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Additional Information

Hyperspaces of a Weightable Quasi-metric Space: Application to Models in the Theory of Computation

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Dedicated to Klaus Weihrauch on his 65th birthday

Abstract: It is well known that both weightable quasi-metrics and the Hausdorff distance provide efficient tools in several areas of Computer Science. This fact suggests, in a natural way, the problem of when the upper and the lower Hausdorff quasi-pseudo-metrics of a weightable quasi-metric space (X, d) are weightable. Here we discuss this problem. Although the answer is negative in general, we show, however, that it is positive for several nice classes of (nonempty) subsets of X. Since the construction of these classes depends, to a large degree, on the specialization order of the quasi-metric d, we are able to apply our results to some distinguished quasi-metric models that appear in theoretical computer science and information theory, like the domain of words, the interval domain and the complexity space.

Key Words: weightable quasi-metric, Hausdorff quasi-pseudo-metric, Pompéiu quasi-pseudo-metric, hyperspace, the specialization order, the information order, the domain of words, the interval domain, the complexity space.

Category: F.0

1 Introduction

It is well known that the Hausdorff metric provides a useful tool, not only in several fields of mathematics but also in image processing ([Huttenlocher et al. 1993, Sendov 04, Zhao et al. 05, etc]), programming language and semantics ([de Bakker and de Vink 96a, de Bakker and de Vink 96b, de Bakker and de Vink 98, etc.]), and computational biology ([Guerra and Pascucci 05, Panchenko and Madej 05, Sikora and Piramuthu 05, etc.]), among others. Recently, and motivated by questions in computer vision, a notion of fuzzy Hausdorff quasi-metric was introduced in [Rodríguez-López et al. 07], while some aspects of the analysis of asymptotic complexity of algorithms in the framework of upper and lower Hausdorff quasi-pseudo-metrics were discussed in [Rodríguez-López et al. 06].

On the other hand, since Matthews introduced in [Matthews 94] the weightable quasi-metric spaces (and the equivalent partial metric spaces) as a mathematical model in the study of denotational semantics of data flow networks, various authors have studied the theory of such spaces and developed new applications of them. In particular, several topological properties of weightable quasi-metric spaces were discussed in [Künzi 93, Künzi and Vajner 94, Oltra et al. 02, etc], while connections of these spaces with domain theory and applications to several fields of computer science and information sciences were given in [Heckmann 99, O'Neill 96, Romaguera, Romaguera and Schellekens 99, Romaguera and Schellekens 05, Schellekens 95, Schellekens 03, Schellekens 04, Waszkiewicz 06, etc.].

Motivated by these facts, we here study the problem of obtaining suitable classes of (nonempty) subsets of a given weightable quasi-metric space for which the upper and/or lower Hausdorff quasi-pseudo-metric is weightable. This is a hard problem because there exist easy examples of metric spaces for which both the upper and lower Hausdorff quasi-pseudo-metrics are not weightable even on the collection of (nonempty) finite sets, as we will show. However, we prove that it is still possible to find positive results for some interesting classes of collections of subsets whose construction depends, to a great part, on the specialization order induced by the given quasi-metric. This fact permits us to successfully apply our constructions to some paradigmatic examples of the theories of computation and information like the domain of words, the interval domain and the complexity space.

2 Preliminaries

In the sequel the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and ω will denote the set of real numbers, the set of nonnegative real numbers, the set of positive integer numbers and the set of nonnegative integer numbers, respectively.

Our basic references for quasi-metric spaces and quasi-uniform spaces are [Fletcher and Lindgren 82] and [Künzi 01], and for general topology it is [Engelking 77]. An excellent discussion on different types of quasi-metrics that appear in the theory of computation may be found in [Seda and Hitzler].

By a quasi-pseudo-metric on a set X we mean a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $d(x, x) = 0$;

(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

Following the modern terminology, a quasi-metric on X is a quasi-pseudo-metric d on X which satisfies the condition

(i') $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$.

By a quasi-(pseudo-)metric space we mean a pair (X, d) such that X is a nonempty set and d is a quasi(pseudo-)metric on X.

Each quasi-pseudo-metric d on X induces a topology τ_d on X which has as a base the family of open balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Observe that if d is a quasi-metric on X, then τ_d is a T_0 topology on X.

Given a quasi-(pseudo-)metric d on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-(pseudo-)metric on X, called the conjugate of d, and the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a (pseudo-)metric on X.

A subset A of a quasi-(pseudo-)metric space (X, d) is called bounded if A is bounded in the (pseudo-)metric space (X, d^s).

The following is an easy but paradigmatic example of a quasi-metric space.

Example 1. Let ℓ be the function defined on $\mathbb{R} \times \mathbb{R}$ by $\ell(x, y) = \max\{x - y, 0\}$. Then ℓ is the so-called lower quasi-metric on $\mathbb{R}$. Note that ℓ^s is the Euclidean metric on $\mathbb{R}$. Denote by u the conjugate quasi-metric of ℓ; then $u(x, y) = \max\{y - x, 0\}$ for all $x, y \in \mathbb{R}$, and u is said to be the upper quasi-metric on $\mathbb{R}$. Note that u^s is the Euclidean metric on $\mathbb{R}$.

If d is a quasi-pseudo-metric on a set X, then the relation $\leq_d$ on X given by $x \leq_d y \Leftrightarrow d(x, y) = 0$, is a preorder on X (i.e., $\leq_d$ is reflexive and transitive).

It is well known, and easy to see, that d is a quasi-metric on a set X if and only if $\leq_d$ is a (partial) order on X (i.e., the preorder $\leq_d$ is antisymmetric, which means that $x \leq_d y$ and $y \leq_d x$, implies $x = y$). In this case, $\leq_d$ is called the specialization order.

Note that in Example 1 the specialization order of ℓ coincides with the usual order on $\mathbb{R}$.

When we model a computational process, is often necessary to build spaces containing both the total objects, i.e., the final results of the computation, as the partial objects appearing in the different stages of the process [Davey and Priestley, p. 5]. This model should therefore include an ingredient that allows us to distinguish these two types of objects. In this context, the notion of a weightable quasi-(pseudo-)metric space, introduced and discussed by Matthews in [Matthews 94], provides a suitable framework.

Let us recall that a quasi-(pseudo-)metric space (X, d) is said to be weightable if there is a function $w : X \to \mathbb{R}^+$ such that

$$d(x, y) + w(x) = d(y, x) + w(y),$$

for all $x, y \in X$. In this case, we say that d is a weightable quasi-(pseudo-)metric, and w is called a weighting function for d. By a weighted quasi-(pseudo-)metric space we mean a triple (X, d, w) such that d is a weightable quasi-(pseudo-)metric on X and w is a weighting function for d.

Obviously, each metric space (X, d) is weightable with weighting function w given by $w(x) = 0$ for all $x \in X$.

From a computational point of view, the weighting function w permits us to distinguish between the total objects and the remaining partial objects; in fact the value of $w(x)$ is used to describe the amount of information contained in x (see, for instance, [Matthews 94, p. 189] and [Spreen, Section 2]). In this direction, it is interesting to note that if $x \in X$ satisfies $w(x) = \inf_{y \in X} w(y)$, then x is maximal with respect to the specialization order, i.e., $x \leq_d y \Rightarrow x = y$.

Remark 1. The restriction to the nonpositive real numbers $\mathbb{R}^-$ of the quasi-metric ℓ of Example 1 is weightable with weighting function w given by $w(x) = -x$ for all $x, y \in \mathbb{R}^-$. Furthermore, the restriction of u to $\mathbb{R}^+$ is weightable with weighting function w given by $w(x) = x$ for all $x, y \in \mathbb{R}^+$. So $(\mathbb{R}^+, u, w)$ is a weighted quasi-metric space.

Next we recall the construction of the Hausdorff quasi-(pseudo-)metric of a given quasi-metric space.

If (X, d) is a quasi-metric space we denote by $\mathcal{P}_0(X)$, $\mathcal{F}_0(X)$, $\mathcal{B}_0(X)$, $\mathcal{C}_0(X)$ and $\mathcal{K}_0(X)$, the collection of all nonempty subsets of X, the collection of all nonempty finite subsets of X, the collection of all nonempty bounded subsets of X, the collection of all nonempty closed subsets of (X, τ_d) and the collection of all nonempty compact subsets of (X, τ_d), respectively. The collection of all nonempty compact subsets in $(X, \tau_{d^{-1}})$ will be denoted by $\mathcal{K}_0^{-1}(X)$.

If A is a subset of X we denote by $cl_d(A)$ the closure of A with respect to τ_d.

For a quasi-pseudo-metric space (X, d), we define

$$\mathcal{C}_\cap(X) = \{cl_d(A) \cap cl_{d^{-1}}(A) : A \in \mathcal{P}_0(X)\}.$$

Remark 2. The following inclusions are obvious: $\mathcal{F}_0(X) \subseteq \mathcal{B}_0(X) \cap \mathcal{K}_0(X) \subseteq \mathcal{P}_0(X)$, and $\mathcal{C}_0(X) \subseteq \mathcal{C}_\cap(X) \subseteq \mathcal{P}_0(X)$. Moreover, if (X, d) is a metric space, then $\mathcal{K}_0(X) \subseteq \mathcal{C}_0(X)$ and $\mathcal{C}_0(X) = \mathcal{C}_\cap(X)$.

Now, for each $A, B \in \mathcal{B}_0(X)$ let

$$H_d^-(A, B) = \sup_{a \in A} d(a, B), \quad H_d^+(A, B) = \sup_{b \in B} d(A, b),$$

and

$$H_d(A, B) = \max\{H_d^-(A, B), H_d^+(A, B)\}.$$

Then each of H_d^-, H_d^+ and H_d is a quasi-pseudo-metric on $\mathcal{B}_0(X)$, called the lower Hausdorff quasi-pseudo-metric of d, the upper Hausdorff quasi-pseudo-metric of d and the Hausdorff quasi-pseudo-metric of d, respectively (compare [Berthiaume 77, Künzi and Ryser 95, Rodríguez-López and Romaguera 02, Rodríguez-López and Romaguera 03, etc]), and for any $\mathcal{A} \subseteq \mathcal{B}_0(X)$, $\mathcal{A} \neq \emptyset$,

we refer to the quasi-pseudo-metric spaces $(\mathcal{A}, H_d^-)$, $(\mathcal{A}, H_d^+)$ and $(\mathcal{A}, H_d)$ as hyperspaces.

Moreover H_d is a quasi-metric on $\mathcal{B}_0(X) \cap \mathcal{C}_\cap(X)$ (compare Lemma 2 of [Künzi and Ryser 95]). In this case we say that H_d is the Hausdorff quasi-metric of d.

Of course, if (X, d) is a metric space, then H_d is the Hausdorff metric of d (on $\mathcal{B}_0(X) \cap \mathcal{C}_0(X)$), and $H_d^+ + H_d^-$ is the so-called Pompéiu metric of d (see, for instance, [Engelking 77, p. 371]). For this reason, if (X, d) is a quasi-metric space, the quasi-pseudo-metric $H_d^+ + H_d^-$ on $\mathcal{B}_0(X)$, will be called the Pompéiu quasi-pseudo-metric of d.

Observe that $(H_d^+ + H_d^-)/2 \leq H_d \leq H_d^+ + H_d^-$ on $\mathcal{B}_0(X)$.

3 Weightability of hyperspaces

In a first attempt to obtain a representative class of (nonempty) subsets of a given weightable quasi-metric space (X, d) for which the Hausdorff quasi-pseudo-metric, or at least the upper or lower Hausdorff quasi-pseudo-metric, is weightable, we focus our attention on $\mathcal{K}_0(X)$ and $\mathcal{K}_0^{-1}(X)$ due to the following fact.

Proposition 3. *Let (X, d, w) be a weighted quasi-metric space. Then:*

(A) For each $A \in \mathcal{K}_0(X)$ there exists $a_0 \in A$ such that $w(a_0) = \sup_{a \in A} w(a)$.

(B) For each $A \in \mathcal{K}_0^{-1}(X)$ there exist $a_1 \in A$ such that $w(a_1) = \inf_{a \in A} w(a)$.

Proof. By [Künzi and Vajner 94, Lemma 3], w is upper semicontinuous on (X, τ_d) and lower semicontinuous on $(X, \tau_{d^{-1}})$. Hence, by [Kolmogorov and Fomin 70, Theorem 2" and its Remark, on p. 111], if $A \in \mathcal{K}_0(X)$, then $\sup_{a \in A} w(a) = w(a_0)$ for some $a_0 \in A$, which shows (A), and if $A \in \mathcal{K}_0^{-1}(X)$, then $\inf_{a \in A} w(a) = w(a_1)$ for some $a_1 \in A$, which shows (B). $\qquad\qquad\qquad\square$

In the light of the above proposition one could expect that H_d^+ (respectively, H_d^-) is weightable on $\mathcal{B}_0(X) \cap \mathcal{K}_0(X)$ (respectively, on $\mathcal{B}_0(X) \cap \mathcal{K}_0^{-1}(X)$), with weighting function $\sup_{a \in A} w(a)$ (respectively, $\inf_{a \in A} w(a)$).

Next we show that, unfortunately, this is not the case. In fact, the following is an example of a compact weighted quasi-metric space (X, d, w) for which the hyperspaces $(\mathcal{F}_0(X), H_d)$, $(\mathcal{F}_0(X), H_d^+)$ and $(\mathcal{F}_0(X), H_d^-)$ are not weightable.

Example 2. Let d be the quasi-metric on $\mathbb{N}$ given by $d(n, n) = 0$ for all $n \in \mathbb{N}$ and $d(n, m) = 1/m$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Clearly τ_d is the cofinite topology on $\mathbb{N}$ (i.e. closed proper sets are the finite subsets of $\mathbb{N}$), so τ_d is a compact T_1 topology on $\mathbb{N}$.

Furthermore the function w defined on $\mathbb{N}$ by $w(n) = 1/n$ for all $n \in \mathbb{N}$, satisfies for $n \neq m$,

$$d(n,m) + w(n) = \frac{1}{m} + \frac{1}{n} = w(m) + d(m,n).$$

Hence $(\mathbb{N}, d, w)$ is a weighted quasi-metric space. Next we show that H_d is not weightable on $\mathcal{F}_0(\mathbb{N})$. Indeed, suppose that there is $W : \mathcal{F}_0(\mathbb{N}) \to \mathbb{R}^+$ such that $H_d(A,B) + W(A) = H_d(B,A) + W(B)$ for all $A, B \in \mathcal{F}_0(\mathbb{N})$. Let $A_k = \{1, 2, ..., k\}$ for all $k \in \mathbb{N}$. Then, for $j < k$, we obtain

$$H_d^+(A_j, A_k) = \sup_{a \in A_k} d(A_j, a) = \sup\{\frac{1}{j+1}, ..., \frac{1}{k}\} = \frac{1}{j+1},$$
$$H_d^-(A_j, A_k) = 0,$$
$$H_d^+(A_k, A_j) = 0, \quad \text{and}$$
$$H_d^-(A_k, A_j) = \sup_{a \in A_k} d(a, A_j) = \frac{1}{j}.$$

Since, by our assumption, $H_d(A_j, A_k) + W(A_j) = H_d(A_k, A_j) + W(A_k)$, we deduce, for $j = 1$, that:

$$\frac{1}{2} + W(A_1) = 1 + W(A_k)$$

for all $k > 1$. It follows that, for $k > 2$ and $j = k - 1$,

$$H_d(A_{k-1}, A_k) = H_d(A_k, A_{k-1}),$$

a contradiction because $H_d(A_{k-1}, A_k) = 1/k$ and $H_d(A_k, A_{k-1}) = 1/(k-1)$. We conclude that $(\mathcal{F}_0(\mathbb{N}), H_d)$ is not weightable and thus $(\mathcal{K}_0(\mathbb{N}), H_d)$ and $(\mathcal{C}_0(\mathbb{N}), H_d)$ are not weightable.

Note also that from the preceding construction it also follows that the quasi-pseudo-metric spaces $(\mathcal{F}_0(\mathbb{N}), H_d^+)$ and $(\mathcal{F}_0(\mathbb{N}), H_d^-)$ are not weightable.

Next we show that if d is the Euclidean metric on $\mathbb{R}$, then both $(\mathcal{F}_0(\mathbb{R}), H_d^+)$ and $(\mathcal{F}_0(\mathbb{R}), H_d^-)$ are not weightable.

Example 3. Let $X = \{0, 1, 2\}$ and let d be the restriction to X of the Euclidean metric on $\mathbb{R}$. Suppose that there is a weighting function W for H_d^+. Put $F_1 = X$ and $F_2 = \{0, 2\}$. Then, we have

$$H_d^+(F_1, F_2) = H_d^+(F_1, \{2\}) = H_d^+(F_2, \{2\}) = 0,$$
$$H_d^+(F_2, F_1) = 1, \quad \text{and} \quad H_d^+(\{2\}, F_1) = H_d^+(\{2\}, F_2) = 2.$$

Hence

$$W(F_1) = 1 + W(F_2), \quad \text{and} \quad W(F_1) = 2 + W(\{2\}) = W(F_2),$$

which is a contradiction. We conclude that $(\mathcal{F}_0(X), H_d^+)$ is not weightable.

Now suppose that there is a weighting function V for H_d^-. Then, for $F_1 = X$ and $F_2 = \{0, 2\}$, we have

$$H_d^-(F_2, F_1) = H_d^-(\{2\}, F_1) = H_d^-(\{2\}, F_2) = 0,$$
$$H_d^-(F_1, F_2) = 1, \quad \text{and} \quad H_d^-(F_1, \{2\}) = H_d^-(F_2, \{2\}) = 2.$$

Hence

$$1 + V(F_1) = V(F_2), \quad \text{and} \quad 2 + V(F_1) = V(\{2\}) = 2 + V(F_2),$$

which is a contradiction. We conclude that $(\mathcal{F}_0(X), H_d^-)$ is not weightable.

In the following we shall prove that, nevertheless, it is still possible to obtain significant classes of (nonempty) subsets of a weightable quasi-metric space (X, d) for which the upper, the lower, and the Pompéiu quasi-pseudo-metric are weightable.

Let (X, d) be a quasi-metric space. For each $A \in \mathcal{P}_0(X)$ let

$$A_{\leq_d} = \{y \in X : y \leq_d a \text{ for all } a \in A\},$$

and put

$$\mathcal{P}_{\leq_d, cl_{d^{-1}}}(X) = \{A \in \mathcal{P}_0(X) : A_{\leq_d} \cap cl_{d^{-1}}(A) \neq \emptyset\}.$$

Note that both $\{x\}$ and $cl_{d^{-1}}(\{x\})$ belong to $\mathcal{P}_{\leq_d, cl_{d^{-1}}}(X)$ for all $x \in X$.

Next we give two easy but interesting instances of subsets of $\mathcal{P}_0(X)$ which belong to $\mathcal{P}_{\leq_d, cl_{d^{-1}}}(X)$.

Example 4. Let $(\mathbb{R}^+, u)$ be the (weightable) quasi-metric space of Remark 1. We observe that a nonempty subset A of $\mathbb{R}^+$ belongs to $\mathcal{P}_{\leq_u, cl_{u^{-1}}}(\mathbb{R}^+)$ if and only if A is a bounded subset of $\mathbb{R}^+$ with respect to the Euclidean metric: Indeed, if $A \in \mathcal{P}_{\leq_u, cl_{u^{-1}}}(\mathbb{R}^+)$, then there is $x_0 \in \mathbb{R}^+$ such that $u(x_0, a) = 0$ for all $a \in A$, i.e., $a \leq x_0$ for all $a \in A$, in the usual order of $\mathbb{R}^+$. Conversely, if A is bounded in $\mathbb{R}^+$, then $x_0 \in A_{\leq_u} \cap cl_{u^{-1}}(A)$, where by x_0 we denote the supremum of A for the usual order.

Example 5. Let (X, d) be a quasi-metric space having a minimum element $\perp$ with respect to the specialization order (these types of spaces are common in theoretical computer science; see Section 4). Then each subset of X containing $\perp$ belongs to $\mathcal{P}_{\leq_d, cl_{d^{-1}}}(X)$. In fact, for any quasi-metric space (X, d), if A is a nonempty subset of X having a minimum element with respect to the specialization order, then $A \in \mathcal{P}_{\leq_d, cl_{d^{-1}}}(X)$.

The following lemmas will be crucial later on.

Lemma 4. *Let (X,d) be a quasi-metric space and let $A \in \mathcal{P}_{\leq_d, cl_{d-1}}(X)$. Then, there is $x_0 \in X$ such that*

$$A_{\leq_d} \cap cl_{d-1}(A) = \{x_0\}.$$

Furthermore x_0 is the infimum of A with respect to $\leq_d$.

Proof. Suppose that there exist $x, y \in A_{\leq_d} \cap cl_{d-1}(A)$. Then $d(x,a) = d(y,a) = 0$ for all $a \in A$, and there exist sequences $(a_n)_n$, $(a'_n)_n$, in A such that $d(a_n, x) \to 0$ and $d(a'_n, y) \to 0$. By the triangle inequality it follows that $d(x,y) = 0$ and $d(y,x) = 0$. So $x = y$.

Finally, suppose that there is $z \in X$ such that $z \leq_d a$ for all $a \in A$. Since $x_0 \in cl_{d-1}(A)$, it follows that $d(z, x_0) = 0$, i.e., $z \leq_d x_0$. This concludes the proof. $\square$

The point x_0 of the above lemma will be denoted by $\inf A$ in the following. If, in addition, $x_0 \in A$, then it will be denoted by $\min A$.

Lemma 5. *Let (X, d, w) be a weighted quasi-metric space. Then, for each $A \in \mathcal{P}_{\leq_d, cl_{d-1}}(X)$ we have*

$$\sup_{a \in A} w(a) = w(\inf A).$$

Furthermore A is bounded.

Proof. Since $w(\inf A) = d(a, \inf A) + w(a)$ for all $a \in A$, it follows that $\sup_{a \in A} w(a) \leq w(\inf A)$. Now let $(a_n)_n$ be a sequence in A such that $d(a_n, \inf A) < 1/n$ for all $n \in \mathbb{N}$. Then $w(\inf A) < 1/n + w(a_n)$ for all $n \in \mathbb{N}$, and, consequently, $\sup_{a \in A} w(a) = w(\inf A)$.

Finally, for each pair $a, a' \in A$, we have that $d(a, a') \leq d(a, \inf A) \leq w(\inf A)$, so that A is bounded. $\square$

Proposition 6. *Let (X, d, w) be a weighted quasi-metric space. Then for each $A, B \in \mathcal{P}_{\leq_d, cl_{d-1}}(X)$ we have*

$$H_d^+(A, B) + w(\inf A) = H_d^+(B, A) + w(\inf B).$$

Hence H_d^+ is weightable on $\mathcal{P}_{\leq_d, cl_{d-1}}(X)$ with weighting function W given by $W(A) = w(\inf A)$, for all $A \in \mathcal{P}_{\leq_d, cl_{d-1}}(X)$.

Proof. We show that $H_d^+(A, B) + w(\inf A) \leq H_d^+(B, A) + w(\inf B)$. Indeed, given $\varepsilon > 0$ there exist $b' \in B$ such that $H_d^+(A, B) < d(A, b') + \varepsilon$. Since, by Lemma 5, $w(\inf A) = \sup_{a \in A} w(a)$, there exists $a' \in A$ such that $w(\inf A) < w(a') + \varepsilon$.

Then, for each $b \in B$ we have

$$d(A, b') \leq d(a', b')$$
$$\leq d(a', b) + d(b, \inf B) + d(\inf B, b')$$
$$= d(b, a') + w(b) - w(a') + w(\inf B) - w(b)$$
$$< d(b, a') + \varepsilon - w(\inf A) + w(\inf B).$$

Consequently

$$d(A, b') + w(\inf A) \leq d(B, a') + w(\inf B) + \varepsilon$$
$$\leq H_d^+(B, A) + w(\inf B) + \varepsilon.$$

We conclude that

$$H_d^+(A, B) + w(\inf A) \leq H_d^+(B, A) + w(\inf B).$$

Interchanging A and B, we obtain that

$$H_d^+(A, B) + w(\inf A) \geq H_d^+(B, A) + w(\inf B).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let (X, d) be a quasi-metric space. For each $A \in \mathcal{P}_0(X)$ let

$$A_{\geq_d} = \{y \in X : a \leq_d y \text{ for all } a \in A\},$$

and put

$$\mathcal{P}_{\geq_d, cl_d}(X) = \{A \in \mathcal{P}_0(X) : A_{\geq_d} \cap cl_d(A) \neq \emptyset\}.$$

Note that both $\{x\}$ and $cl_d(\{x\})$ belong to $\mathcal{P}_{\geq_d, cl_d}(X)$ for all $x \in X$.

Since $\mathcal{P}_{\geq_d, cl_d}(X) = \mathcal{P}_{\leq_{d^{-1}}, cl_d}(X)$, we deduce from Lemma 4 the following.

Lemma 7. *Let (X,d) be a quasi-metric space and let $A \in \mathcal{P}_{\geq_d, cl_d}(X)$. Then, there is $y_0 \in X$ such that*

$$A_{\geq_d} \cap cl_d(A) = \{y_0\}.$$

Furthermore y_0 is the supremum of A with respect to $\leq_d$.

In the light of the above lemma, the point y_0 will be denoted by $\sup A$ in the following. If, in addition, $y_0 \in A$, then it will be denoted by $\max A$.

Remark 8. We give an easy example which shows that in contrast to Lemma 5, the fact that $A \in \mathcal{P}_{\geq_d, cl_d}(X)$, does not imply that A be bounded. Indeed, consider the weighted quasi-metric space $(\mathbb{R}^+, u, w)$ of Remark 1. Clearly $\mathbb{R}^+ \in \mathcal{P}_{\geq_u, cl_u}(\mathbb{R}^+)$ because $x \leq_u 0$ for all $x \in \mathbb{R}^+$. However $\mathbb{R}^+$ is not bounded with respect to the Euclidean metric.

However, we can obtain, similarly to Lemma 5, the following fact.

Lemma 9. *Let (X, d, w) be a weighted quasi-metric space. Then, for each $A \in \mathcal{P}_{\geq_d, cl_d}(X)$ we have*

$$\inf_{a \in A} w(a) = w(\sup A).$$

Proposition 10. *Let (X, d, w) be a weighted quasi-metric space. Then for each $A, B \in \mathcal{P}_{\geq_d, cl_d}(X)$ such that A and B are bounded, we have*

$$H_d^-(A, B) + w(\sup A) = H_d^-(B, A) + w(\sup A).$$

Hence H_d^- is weightable on $\mathcal{P}_{\geq_d, cl_d}(X)$ with weighting function W given by $W(A) = w(\sup A)$, for all $A \in \mathcal{P}_{\geq_d, cl_d}(X)$.

Proof. We show that $H_d^-(A, B) + w(\sup A) \leq H_d^-(B, A) + w(\sup B)$. Indeed, given $\varepsilon > 0$, there exists $a' \in A$ such that $H_d^-(A, B) < d(a', B) + \varepsilon$. By Lemma 9, there exists $b' \in B$ such that $w(b') < w(\sup B) + \varepsilon$. Then, for each $a \in A$ we have

$$
\begin{aligned}
d(a', B) &\leq d(a', b') \\
&\leq d(a', \sup A) + d(\sup A, a) + d(a, b') \\
&= d(a, \sup A) + w(a) - w(\sup A) + d(b', a) + w(b') - w(a) \\
&< d(b', a) - w(\sup A) + w(\sup B) + \varepsilon.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
d(a', B) + w(\sup A) &\leq d(b', A) + w(\sup B) + \varepsilon \\
&\leq H_d^-(B, A) + w(\sup B) + \varepsilon.
\end{aligned}
$$

We conclude that

$$H_d^-(A, B) + w(\sup A) \leq H_d^-(B, A) + w(\sup A).$$

Interchanging A and B we obtain,

$$H_d^-(A, B) + w(\sup A) \geq H_d^-(B, A) + w(\sup A).$$

This completes the proof. $\square$

Remark 11. It follows from Example 4 and Propositions 6 and 10 that both H_u^+ and H_u^- are weightable on the collection of all nonempty bounded subsets of $\mathbb{R}^+$.

Note that if $\mathcal{A}_0$ is a collection of nonempty subsets of a quasi-metric space (X, d) such that there exist functions $W_1, W_2 : \mathcal{A}_0 \to \mathbb{R}^+$ satisfying

$$H_d^+(A, B) + W_1(A) = H_d^+(B, A) + W_1(B), \quad \text{and}$$
$$H_d^-(A, B) + W_2(A) = H_d^-(B, A) + W_2(B),$$

for all $A, B \in \mathcal{A}_0$, then the Pompéiu quasi-pseudo-metric on $\mathcal{A}_0$ is weightable with weighting function $W_1 + W_2$.

Combining this fact with Lemma 5 and Propositions 6 and 10, we deduce the following result.

Proposition 12. *Let (X, d, w) be a weighted quasi-metric space. Then $H_d^+ + H_d^-$ is weightable on $\mathcal{P}_{\leq_d, cl_{d-1}}(X) \cap \mathcal{P}_{\geq_d, cl_d}(X)$, with weighting function W given by*

$$W(A) = w(\inf A) + w(\sup A),$$

for all $A \in \mathcal{P}_{\leq_d, cl_{d-1}}(X) \cap \mathcal{P}_{\geq_d, cl_d}(X)$.

There exist some interesting special cases of the preceding results. If (X, d, w) is a weighted quasi-metric space we define

$$\mathcal{P}_{\leq_d, \min}(X) = \{A \in \mathcal{P}_0(X) : \text{there is } \min A \text{ with respect to } \leq_d\},$$

and

$$\mathcal{P}_{\geq_d, \max}(X) = \{A \in \mathcal{P}_0(X) : \text{there is } \max A \text{ with respect to } \leq_d\}.$$

Note that $\{x\} \in \mathcal{P}_{\leq_d, \min}(X) \cap \mathcal{P}_{\geq_d, \max}(X)$ for all $x \in X$. Moreover $\mathcal{P}_{\leq_d, \min}(X) \subseteq \mathcal{P}_{\leq_d, cl_{d-1}}(X)$ and $\mathcal{P}_{\geq_d, \max}(X) \subseteq \mathcal{P}_{\geq_d, cl_d}(X)$.

Proposition 13. *Let (X, d, w) be a weighted quasi-metric space. Then:*
(A) For each $A, B \in \mathcal{P}_{\leq_d, \min}(X)$, we have

$$H_d^+(A, B) = d(\min A, \min B),$$

and
$$H_d^+(A, B) + w(\min A) = H_d^+(B, A) + w(\min B).$$

(B) For each $A, B \in \mathcal{P}_{\geq_d, \max}(X)$ such that A and B are bounded, we have

$$H_d^-(A, B) = d(\max A, \max B),$$

and
$$H_d^-(A, B) + w(\max A) = H_d^-(B, A) + w(\max B).$$

Proof. Part (A): For each $b \in B$ we have $d(\min A, b) = d(A, b)$. Since $d(\min A, b) \leq d(\min A, \min B)$ for all $b \in B$, it follows that

$$H_d^+(A, B) = \sup_{b \in B} d(\min A, b) = d(\min A, \min B).$$

Finally, the equality

$$H_d^+(A, B) + w(\min A) = H_d^+(B, A) + w(\min B).$$

follows from Proposition 6.

Part (B): For each $a \in A$ we have $d(a, \max B) = d(a, B)$. Since $d(a, \max B) \leq d(\max A, \max B)$ for all $a \in A$, it follows that

$$H_d^-(A, B) = \sup_{a \in A} d(a, \max B) = d(\max A, \max B).$$

Finally, the equality

$$H_d^-(A, B) + w(\max A) = H_d^-(B, A) + w(\max B).$$

follows from Proposition 10. $\qquad\square$

Corollary 14. *Let (X, d, w) be a weighted quasi-pseudo-metric space. Then $H_d^+ + H_d^-$ is weightable on $\mathcal{P}_{\leq_d,\min}(X) \cap \mathcal{P}_{\geq_d,\max}(X)$ with weighting function W given by*

$$W(A) = w(\min A) + w(\max A),$$

for all $A \in \mathcal{P}_{\leq_d,\min}(X) \cap \mathcal{P}_{\geq_d,\max}(X)$.

Related to the above corollary, we present an example of a weighted quasi-metric space (X, d, w) such that H_d is not weightable on $\mathcal{P}_{\leq_d,\min}(X) \cap \mathcal{P}_{\geq_d,\max}(X)$. To this end the following auxiliary result, obtained in [Schellekens 96, Lemma 8], will be useful.

Lemma 15. *If (X, d) is a weightable quasi-pseudo-metric space such that $(X, \leq_d)$ has a maximum x_0, then its weighting functions are exactly the functions $w_{x_0} + c$ where $w_{x_0}(x) = d(x_0, x)$ for all $x \in X$ and $c \in \mathbb{R}^+$.*

Example 6. Consider the weightable quasi-metric space $(\mathbb{R}^+, u)$ and suppose that H_u has a weighting function W on $\mathcal{P}_{\leq_u,\min}(\mathbb{R}^+) \cap \mathcal{P}_{\geq_u,\max}(\mathbb{R}^+)$. By Proposition 13, we have that for each $A \in \mathcal{P}_{\leq_u,\min}(\mathbb{R}^+) \cap \mathcal{P}_{\geq_u,\max}(\mathbb{R}^+)$,

$$H_u(A, \{0\}) = \max\{u(\min A, 0), u(\max A, 0)\} = 0.$$

So 0 is the maximum of $(\mathcal{P}_{\leq_u,\min}(\mathbb{R}^+) \cap \mathcal{P}_{\geq_u,\max}(\mathbb{R}^+), \leq_{H_u})$. Hence, by Lemma 15, for each $A \in \mathcal{P}_{\leq_u,\min}(\mathbb{R}^+) \cap \mathcal{P}_{\geq_u,\max}(\mathbb{R}^+)$, $W(A) = H_u(\{0\}, A) + c$, where $c \in \mathbb{R}^+$.

Now, put $A = [0,1]$ and $B = \{1\}$, and note that $A, B \in \mathcal{P}_{\leq_u,\min}(\mathbb{R}^+) \cap \mathcal{P}_{\geq_u,\max}(\mathbb{R}^+)$ (in fact $1 = \min A$ and $0 = \max A$). Then

$$H_u(A, B) = \max\{u(1,1), u(0,1)\} = 1 \quad \text{and}$$
$$H_u(B, A) = \max\{u(1,1), u(1,0)\} = 0.$$

So $1 + H_u(\{0\}, A) = H_u(\{0\}, B)$, which provides a contradiction because $H_u(\{0\}, A) = H_u(\{0\}, B) = 1$.

4   Further examples

In this section we shall apply the results obtained in the above section to some interesting examples of quasi-metric spaces for which the induced topology is T_0 but non T_1, like the domain of words, the interval domain and the complexity space.

Let us recall that the domain of words yields an appropriate setting to model, for instance, streams of information in Kahn's model of parallel computation ([Kahn 74, Matthews 94]), as well as several computational processes in denotational semantics, program correctness and control flow semantics ([de Bakker and de Vink 96a, de Bakker and de Vink 96b, de Bakker and de Vink 98]). Recently, it was applied in [Romaguera and Valero 08] to model certain processes which arise in a natural way in symbolic and numerical computation. On the other hand, it is well known that the interval domain provides a suitable framework to model both computational algorithms on the unit interval and zero finding methods in numerical analysis (see, for instance, [Escardo 98, Lawson 98, Martin 02]). Finally, we recall that the so-called complexity (quasimetric) space was introduced by Schellekens ([Schellekens 95]) in order to construct a topological framework for the complexity analysis of programs and algorithms. Further contributions to the study of this space and its applications, as well as to other related spaces, may be found in [Romaguera and Schellekens 99, Romaguera and Schellekens 05, Romaguera and Valero 08, Schellekens 96, etc].

Example 7. The domain of words Σ^∞ ([Künzi 01, Matthews 94, Perrin and Pin 04, Romaguera and Schellekens 05, Schellekens 03, etc]) consists of all finite and infinite sequences ("words") over a nonempty set ("alphabet") Σ, ordered by the information order $\sqsubseteq$ on Σ^∞ ([Davey and Priestley 90, Example 1.6]), i.e., $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y, where we assume that the empty sequence ϕ is an element of Σ^∞.

For each $x, y \in \Sigma^\infty$ denote by $x \sqcap y$ the longest common prefix of x and y, and for each $x \in \Sigma^\infty$ denote by $\ell(x)$ the length of x. Thus $\ell(x) \in [1, \infty]$ whenever

$x \neq \phi$, and $\ell(\phi) = 0$. It was observed in [Künzi 01, Example 8 (b)] that the function $d : \Sigma^\infty \times \Sigma^\infty \to \mathbb{R}^+$ given by

$$d(x, y) = 2^{-\ell(x \sqcap y)} - 2^{-\ell(x)},$$

is a weightable bounded quasi-metric on Σ^∞ with weighting function w given by $w(x) = 2^{-\ell(x)}$ for all $x \in \Sigma^\infty$.

Note that the specialization order $\leq_d$ coincides with $\sqsubseteq$.

Since from a computational point of view, the fact that $x \sqsubseteq y$ is interpreted as the element y contains all the information provided by x, we focus our attention on the class $\mathcal{P}_{\geq_d, cl_d}(\Sigma^\infty)$ because, by Lemma 7, each member of this class of subsets has a supremum with respect to $\leq_d$ and hence with respect to $\sqsubseteq$.

In fact, it is immediate to show that $\mathcal{P}_{\geq_d, cl_d}(\Sigma^\infty)$ consists of all nonempty subsets A of Σ^∞ having a supremum with respect to $\sqsubseteq$, which makes relevant both the class $\mathcal{P}_{\geq_d, cl_d}(\Sigma^\infty)$ and the lower Hausdorff quasi-pseudo-metric H_d^-, in this context. In particular, it follows from Proposition 10 that the hyperspace $(\mathcal{P}_{\geq_d, cl_d}(\Sigma^\infty), H_d^-)$ is weightable with weighting function W given by $W(A) = 2^{-\ell(\sup A)}$ for all $A \in \mathcal{P}_{\geq_d, cl_d}(\Sigma^\infty)$. We illustrate this situation with some particular instances.

Let $x, y \in \Sigma^\infty$ with $x \sqsubseteq y$. Let $A = \{z \in \Sigma^\infty : z \sqsubseteq x\}$ and $B = \{z \in \Sigma^\infty : z \sqsubseteq y\}$. By Proposition 10, $W(A) = 2^{-\ell(x)}$ and $W(B) = 2^{-\ell(y)}$. Moreover, since $A \subseteq B$, we have $H_d^-(A, B) = 0$ (which is reasonable because the information provided by A is contained in the information provided by B), and, by Proposition 13 (B), $H_d^-(B, A) = d(y, x) = W(A) - W(B) = 2^{-\ell(x)} - 2^{-\ell(y)}$.

Now assume $A = \{z \in \Sigma^\infty : z \sqsubseteq x, z \neq x\}$ and $B = \{z \in \Sigma^\infty : z \sqsubseteq x\}$ with $\ell(x) = \infty$. In this case we have $W(A) = W(B) = 0$ and $H_d^-(A, B) = H_d^-(B, A) = 0$, which is reasonable because, from a computational point of view, the information provided by A and B coincide.

Example 8. The interval domain $I([0, 1])$ ([Escardo 98, Lawson 98, Matthews 94]) consists of the nonempty compact intervals of $[0, 1]$ ordered by reverse inclusion, i.e., $[a, b] \sqsubseteq [c, d] \Leftrightarrow [a, b] \supseteq [c, d]$. In particular, points of [0,1] are identified with the singleton intervals. Then, the function d defined on $I([0, 1]) \times I([0, 1])$ by

$$d([a, b], [c, d]) = (b \vee d) - (a \wedge c) - (b - a),$$

is a weightable bounded quasi-metric on $I([0, 1])$ with weighting function w given by $w([a, b]) = b - a$, for all $[a, b] \in I([0, 1])$ (compare [Matthews 94, Romaguera and Schellekens 05, Schellekens 03, etc]).

In this case, $\sqsubseteq$ is also called the information order because, following [Lawson 98, Example 1.1], successful algorithms for computing a number $r \in [0, 1]$, compute small intervals containing r in such a way that smaller intervals give more information about r, and thus the order of reverse inclusion can be viewed as

an information ordering (see also [Davey and Priestley 90, Example 1.6] and [Gierz et al. 03, Example I-1.26.1]).

As in Example 7 above, we have that the specialization order $\leq_d$ coincides with $\sqsubseteq$, and thus $\mathcal{P}_{\geq_d, cl_d}(I([0,1]))$ consists of all nonempty subsets of $I([0,1])$ having a supremum with respect to $\sqsubseteq$. By Proposition 10, the hyperspace $(\mathcal{P}_{\geq_d, cl_d}(I([0,1])), H_d^-)$ is weightable with weighting function W given by $W(A) = b - a$, for all $A \in \mathcal{P}_{\geq_d, cl_d}(I([0,1]))$, where $\sup A = [a, b]$.

Example 9. The complexity quasi-metric space [Schellekens 95] is the pair $(\mathcal{C}, d_\mathcal{C})$, where

$$\mathcal{C} = \{f : \omega \to (0, \infty] : \sum_{n=0}^{+\infty} 2^{-n} \frac{1}{f(n)} < \infty\},$$

and $d_\mathcal{C}$ is the quasi-metric on $\mathcal{C}$ defined by

$$d_\mathcal{C}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \left[\max\left(\frac{1}{g(n)} - \frac{1}{f(n)}, 0 \right) \right].$$

Furthermore, $(\mathcal{C}, d_\mathcal{C})$ is weightable with weighting function $w_\mathcal{C}$ given by $w_\mathcal{C}(f) = \sum_{n=0}^{\infty}(2^{-n}/f(n))$ for all $f \in \mathcal{C}$.

According to [Schellekens 95], given two functions $f, g \in \mathcal{C}$ the numerical value $d_\mathcal{C}(f, g)$ (the "complexity distance" from f to g) can be interpreted as the relative progress made in lowering the complexity by replacing any program P with complexity function f by any program Q with complexity function g. Therefore, if $f \neq g$, the condition $d_\mathcal{C}(f, g) = 0$ can be assumed as f is "more efficient" than g on all inputs, because in this case we have that $f(n) \leq g(n)$ for all $n \in \omega$. Hence, if we denote by $\leq_p$ the usual pointwise order on $\mathcal{C}$, it follows that $f \leq_p g \Leftrightarrow d_\mathcal{C}(f, g) = 0$, and thus the specialization order of $d_\mathcal{C}$ coincides with $\leq_p$.

Now for each $f \in \mathcal{C}$, put $f_{\leq_P} = \{h \in \mathcal{C} : h \leq_p f\}$ and $f_{\geq_P} = \{h \in \mathcal{C} : f \leq_p h\}$. Then $f_{\leq_P} \in \mathcal{P}_{\geq_{d_\mathcal{C}}, \max}(\mathcal{C})$ and $f_{\geq_P} \in \mathcal{P}_{\leq_{d_\mathcal{C}}, \min}(\mathcal{C})$. By Proposition 13, for each $f, g \in \mathcal{C}$,
$$H_d^-(f_{\leq_P}, g_{\leq_P}) = H_d^+(f_{\geq_P}, g_{\geq_P}) = d_\mathcal{C}(f, g).$$

In particular, if $f \leq_p g$, i.e., if f is "more efficient" than g on all inputs, we obtain $H_d^-(f_{\leq_P}, g_{\leq_P}) = H_d^+(f_{\geq_P}, g_{\geq_P}) = 0$, as one could expect because from $f \leq_p g$ it follows that $f_{\leq_P} \subseteq g_{\leq_P}$ and $g_{\geq_P} \subseteq f_{\geq_P}$. So, by Proposition 13 we deduce that, in this case, $H_d^-(g_{\leq_P}, f_{\leq_P}) = H_d^+(g_{\geq_P}, f_{\geq_P}) = w_\mathcal{C}(f) - w_\mathcal{C}(g)$.

5 Acknowledgements

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