On a partial order defined by the weighted Moore-Penrose inverse[*]

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Abstract

The weighted Moore-Penrose inverse of a matrix can be used to define a partial order on the set of $m \times n$ complex matrices and to introduce the concept of weighted-EP matrices. In this paper we study the weighted star partial order on the set of weighted-EP matrices. In addition, some properties that relate the eigenprojection at zero with the weighted star partial order are obtained.

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1   Introduction and Background

Partial orders on matrices have been considered by several authors and they are defined by using different generalized inverses. Some classic references about matrices, generalized inverses and partial orders on matrices can be found, for instance, in [4, 5, 13, 14, 16, 20]. Applications of partial orders in areas such as statistics, generalized inverses, electrical networks, etc. can be found in [2, 3, 14, 15, 17].

Let $\mathbb{C}^{m \times n}$ denote the space of complex $m \times n$ matrices; in particular, I_n denotes the $n \times n$ identity matrix. The symbols A^*, A^{-1}, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the conjugate transpose, the

inverse ($m = n$), the range and the null space of a matrix $A \in \mathbb{C}^{m \times n}$, respectively. The symbol $A \oplus B$ denotes the direct sum of two square matrices A and B.

The index of a matrix $A \in \mathbb{C}^{n \times n}$, denoted by $\mathrm{ind}(A)$, is the smallest nonnegative integer k such that $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$. The index of a nonsingular matrix A is 0. For a given integer $k \geq 0$, let $\mathbb{C}_k^n$ denote the set of all matrices $A \in \mathbb{C}^{n \times n}$ of index k. The symbol $A^\#$ stands for the group inverse of A, which exists if and only if A has index at most 1. The group inverse of $A \in \mathbb{C}^{n \times n}$ is the unique matrix $A^\# \in \mathbb{C}^{n \times n}$ that satisfies $AA^\# A = A$, $A^\# AA^\# = A^\#$ and $AA^\# = A^\# A$. More generally, for every matrix $A \in \mathbb{C}_k^n$, there exists a unique matrix $A^D \in \mathbb{C}^{n \times n}$, called the Drazin inverse of A, such that $A^D AA^D = A^D$, $AA^D = A^D A$ and $A^{k+1} A^D = A^k$. If $k \leq 1$ then $A^D = A^\#$.

In [6, 7, 8, 9], the authors worked with the concept of eigenprojection at zero defined as $A^\pi = I - AA^D$. Specifically, for a matrix $A \in \mathbb{C}^{n \times n}$ of index at most 1, its eigenprojection at zero becomes $A^\pi = I - AA^\#$, which is the projector onto $\mathcal{N}(A)$ along $\mathcal{R}(A)$.

For every matrix $A \in \mathbb{C}^{m \times n}$, there is a unique matrix $X \in \mathbb{C}^{n \times m}$, called the Moore-Penrose inverse of A and denoted by $A^\dagger$, which satisfies the four conditions: $AXA = A$, $XAX = X$, $(AX)^* = AX$, $(XA)^* = XA$.

Let $A, B \in \mathbb{C}^{m \times n}$. It is said that A is below B under the star partial order [10, 1] if $A^* A = A^* B$ and $AA^* = BA^*$ (or equivalently if $A^\dagger A = A^\dagger B$ and $AA^\dagger = BA^\dagger$). It is denoted by $A \leq^* B$.

A square matrix A is called EP if the projectors $AA^\dagger$ and $A^\dagger A$ are equal (or equivalently $\mathcal{R}(A^*) = \mathcal{R}(A)$). A characterization for a square matrix A to be EP is $A^\dagger = A^\#$. Several representations were given for these matrices [5]. Moreover, Tian et al. summarize thirty five characterizations of EP matrices in [18].

Throughout this paper, $M \in \mathbb{C}^{m \times m}$ and $N \in \mathbb{C}^{n \times n}$ will denote two Hermitian positive definite matrices.

For every matrix $A \in \mathbb{C}^{m \times n}$, we recall that there is a unique matrix $X \in \mathbb{C}^{n \times m}$, called the weighted Moore-Penrose inverse of A with respect to matrices M and N and denoted by $A^{\dagger(M,N)}$, which satisfies the four conditions: $AXA = A$, $XAX = X$, $(MAX)^* = MAX$, $(NXA)^* = NXA$.

The following properties on the weighted Moore-Penrose inverse will often be used [4, 5, 18, 20, 21].

Let $M^{1/2}$ and $N^{1/2}$ denote the square roots of M and N, respectively. Then

$$A^{\dagger(M,N)} = N^{-1/2} \left(M^{1/2} A N^{-1/2} \right)^\dagger M^{1/2}. \tag{1}$$

The M-weighted inner product in $\mathbb{C}^m$ is defined by $\langle x, y \rangle_M = y^* M x$, for all $x, y \in \mathbb{C}^m$. If S is a subspace of $\mathbb{C}^m$, then $S^{\perp_M}$ denotes its orthogonal complementary subspace with respect to the M-weighted inner product. When $M = I_m$, we write $\perp$ instead of $\perp_{I_m}$. A matrix $P \in \mathbb{C}^{m \times m}$ is called an M-orthogonal projector if $P^2 = P$ and $(MP)^* = MP$. Analogously, similar definitions can be considered for the matrix N.

A matrix $A \in \mathbb{C}^{n \times n}$ is called weighted-EP with respect to matrices M and N (for short, an $EP_{(M,N)}$ matrix) if $AA^{\dagger(M,N)} = A^{\dagger(M,N)}A$ [18]. For a given $A \in \mathbb{C}^{n \times n}$, it is also well known [18] that the following conditions are equivalent:

(w.1) A is $EP_{(M,N)}$.

(w.2) $\operatorname{rank}(A) = \operatorname{rank}(A^2)$ and $A^{\#} = A^{\dagger(M,N)}$.

(w.3) A is both $EP_{(M,M)}$ and $EP_{(N,N)}$.

(w.4) MA and AN^{-1} are both EP.

(w.5) $\operatorname{rank}(A) = \operatorname{rank}(A^2)$ and $AA^{\#} = AA^{\dagger(M,M)} = AA^{\dagger(N,N)}$.

We denote $\mathcal{EP}_{(M,N)} = \{A \in \mathbb{C}^{n \times n} : A \text{ is } EP_{(M,N)}\}$.

The weighted Moore-Penrose inverse of A with respect to the matrices $M = I_m$ and $N = I_n$ is the classical Moore-Penrose inverse of A. In this case, we write EP and $\mathcal{EP}$ instead of $EP_{(I_m, I_n)}$ and $\mathcal{EP}_{(I_m, I_n)}$, respectively.

The weighted conjugate transpose matrix of $A \in \mathbb{C}^{m \times n}$ was defined in [20, 21] by

$$A^{\circledast(M,N)} = N^{-1} A^* M.$$

Moreover, when $m = n$ we say that A is (M, N)-Hermitian if $A^{\circledast(M,N)} = A$.

Lemma 1.1 *[18, 20] Let $A \in \mathbb{C}^{m \times n}$ and S_m, S_n be subsets of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Then*

(a) $\mathcal{R}(AA^{\dagger(M,N)}) = \mathcal{R}(A)$ and $\mathcal{N}(AA^{\dagger(M,N)}) = \mathcal{N}(A^{\circledast(M,N)})$.

(b) $\mathcal{R}(A^{\dagger(M,N)}A) = \mathcal{R}(A^{\circledast(M,N)})$ and $\mathcal{N}(A^{\dagger(M,N)}A) = \mathcal{N}(A)$.

(c) $S_m^{\perp_M} = M^{-1/2}(M^{1/2}S_m)^{\perp}$ and $S_n^{\perp_N} = N^{-1/2}(N^{1/2}S_n)^{\perp}$.

(d) $\mathcal{N}(A^{\circledast(M,N)}) = (\mathcal{R}(A))^{\perp_M}$.

(e) $\mathcal{R}(A^{\circledast(M,N)}) = (\mathcal{N}(A))^{\perp_N}$.

(f) $AA^{\dagger(M,N)}$ *is an M-orthogonal projector onto $\mathcal{R}(A)$ along $\mathcal{N}(A^{\circledast(M,N)})$.*

(g) $A^{\dagger(M,N)}A$ *is an N-orthogonal projector onto $\mathcal{R}(A^{\circledast(M,N)})$ along $\mathcal{N}(A)$.*

(h) $\mathbb{C}^m = M^{1/2}\mathcal{R}(A) \oplus^\perp M^{1/2}\mathcal{N}(A^{\circledast(M,N)}) = M^{1/2}\mathcal{R}(A) \oplus^\perp M^{-1/2}\mathcal{N}(A^*).$

(i) $\mathbb{C}^n = N^{1/2}\mathcal{R}(A^{\circledast(M,N)}) \oplus^\perp N^{1/2}\mathcal{N}(A) = N^{-1/2}\mathcal{R}(A^*) \oplus^\perp N^{1/2}\mathcal{N}(A).$

(j) $\mathcal{R}(A^{\dagger(M,N)}) = \mathcal{R}(A^{\dagger(M,N)}A) = \mathcal{R}(N^{-1}A^*)$ *and* $\mathcal{R}((A^{\dagger(M,N)})^*) = \mathcal{R}((AA^{\dagger(M,N)})^*) = \mathcal{R}(MA).$

This paper is organized as follows. In Section 2 some properties for $EP_{(M,N)}$ matrices are given. Specifically, the weighted star partial order with respect to the matrices M and N is studied in the class of $EP_{(M,N)}$ matrices. We also obtain characterizations for predecessors and successors of a given $EP_{(M,M)}$ matrix. Finally, in Section 3, it is proved that the class of $EP_{(M,M)}$ matrices is closed under eigenprojections at zero and this eigenprojection is related to the weighted star partial order for $EP_{(M,M)}$ matrices.

2   Some properties of $EP_{(M,N)}$ matrices

In this section, some properties of $EP_{(M,N)}$ matrices are given. The weighted star partial order with respect to the matrices M and N is defined for $EP_{(M,N)}$ matrices and some characterizations for predecessors and successors of a given matrix are obtained.

For every matrix $A \in \mathbb{C}^{m \times n}$, we write

$$\Psi_{(M,N)}(A) = M^{1/2}AN^{-1/2}.$$

Lemma 2.1 *Let $A, B \in \mathbb{C}^{m \times n}$. The following conditions are equivalent:*

(a) $A^{\circledast(M,N)}A = A^{\circledast(M,N)}B$ *and* $AA^{\circledast(M,N)} = BA^{\circledast(M,N)}$.

(b) $A^{\dagger(M,N)}A = A^{\dagger(M,N)}B$ *and* $AA^{\dagger(M,N)} = BA^{\dagger(M,N)}$.

(c) $\Psi_{(M,N)}(A) \leq^* \Psi_{(M,N)}(B)$.

Proof. (a) $\Longleftrightarrow$ (c) The equality $A^{\circledast(M,N)}A = A^{\circledast(M,N)}B$ holds if and only if $N^{-1}A^*MA = N^{-1}A^*MB$. Pre-multiplying by $N^{1/2}$ and post-multiplying by $N^{-1/2}$ the last equality is equivalent to $(N^{-1/2}A^*M^{1/2})(M^{1/2}AN^{-1/2}) = (N^{-1/2}A^*M^{1/2})(M^{1/2}BN^{-1/2})$, that is

$$(\Psi_{(M,N)}(A))^*\Psi_{(M,N)}(A) = (\Psi_{(M,N)}(A))^*\Psi_{(M,N)}(B). \tag{2}$$

Similarly, we get the equivalence between the second equality in (a) and

$$\Psi_{(M,N)}(A)(\Psi_{(M,N)}(A))^* = \Psi_{(M,N)}(B)(\Psi_{(M,N)}(A))^*. \tag{3}$$

Finally, the equalities (2) and (3) can be rewritten as $\Psi_{(M,N)}(A) \leq^* \Psi_{(M,N)}(B)$.

(b) $\Longleftrightarrow$ (c) From (1), it is easy to see that $A^{\dagger(M,N)}A = A^{\dagger(M,N)}B$ holds if and only if $N^{-1/2}(\Psi_{(M,N)}(A))^{\dagger}M^{1/2}A = N^{-1/2}(\Psi_{(M,N)}(A))^{\dagger}M^{1/2}B$. Pre-multiplying by $N^{1/2}$ and post-multiplying by $N^{-1/2}$ the last equality we get

$$(\Psi_{(M,N)}(A))^{\dagger}\Psi_{(M,N)}(A) = (\Psi_{(M,N)}(A))^{\dagger}\Psi_{(M,N)}(B). \tag{4}$$

Similarly, we get the equivalence between the second equality in (b) and

$$\Psi_{(M,N)}(A)(\Psi_{(M,N)}(A))^{\dagger} = \Psi_{(M,N)}(B)(\Psi_{(M,N)}(A))^{\dagger}. \tag{5}$$

Finally, the equalities (4) and (5) can be rewritten as $\Psi_{(M,N)}(A) \leq^* \Psi_{(M,N)}(B)$. ∎

Hence, the weighted star partial order with respect to the matrices M and N, for short the (M, N)-star partial order, is defined as follows.

Definition 2.1 *For two given matrices $A, B \in \mathbb{C}^{m\times n}$, it is said that A is below B under the (M, N)-star partial order and denoted by $A \leq^{\circledast(M,N)} B$, if one of the equivalent conditions in Lemma 2.1 holds.*

Since $\leq^*$ is a partial order on $\mathbb{C}^{m\times n}$, we can assure that the binary relation above defined is a partial order as well.

The following result is a version for $EP_{(M,N)}$ matrices of the Theorem 4.3.1 in [5] and a result by Katz [12].

Theorem 2.1 *Let $A \in \mathbb{C}^{m\times n}$. If $m = n$ then the following conditions are equivalent:*

(a) A is $EP_{(M,N)}$.

(b) $\mathcal{R}(A^{\circledast(M,N)}) = \mathcal{R}(A)$ and $\mathcal{N}(A^{\circledast(M,N)}) = \mathcal{N}(A)$.

(c) $\mathbb{C}^n = \mathcal{R}(A) \oplus \mathcal{N}(A)$, $(\mathcal{R}(A))^{\perp_M} = \mathcal{N}(A)$ and $(\mathcal{N}(A))^{\perp_N} = \mathcal{R}(A)$.

(d) There exist nonsingular matrices $P, Q \in \mathbb{C}^{n\times n}$ such that $A^{\circledast(M,N)} = AP$ and $A^{\circledast(M,N)} = QA$.

(e) There exist matrices $X, Y \in \mathbb{C}^{n\times n}$ such that $A^{\circledast(M,N)} = AX$ and $A^{\circledast(M,N)} = YA$.

Proof. (a) $\Longleftrightarrow$ (b) It is clear that $\mathcal{N}(A^{\circledast(M,N)}) = \mathcal{N}(A^*M) = \mathcal{N}((MA)^*)$ and $\mathcal{R}(A^{\circledast(M,N)}) = \mathcal{R}(N^{-1}A^*) = \mathcal{R}((AN^{-1})^*)$. By (w.4), A is $EP_{(M,N)}$ if and only if MA and AN^{-1} are EP, which is equivalent to $\mathcal{N}(A^{\circledast(M,N)}) = \mathcal{N}(MA) = \mathcal{N}(A)$ and $\mathcal{R}(A^{\circledast(M,N)}) = \mathcal{R}(AN^{-1}) = \mathcal{R}(A)$.

(a) $\Longrightarrow$ (c) It follows directly from Lemma 1.1 (f), (g), (d) and (e).

(c) $\Longrightarrow$ (a) Since $\mathbb{C}^n = \mathcal{R}(A) \oplus \mathcal{N}(A)$, there exists a unique projector P such that $\mathcal{R}(P) = \mathcal{R}(A)$ and $\mathcal{N}(P) = \mathcal{N}(A)$. Moreover, from $(\mathcal{R}(A))^{\perp_M} = \mathcal{N}(A)$ and Lemma 1.1 (d) we have $\mathcal{N}(A^{\circledast(M,N)}) = \mathcal{N}(A)$. Thus, Lemma 1.1 (f) assures that $P = AA^{\dagger(M,N)}$. Analogously, from $(\mathcal{N}(A))^{\perp_N} = \mathcal{R}(A)$ and Lemma 1.1 (e) we get $\mathcal{R}(A^{\circledast(M,N)}) = \mathcal{R}(A)$. Hence, by Lemma 1.1 (g), $P = A^{\dagger(M,N)}A$.

(b) $\Longleftrightarrow$ (d) It follows from basic results on equivalent matrices [13].

(a) $\Longrightarrow$ (e) The equality $A = AN^{-1}(NA^{\dagger(M,N)}A)$ implies $A^* = (NA^{\dagger(M,N)}A)^*N^{-1}A^* = NA^{\dagger(M,N)}AN^{-1}A^*$. Then $A^{\circledast(M,N)} = N^{-1}A^*M = A^{\dagger(M,N)}AN^{-1}A^*M = A^{\dagger(M,N)}AA^{\circledast(M,N)}$. By (a) we get $A^{\circledast(M,N)} = AX$ taking $X = A^{\dagger(M,N)}A^{\circledast(M,N)}$. Similarly, we can prove that $A^{\circledast(M,N)} = A^{\circledast(M,N)}AA^{\dagger(M,N)}$. Again, by (a) we have $A^{\circledast(M,N)} = YA$ taking $Y = A^{\circledast(M,N)}A^{\dagger(M,N)}$.

(e) $\Longrightarrow$ (b) Suppose that $A^{\circledast(M,N)} = AX$ and $A^{\circledast(M,N)} = YA$ for some $X, Y \in \mathbb{C}^{n\times n}$. Then $\mathcal{R}(N^{-1}A^*) = \mathcal{R}(A^{\circledast(M,N)}) = \mathcal{R}(AX) \subseteq \mathcal{R}(A)$. Since $\operatorname{rank}(N^{-1}A^*) = \operatorname{rank}(A)$, we get $\operatorname{rank}(A^{\circledast(M,N)}) = \operatorname{rank}(A)$. So, $\mathcal{R}(A^{\circledast(M,N)}) = \mathcal{R}(A)$. Similarly, we have $\mathcal{N}(A^{\circledast(M,N)}) = \mathcal{N}(A)$. ∎

Notice that, using Lemma 1.1 (h) and (i), the condition (c) of Theorem 2.1 can be rewritten as
$$\mathbb{C}^n = M^{1/2}\mathcal{R}(A) \oplus^{\perp} M^{1/2}\mathcal{N}(A) = N^{1/2}\mathcal{R}(A) \oplus^{\perp} N^{1/2}\mathcal{N}(A).$$

Lemma 2.2 *If $m = n$ then $\mathcal{EP}_{(M,N)} = \mathcal{EP}_{(M,M)} \cap \mathcal{EP}_{(N,N)}$.*

Proof. The equality follows from the equivalences between (w.1) and (w.3). ∎

Remark 2.1 Notice that from (w.2) and (w.3) we have that the statement $A \in \mathcal{EP}_{(M,N)}$ implies $A^{\#} = A^{\dagger(M,N)} = A^{\dagger(M,M)} = A^{\dagger(N,N)}$ by the uniqueness of the group inverse. Conversely, it is immediate that $A^{\#} = A^{\dagger(M,M)} = A^{\dagger(N,N)}$ implies (w.5). This leads to the following equivalent condition for a matrix A to be $EP_{(M,N)}$:

(w.6) $\operatorname{rank}(A) = \operatorname{rank}(A^2)$ and $A^{\#} = A^{\dagger(M,M)} = A^{\dagger(N,N)}$.

Notice that MA and AN^{-1} are not EP simultaneously in general, unless $A \in \mathcal{EP}_{(M,N)}$. For

example, if we consider the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \tag{6}$$

it is easy to see that MA is EP but AN^{-1} is not EP.

From now on, we will consider $M = N \in \mathbb{C}^{n \times n}$ a Hermitian positive definite matrix. Moreover, a, b stand for the rank of the matrices $A, B \in \mathbb{C}^{n \times n}$, respectively.

Remark 2.2 Let $A \in \mathbb{C}^{n \times n}$. The following statements hold:

(i) $(A^{\dagger(M,M)})^* = M^{1/2} \left(M^{-1/2} A^* M^{1/2} \right)^{\dagger} M^{-1/2}$.

(ii) $(A^{\dagger(M,M)})^{\circledast(M,M)} = \left(A^{\circledast(M,M)} \right)^{\dagger(M,M)}$.

(iii) The conditions MA is EP and AM^{-1} is EP are equivalent, i.e., the statement (w.4) becomes MA is EP when $M = N$.

(iv) A is an idempotent (M, M)-Hermitian matrix if and only if A is an M-orthogonal projector.

Indeed, items (i) and (ii) follow from (1). In order to prove (iii), suppose that MA is EP. Then $\mathcal{R}(A^*) = \mathcal{R}(A^*M) = \mathcal{R}((MA)^*) = \mathcal{R}(MA) = M\mathcal{R}(A)$. Hence, $\mathcal{R}(AM^{-1}) = \mathcal{R}(A) = M^{-1}\mathcal{R}(A^*) = \mathcal{R}(M^{-1}A^*) = \mathcal{R}((AM^{-1})^*)$, that is AM^{-1} is EP. The converse is similar. Item (iv) follows from definitions.

The above Remark (iii), Remark 2.1 and Theorem 3.5 in [18] allow us to present some characterizations for weighted-EP matrices when $M = N$.

Proposition 2.1 *Let $A \in \mathbb{C}^{n \times n}$. The following conditions are equivalent:*

(a) A is $EP_{(M,M)}$.

(b) $\operatorname{rank}(A) = \operatorname{rank}(A^2)$ and $A^{\#} = A^{\dagger(M,M)}$.

(c) MA is EP.

(d) AM^{-1} is EP.

(e) $\mathcal{R}(MA) = \mathcal{R}(A^*)$.

(f) $\mathcal{R}(M^{-1}A^*) = \mathcal{R}(A)$.

(g) $\mathcal{R}(A^{\dagger(M,M)}) = \mathcal{R}(A)$.

(h) $\mathcal{R}((A^{\dagger(M,M)})^*) = \mathcal{R}(A^*)$.

(i) $\mathcal{N}((MA)^*) = \mathcal{N}(A)$.

(j) $\mathcal{N}(AM^{-1}) = \mathcal{N}(A^*)$.

(k) $\mathbb{C}^n = \mathcal{R}(M^{-1}A^*) \oplus^{\perp} \mathcal{N}(A^*)$.

(l) $\mathbb{C}^n = \mathcal{R}(A^*) \oplus^{\perp} \mathcal{N}(A^*M)$.

(m) $\mathbb{C}^n = \mathcal{R}(MA) \oplus^{\perp} \mathcal{N}(A)$.

(n) $\mathbb{C}^n = \mathcal{R}(A) \oplus^{\perp} \mathcal{N}(AM^{-1})$.

(o) $r(A) = r(A^2)$ and $MAA^{\#}$ is Hermitian.

Proof.

(c) $\iff$ (e) MA is EP if and only if $\mathcal{R}(MA) = \mathcal{R}((MA)^*) = \mathcal{R}(A^*M) = \mathcal{R}(A^*)$.

(c) $\iff$ (d) and (a) $\iff$ (c) follow from (w.1), (w.4) and Remark 2.2 (iii).

(e) $\iff$ (f) We have seen that $\mathcal{R}(MA) = \mathcal{R}(A^*)$ is equivalent to MA is EP (see (c) $\iff$ (e)). By using the equivalence (c) $\iff$ (d), it remains to prove the equivalence between (f) and the condition that AM^{-1} is EP. Indeed, AM^{-1} is EP if and only if $\mathcal{R}(M^{-1}A^*) = \mathcal{R}((AM^{-1})^*) = \mathcal{R}(AM^{-1}) = \mathcal{R}(A)$.

(f) $\iff$ (g) and (e) $\iff$ (h) follow from Lemma 1.1 (j).

(c) $\iff$ (i) It follows from $\mathcal{N}(A) = \mathcal{N}(MA)$.

(e) $\implies$ (l) The decomposition $\mathbb{C}^n = \mathcal{R}(MA) \oplus^{\perp} \mathcal{N}((MA)^*)$ is always true. Now, the implication is direct.

(d) $\implies$ (k) It is similar to (e) $\implies$ (l) by using the decomposition $\mathbb{C}^n = \mathcal{R}(AM^{-1}) \oplus^{\perp} \mathcal{N}((AM^{-1})^*)$.

(k) $\implies$ (d) From $\mathbb{C}^n = \mathcal{R}(M^{-1}A^*) \oplus^{\perp} \mathcal{N}(A^*)$ it is easy to see that $\mathbb{C}^n = \mathcal{R}((AM^{-1})^*) \oplus^{\perp} \mathcal{N}((AM^{-1})^*)$, that is, $(AM^{-1})^*$ is EP. Then, AM^{-1} is EP.

(l) $\implies$ (c) It is similar to (k) $\implies$ (d) by using that the decomposition $\mathbb{C}^n = \mathcal{R}(A^*) \oplus^{\perp} \mathcal{N}(A^*M)$ implies $\mathbb{C}^n = \mathcal{R}((MA)^*) \oplus^{\perp} \mathcal{N}((MA)^*)$.

(i) $\implies$ (m) It follows from the identity $\mathbb{C}^n = \mathcal{R}(MA) \oplus^{\perp} \mathcal{N}((MA)^*)$.

(m) $\implies$ (c) Since $\mathbb{C}^n = \mathcal{R}(MA) \oplus^{\perp} \mathcal{N}(A) = \mathcal{R}(MA) \oplus^{\perp} \mathcal{N}(MA)$, we get that MA is EP.

(d) $\iff$ (n) From $\mathbb{C}^n = \mathcal{R}(AM^{-1}) \oplus^{\perp} \mathcal{N}((AM^{-1})^*)$ and (d) we have $\mathbb{C}^n = \mathcal{R}(A) \oplus^{\perp} \mathcal{N}(AM^{-1})$, which is (n). The converse is trivial.

(d) $\Longleftrightarrow$ (j) Since $\mathcal{N}(A^*) = \mathcal{N}(M^{-1}A^*) = \mathcal{N}((AM^{-1})^*)$, it is clear that AM^{-1} is EP if and only if $\mathcal{N}(A^*) = \mathcal{N}(AM^{-1})$.

(b) $\Longrightarrow$ (o) Since $A^\# = A^{\dagger(M,M)}$, we get $MAA^\# = MAA^{\dagger(M,M)} = (MAA^{\dagger(M,M)})^* = (MAA^\#)^*$.

(o) $\Longrightarrow$ (b) By hypothesis, $AA^\#A = A$, $A^\#AA^\# = A^\#$, $(MAA^\#)^* = (MAA^\#)$ and $(MA^\#A)^* = (MA^\#A)$ hold. The uniqueness of the weighted Moore-Penrose inverse assures that $A^\# = A^{\dagger(M,M)}$.

(b)$\Longleftrightarrow$ (a) It is immediate from (w.6). $\blacksquare$

The characterizations presented in Theorem 2.1 can be simplified when $M = N$ as follows. Moreover, a representation for an $EP_{(M,M)}$ matrix is provided.

Theorem 2.2 *Let $A \in \mathbb{C}^{n \times n}$. The following conditions are equivalent:*

(a) $A \in \mathcal{EP}_{(M,M)}$.

(b) $\Psi_{(M,M)}(A) \in \mathcal{EP}$.

(c) There exist a unitary matrix $U_A \in \mathbb{C}^{n \times n}$ and a nonsingular matrix $C_A \in \mathbb{C}^{a \times a}$ such that

$$A = M^{-1/2}U_A(C_A \oplus O)U_A^*M^{1/2}. \tag{7}$$

(d) $\mathcal{R}(A^{\circledast(M,M)}) = \mathcal{R}(A)$.

(e) $\mathcal{N}(A^{\circledast(M,M)}) = \mathcal{N}(A)$.

(f) $\mathbb{C}^n = \mathcal{R}(A) \oplus^{\perp_M} \mathcal{N}(A)$.

(g) There exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that $A^{\circledast(M,M)} = AP$.

(h) There exists a nonsingular matrix $Q \in \mathbb{C}^{n \times n}$ such that $A^{\circledast(M,M)} = QA$.

(i) There exists a matrix $X \in \mathbb{C}^{n \times n}$ such that $A^{\circledast(M,M)} = AX$.

(j) There exists a matrix $Y \in \mathbb{C}^{n \times n}$ such that $A^{\circledast(M,M)} = YA$.

If either (and hence all) of these statements hold then

$$A^{\dagger(M,M)} = M^{-1/2}U_A(C_A^{-1} \oplus O)U_A^*M^{1/2} \tag{8}$$

and

$$A^{\circledast(M,M)} = M^{-1/2}U_A(C_A^* \oplus O)U_A^*M^{1/2}.$$

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Proof. (a) $\Longleftrightarrow$ (b) From (1), we get $(\Psi_{(M,M)}(A))^\dagger = (\Psi_{(M,M)}(A^{\dagger(M,M)}))$. Then

$$\Psi_{(M,M)}(A)(\Psi_{(M,M)}(A))^\dagger = M^{1/2}AA^{\dagger(M,M)}M^{-1/2}$$

and

$$(\Psi_{(M,M)}(A))^\dagger \Psi_{(M,M)}(A) = M^{1/2}A^{\dagger(M,M)}AM^{-1/2}.$$

Now, the conclusion is evident.

(b) $\Longleftrightarrow$ (c) It follows as a direct application of Theorem 4.3.1 in [5] and the definition of $\Psi_{(M,M)}$.

The equivalences (a) $\Longleftrightarrow$ (d) and (a) $\Longleftrightarrow$ (e) follow from Proposition 2.1, items (f) and (i), respectively.

(e) $\Longrightarrow$ (f) By Lemma 1.1 (f), we get $\mathbb{C}^n = \mathcal{R}(A) \oplus \mathcal{N}(A^{\circledast(M,M)})$ and $\mathcal{R}(A)^{\perp_M} = \mathcal{N}(A^{\circledast(M,M)})$, then $\mathbb{C}^n = \mathcal{R}(A) \oplus^{\perp_M} \mathcal{N}(A)$ follows directly from (e).

(f) $\Longrightarrow$ (e) Since $\mathcal{R}(A)^{\perp_M} = \mathcal{N}(A)$, from Lemma 1.1 (d) we have $\mathcal{R}(A)^{\perp_M} = \mathcal{N}(A^{\circledast(M,M)})$. Then $\mathcal{N}(A^{\circledast(M,M)}) = \mathcal{N}(A)$.

The implications (a) $\Longrightarrow$ (i), (a) $\Longrightarrow$ (j), (d) $\Longleftrightarrow$ (g), (e) $\Longleftrightarrow$ (h), (i) $\Longrightarrow$ (d) and (j) $\Longrightarrow$ (e) follow directly from Theorem 2.1. $\blacksquare$

Since the function $\Psi_{(M,M)} : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ defined by $\Psi_{(M,M)}(A) = M^{1/2}AM^{-1/2}$ is bijective, Theorem 2.2 assures that

$$\Psi_{(M,M)}\left(\mathcal{EP}_{(M,M)}\right) = \mathcal{EP}. \tag{9}$$

Next, we will characterize predecessors and successors of an $EP_{(M,M)}$ matrix under the (M,M)-star partial order.

Notice that, from Theorem 2.2, if $B \in \mathbb{C}^{n \times n}$ is $EP_{(M,M)}$ then there exist a unitary matrix $U_B \in \mathbb{C}^{n \times n}$ and a nonsingular matrix $C_B \in \mathbb{C}^{b \times b}$ such that

$$B = M^{-1/2}U_B(C_B \oplus O)U_B^* M^{1/2}. \tag{10}$$

Theorem 2.3 *Let $B \in \mathbb{C}^{n \times n}$ be a non-zero $EP_{(M,M)}$ matrix written as in (10). The following conditions are equivalent:*

(a) *There exists $A \in \mathbb{C}^{n \times n}$ such that $A \leq^{\circledast(M,M)} B$.*

(b) *There exists $X \in \mathbb{C}^{b \times b}$ such that $A = M^{-1/2}U_B(X \oplus O)U_B^* M^{1/2}$ with $X \leq^* C_B$.*

Proof. Assume that $A \leq^{\circledast(M,M)} B$ with $B \in \mathcal{EP}_{(M,M)}$. By Lemma 2.1, $\Psi_{(M,M)}(A) \leq^*$ $\Psi_{(M,M)}(B)$. Moreover, Theorem 2.2 assures that $\Psi_{(M,M)}(B)$ is EP. By Theorem 3.1 in [11], there exists $X \in \mathbb{C}^{b \times b}$ such that $A = M^{-1/2}U_B(X \oplus O)U_A^* M^{1/2}$ and $X \leq^* C_B$. Then (a) $\Longrightarrow$ (b) converse follows by some algebraic manipulations. ∎

Theorem 2.4 Let $A \in \mathbb{C}^{n \times n}$ be a non-zero $EP_{(M,M)}$ matrix written as in (7). The following conditions are equivalent:

(a) There exists $B \in \mathbb{C}^{n \times n}$ such that $A \leq^{\circledast(M,M)} B$.

(b) There exists $T \in \mathbb{C}^{(n-a) \times (n-a)}$ such that $B = M^{-1/2}U_A(C_A \oplus T)U_A^* M^{1/2}$.

Proof. Assume $A \leq^{\circledast(M,M)} B$ with $A \in \mathcal{EP}_{(M,M)}$. By Lemma 2.1, $\Psi_{(M,M)}(A) \leq^* \Psi_{(M,M)}(B)$. Moreover, Theorem 2.2 assures that $\Psi_{(M,M)}(A)$ is EP. By Theorem 3.3 in [11], there exists $T \in \mathbb{C}^{(n-a) \times (n-a)}$ such that $B = M^{-1/2}U_A(C_A \oplus T)U_A^* M^{1/2}$. Then (a) $\Longrightarrow$ (b) is shown. The converse can be obtained in a similar way. ∎

Theorem 3.5 in [11] and the last two theorems provide the following corollary.

Corollary 2.1 Let $A, B \in \mathbb{C}^{n \times n}$ be $EP_{(M,M)}$ matrices, $A \neq O$. The following conditions are equivalent:

(a) $A \leq^{\circledast(M,M)} B$.

(b) There exist $V \in \mathbb{C}^{n \times n}$, $C \in \mathbb{C}^{a \times a}$ and $T \in \mathbb{C}^{(b-a) \times (b-a)}$ such that $A = M^{-1/2}V(C \oplus O \oplus O)V^* M^{1/2}$ and $B = M^{-1/2}V(C \oplus T \oplus O)V^* M^{1/2}$, where V is unitary, C is nonsingular and T is nonsingular or $T = O$.

3 On the eigenprojection at zero

It is well known that $\mathcal{EP} \subseteq \mathbb{C}_0^n \cup \mathbb{C}_1^n$ and moreover $\mathrm{ind}(A) = \mathrm{ind}(\Psi_{(M,M)}(A))$ for every $A \in \mathbb{C}^{n \times n}$. From (9) it is clear that $\mathcal{EP}_{(M,M)} \subseteq \mathbb{C}_0^n \cup \mathbb{C}_1^n$. If $A \in \mathcal{EP}_{(M,M)}$ is written as in (7) then by Proposition 2.1 (b), $A^\pi = I - AA^{\dagger(M,M)}$ and, by (8), we have

$$A^\pi = M^{-1/2}U_A(O \oplus I_{n-a})U_A^* M^{1/2}. \tag{11}$$

Lemma 3.1 Let $A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$. Then

(a) $\Psi_{(M,M)}(A^{\#}) = (\Psi_{(M,M)}(A))^{\#}$ and $\Psi^{-1}_{(M,M)}(A^{\#}) = (\Psi^{-1}_{(M,M)}(A))^{\#}$.

(b) $\Psi_{(M,M)}(A^{\pi}) = (\Psi_{(M,M)}(A))^{\pi}$ and $\Psi^{-1}_{(M,M)}(A^{\pi}) = (\Psi^{-1}_{(M,M)}(A))^{\pi}$.

(c) $(\Psi_{(M,M)}(A))^{*} = \Psi^{-1}_{(M,M)}(A^{*})$ and $(\Psi^{-1}_{(M,M)}(A))^{*} = \Psi_{(M,M)}(A^{*})$.

(d) A is (M,M)-Hermitian if and only if $\Psi_{(M,M)}(A)$ is a Hermitian matrix.

(e) A is an M-orthogonal projector if and only if $\Psi_{(M,M)}(A)$ is an orthogonal projector.

Proof. (a) It follows from definition of group inverse.

(b) From (a) we get $\Psi_{(M,M)}(A)(\Psi_{(M,M)}(A))^{\#} = \Psi_{(M,M)}(AA^{\#})$. Then $(\Psi_{(M,M)}(A))^{\pi} = I_n - \Psi_{(M,M)}(A)(\Psi_{(M,M)}(A))^{\#} = M^{1/2}A^{\pi}M^{-1/2} = \Psi_{(M,M)}(A^{\pi})$.

(c) It follows by definition.

(d) By definition, A is (M,M)-Hermitian when $M^{-1}A^{*}M = A$. This equality is equivalent to $\Psi_{(M,M)}(A) = M^{-1/2}A^{*}M^{1/2} = (\Psi_{(M,M)}(A))^{*}$, that is $\Psi_{(M,M)}(A)$ is Hermitian.

(e) It is a consequence of (d) and Remark 2.2 (iv). ■

However, in general $\Psi_{(M,M)}(A^{\dagger}) \neq (\Psi_{(M,M)}(A))^{\dagger}$. Indeed, the matrices A and M given in (6) provide a counterexample.

Lemma 3.2 Let $A \in \mathbb{C}^{n}_{0} \cup \mathbb{C}^{n}_{1}$.

(a) If A is (M,M)-Hermitian then $A \in \mathcal{EP}_{(M,M)}$.

(b) If $A \in \mathcal{EP}_{(M,M)}$ then A^{π} is (M,M)-Hermitian. Hence, A^{π} is an M-orthogonal projector onto $M^{-1}\mathcal{N}(A^{*})$ along $\mathcal{R}(A)$.

(c) If $A^{\pi} \in \mathcal{EP}_{(M,M)}$ then $A \in \mathcal{EP}_{(M,M)}$.

Proof.

(a) If $M^{-1}A^{*}M = A$ then $(MA)^{*} = MA$, that is MA is EP. Hence, $A \in \mathcal{EP}_{(M,M)}$ by Proposition 2.1.

(b) Since $A \in \mathcal{EP}_{(M,M)}$, Theorem 2.2 implies that $\Psi_{(M,M)}(A)$ is EP. Thus, $\Psi_{(M,M)}(A^{\pi}) = (\Psi_{(M,M)}(A))^{\pi}$ is Hermitian by Lemma 3.1 and Lemma 4.3 in [11]. Therefore, applying Lemma 3.1 (d) we get that A^{π} is an (M,M)-Hermitian matrix. Furthermore, it is well known that A^{π} projects onto $\mathcal{N}(A)$ along $\mathcal{R}(A)$. From Proposition 2.1 (i) we have $\mathcal{N}(A) = \mathcal{N}((MA)^{*}) = \mathcal{N}(A^{*}M) = M^{-1}\mathcal{N}(A^{*})$. Moreover, $\mathcal{N}(A^{*}M) = \mathcal{N}(A^{\oplus(M,M)})$. Thus the M-orthogonality of A^{π} follows from Lemma 1.1 (d).

(c) Using Theorem 2.2 and Lemma 3.1 (a) we get that $(\Psi_{(M,M)}(A))^\pi$ is EP. From Lemma 4.3 in [11], $\Psi_{(M,M)}(A)$ is EP. Hence, A is $EP_{(M,M)}$ by Theorem 2.2. ∎

We define the function $f : \mathbb{C}_0^n \cup \mathbb{C}_1^n \longrightarrow \mathbb{C}_0^n \cup \mathbb{C}_1^n$ by $f(A) = A^\pi$ for each $A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$.

Lemma 3.3 Let $A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$ be an $EP_{(M,M)}$ matrix. Then $f(f(A)) = A$ if and only if A is an M-orthogonal projector.

Proof. By Theorem 2.2, $\Psi_{(M,M)}(A)$ is EP. Applying Lemma 3.1 (b), it is clear that $f(f(A)) = A$ is equivalent to $f(f(\Psi_{(M,M)}(A))) = \Psi_{(M,M)}(A)$. By Remark 4.1 in [11], this last equality holds if and only if $\Psi_{(M,M)}(A)$ is an orthogonal projector. Now, the proof follows applying Lemma 3.1 (e). ∎

Let us consider the sets

$$\mathcal{EEP}_{(M,M)} = \{A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n : f(A) \in \mathcal{EP}_{(M,M)}\}$$

and

$$\mathcal{EEP}_{(M,M)}^0 = \{A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n : f(A) \in \mathcal{EP}_{(M,M)} \text{ and } f(A) \neq O\}.$$

The next result provides a characterization for $EP_{(M,M)}$ matrices.

Theorem 3.1 *The following statements hold.*

(a) $\mathcal{EEP}_{(M,M)} = \mathcal{EP}_{(M,M)}.$

(b) $\mathcal{EEP}_{(M,M)}^0 = \mathcal{EP}_{(M,M)} \cap \mathbb{C}_1^n.$

Proof.

(a) We have shown that $A \in \mathcal{EP}_{(M,M)}$ if and only if $\Psi_{(M,M)}(A) \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$ and $f(\Psi_{(M,M)}(A)) = \Psi_{(M,M)}(f(A)) \in \mathcal{EP}$. These last conditions are equivalent to $A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$ and $f(A) \in \mathcal{EP}_{(M,M)}$, that is $A \in \mathcal{EEP}_{(M,M)}$.

(b) Let $A \in \mathcal{EP}_{(M,M)}$ be a matrix having $\mathrm{ind}(A) = 1$. From Lemma 3.2 (b) and (a), $f(A) \in \mathcal{EP}_{(M,M)}$ and A is a singular matrix, that is $f(A) \neq O$. Thus $A \in \mathcal{EEP}_{(M,M)}^0$. Therefore $\mathcal{EP}_{(M,M)} \cap \mathbb{C}_1^n \subseteq \mathcal{EEP}_{(M,M)}^0$. In order to see the other inclusion, let A be such that $f(A) \in \mathcal{EP}_{(M,M)}$ and $f(A) \neq O$. Then $\mathrm{ind}(A) = 1$ and A is $EP_{(M,M)}$ by Lemma 3.2 (c). Hence, $\mathcal{EEP}_{(M,M)}^0 \subseteq \mathcal{EP}_{(M,M)} \cap \mathbb{C}_1^n$.

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■

Remark 3.1 Let $M\text{-}\mathcal{OP}_n$ be the set of all M-orthogonal projectors of size $n \times n$. From Lemma 3.3, Lemma 3.2, Proposition 2.1 and Theorem 3.1 we obtain:

(a) $f(\mathcal{EEP}^0_{(M,M)} - \{O\}) = M\text{-}\mathcal{OP}_n \cap (\mathbb{C}^n_1 - \{O\}) = (M\text{-}\mathcal{OP}_n \cap \mathcal{EEP}^0_{(M,M)}) - \{O\}$.

(b) $f(\mathcal{EEP}^0_{(M,M)}) = (M\text{-}\mathcal{OP}_n \cap (\mathbb{C}^n_1 - \{O\})) \cup \{I_n\} = ((M\text{-}\mathcal{OP}_n \cap \mathcal{EEP}^0_{(M,M)}) - \{O\}) \cup \{I_n\}$.

From Lemma 3.2 (a) and (b), it is clear that $f(\mathcal{EP}_{(M,M)}) \subseteq \mathcal{EP}_{(M,M)}$, but in general the equality is not true as the matrices $M = \operatorname{diag}(1,3)$ and $A = \operatorname{diag}(2,0)$ show. In fact, the matrix $MA = \operatorname{diag}(2,0)$ is $\mathcal{EP}$, therefore A is $\mathcal{EP}_{(M,M)}$. Let us suppose that there exists $B \in \mathbb{C}^{2\times2} \cap \mathcal{EP}_{(M,M)}$ such that $A = f(B)$. Denoting by $\sigma(A)$ the spectrum of A, we have that $2 \in \sigma(A) = \sigma(f(B))$, which is a contradiction because $f(B)$ is a projector. Therefore, $\mathcal{EP}_{(M,M)} \nsubseteq f(\mathcal{EP}_{(M,M)})$.

Let the function $g : \mathcal{EP} \longrightarrow \mathcal{EP}$ be the restriction of the function f to the set $\mathcal{EP}$ and $h : \mathcal{EP}_{(M,M)} \longrightarrow \mathcal{EP}_{(M,M)}$ be the restriction of the function f to the set $\mathcal{EP}_{(M,M)}$. It is clear that g is well defined and, by Lemma 3.2 (a) and (b), h also is. Notice that $\Psi_{(M,M)} \circ h = g \circ \Psi_{(M,M)}$ on $\mathcal{EP}_{(M,M)}$. It is evident that h is not surjective. Moreover, h is not injective as the matrices

$$M = \operatorname{diag}(1,3), \qquad A = \operatorname{diag}(2,0), \qquad B = \operatorname{diag}(3,0) \tag{12}$$

show.

The next lemma characterizes the interval

$$[O, h(A)] = \{B \in \mathbb{C}^n_0 \cup \mathbb{C}^n_1 : O \leq^{\circledast(M,M)} B \leq^{\circledast(M,M)} h(A)\}.$$

for some given matrix $A \in \mathcal{EP}_{(M,M)}$.

Lemma 3.4 Let $A \in \mathbb{C}^{n\times n}$ be an $EP_{(M,M)}$ matrix written as in (7). Then

$$[O, h(A)] = \left\{ M^{-1/2}U_A(O \oplus T)U_A^*M^{1/2} : T \in \mathcal{OP}_{n-a} \right\} \subseteq M\text{-}\mathcal{OP}_n.$$

Proof. Let $A = M^{-1/2}U_A(C_A \oplus O)U_A^*M^{1/2}$ with $U_A \in \mathbb{C}^{n\times n}$ unitary and $C_A \in \mathbb{C}^{a\times a}$ nonsingular. Thus, $\Psi_{(M,M)}(A) = U_A(C_A \oplus O)U_A^*$.

Let $B \in [O, h(A)]$, that is $O \leq^{\circledast(M,M)} B \leq^{\circledast(M,M)} h(A)$. By Lemma 2.1, $O = \Psi_{(M,M)}(O) \leq^*$ $\Psi_{(M,M)}(B) \leq^* \Psi_{(M,M)}(h(A))$ holds. Since $\Psi_{(M,M)}(A)$ is EP, from Lemma 3.1 (b) we obtain

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$O \leq^* \Psi_{(M,M)}(B) \leq^* g(\Psi_{(M,M)}(A))$. Applying Lemma 4.5 in [11] we get that $\Psi_{(M,M)}(B)$ is an orthogonal proyector of size $n \times n$ given by $\Psi_{(M,M)}(B) = U_A(O \oplus T)U_A^*$, with $T \in \mathcal{OP}_{n-a}$. From Lemma 3.1 (e), $B = M^{-1/2}U_A(O \oplus T)U_A^* M^{1/2}$ is a M-orthogonal projector of size $n \times n$. ∎

Theorem 3.2 *The function h defined above is monotone decreasing.*

Proof. Let $A, B \in \mathbb{C}^{n \times n}$ be $EP_{(M,M)}$ matrices such that $A \leq^{\circledast(M,M)} B$. Then, $\Psi_{(M,M)}(A) \leq^*$ $\Psi_{(M,M)}(B)$ by Lemma 2.1. From Theorem 4.2 in [11], $g(\Psi_{(M,M)}(B)) \leq^* g(\Psi_{(M,M)}(A))$, which is equivalent to $\Psi_{(M,M)}(h(B))) \leq^* \Psi_{(M,M)}(h(A)))$ by Lemma 3.1 (b). Finally, by Lemma 2.1 we get $h(B) \leq^{\circledast(M,M)} h(A)$. ∎

However, considering the matrices given in (12) we have that A and B are $EP_{(M,M)}$ such that $h(A) \leq^{\circledast(M,M)} h(B)$ but $B \not\leq^{\circledast(M,M)} A$. In the following result we state the converse of Theorem 3.2 for a smaller class of matrices.

Theorem 3.3 *Let $A, B \in M\text{-}\mathcal{OP}_n$. If $h(B) \leq^{\circledast(M,M)} h(A)$ then $A \leq^{\circledast(M,M)} B$.*

Proof. Since A and B are M-orthogonal projectors, $\Psi_{(M,M)}(A)$ and $\Psi_{(M,M)}(B)$ are orthogonal projectors. If $h(B) \leq^{\circledast(M,M)} h(A)$ then $\Psi_{(M,M)}(h(B)) \leq^* \Psi_{(M,M)}(h(A))$, which is equivalent to $g(\Psi_{(M,M)}(B)) \leq^* g(\Psi_{(M,M)}(A))$ by Lemma 3.1 (b). By Theorem 4.3 in [11] we obtain $\Psi_{(M,M)}(A) \leq^* \Psi_{(M,M)}(B)$, that is $A \leq^{\circledast(M,M)} B$. ∎

Theorem 3.4 *Let $A \in \mathbb{C}^{n \times n}$ be an $EP_{(M,M)}$ matrix and $B \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$ such that $A \leq^{\circledast(M,M)} B$. Then $f(B) \leq^{\circledast(M,M)} f(A)$ if and only if $f(B) \in M\text{-}\mathcal{OP}_n$.*

Proof. By definition $f(B) \leq^{\circledast(M,M)} f(A)$ means $\Psi_{(M,M)}(f(B)) \leq^* \Psi_{(M,M)}(f(A))$, and from Lemma 3.1 (b), it is equivalent to $f(\Psi_{(M,M)}(B)) \leq^* f(\Psi_{(M,M)}(A))$. Since $f(\Psi_{(M,M)}(A)) \in \mathcal{EP}$, using Theorem 4.4 in [11] and Lemma 3.1 (b), the last inequality holds if and only if $\Psi_{(M,M)}(f(B))$ is an orthogonal projector, that is, $f(B)$ is an M-orthogonal projector. ∎

We close this section with the following two remarks.

Remark 3.2 We can consider all the linear combinations $C_{\alpha,\beta} = \alpha A + \beta B$, $\alpha, \beta \in \mathbb{C}$, between two given $EP_{(M,M)}$ matrices A and B in $\mathbb{C}^{n \times n}$ [19]. A similar result to Theorem 4.7 in [11] can be stated for $EP_{(M,M)}$ where the (M, M)-star partial order is used to compare the following

pairs of matrices: A and $C_{\alpha,\beta}$, $C_{\alpha,\beta}$ and B, $C_{\alpha,\beta}$ and $C_{\gamma,\delta}$, $f(C_{\alpha,\beta})$ and $f(A)$, $f(B)$ and $f(C_{\alpha,\beta})$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

Remark 3.3 Let $A, B \in \mathbb{C}^{n \times n}$. Let $A = A_1 + A_2$ and $B = B_1 + B_2$ be the core-nilpotent decompositions of A and B respectively, where A_1 is core part of A, B_1 is core part of B, A_2 is nilpotent part of A and B_2 is nilpotent part of B. In [14], it was defined that $A \leq^d B$ if and only if $A_1^{\#} A_1 = A_1^{\#} B_1$ and $A_1 A_1^{\#} = B_1 A_1^{\#}$. This binary relation $\leq^d$ is a pre-order. Some algebraic manipulations allow us to prove that $A \leq^d B$ implies $B^{\pi} = I - BB^D \leq^d I - AA^D = A^{\pi}$.

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