J\}$, where $J \subset]0, 1[$, or M is stationary (Definition 2) then the above assertion holds. This study is related with s-fuzzy metric spaces (spaces in which convergent sequences are s-convergent) and principal fuzzy metric spaces (spaces in which p-convergent sequences are convergent).

As another contribution to the study of concepts related to convergence of sequences appeared in the literature, we have answered two open questions involving the concepts of standard Cauchy [74, Definition 3] and standard convergent sequence [69, Definition 17]. As a consequence of these responses we establish conditions for which a pair of concepts related to convergence and Cauchynes, respectively, should be fulfil for being considered *compatible*.

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