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Best proximity points for cyclical contractive mappings

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Abstract

We consider p-cyclic mappings and prove an analogous result to Edelstien contractive theorem for best proximity points. Also we give similar results satisfying Boyd-Wong and Geraghty contractive conditions.

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1. Introduction

Best proximity theorems has evoked considerable interest in recent years following the results of [1], where the authors investigate the existence of an element x satisfying $d(x, Tx) = d(A, B) = \inf\{d(x, y)/x \in A, y \in B\}$ for the map $T : A \cup B \to A \cup B$ satisfying $T(A) \subset B$ and $T(B) \subset A$. In [1] the authors proved a Banach contraction type result in a uniformly convex Banach space setting, which was extended by Di Bari et. al. [4] for cyclic Meir-Keeler contractions. Karpagam et. al. [7] and Vetro [3] considered p-cyclic mappings $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ satisfying $T(A_i) \subset A_{i+1}$, for $1 \leq i \leq p$ and $A_{p+i} = A_i$ and they explored the existence of the best proximity point $x \in A_i$ satisfying $d(x, Tx) = d(A_i, A_{i+1})$. In fact, p-cyclic mappings were first considered by Kirk et. al. [8] in which they discussed fixed point theorems for mappings satisfying the contraction condition. They have also considered extensions of fixed point theorems of Edelstien [5], Boyd-Wong [2] and Geraghty [6].

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In this paper we give analogous results to the above fixed point theorems using cyclical contractive conditions which does not force $\cap_{i=1}^{p} A_i \neq \varnothing$ as in [7] and thereby we investigate the existence of best proximity point $x \in A_i$ satisfying $d(x, Tx) = d(A_i, A_{i+1})$. The contractive conditions given in this paper behave differently from the ones used in [7] and [3], in the sense that the nonexpansive implication is nontrivial as we shall see in section 3.

2. Basic definitions and results

In this section we give some basic concepts related to our results. Given two nonempty subsets A and B of a metric space X, the following notations and definitions are used in the sequel.

$$\begin{aligned} d(A,B) &= \inf\{d(x,y) : x \in A, y \in B\}; \\ d(x,A) &= \inf\{d(x,y) : y \in A\} \\ A_0 &= \{x \in A : d(x,y') = d(A,B) \text{ for some } y' \in B\}; \\ B_0 &= \{y \in B : d(x',y) = d(A,B) \text{ for some } x' \in A\}; \\ P_A(x) &= \{y \in A : d(x,y) = d(x,A)\}. \end{aligned}$$

A Banach space X is said to be

(a) uniformly convex if there exists a strictly increasing function $\delta : (0,2] \to [0,1]$ such that for all $x, y, p \in X, R > 0$ and $r \in [0, 2R]$:

$$\|x-p\| \leq R, \|y-p\| \leq R, \|x-y\| \geq r \Rightarrow \left\|\frac{x+y}{2} - p\right\| \leq \Big(1 - \delta\Big(\frac{r}{R}\Big)\Big)R;$$

(b) strictly convex if for all $x, y, p \in X$ and $R > 0$:

$$\|x-p\| \leq R, \|y-p\| \leq R, x \neq y \Rightarrow \left\|\frac{x+y}{2} - p\right\| < R.$$

Definition 2.1 ([7]). Let $A_1, A_2, \dots, A_p$ be nonempty subsets of a metric space X. Then $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is called p-cyclic mapping if $T(A_i) \subset A_{i+1}$ for $i = 1, 2, \dots, p$, where $A_{p+i} = A_i$. A point $x \in \cup_{i=1}^{p} A_i$ is said to be a best proximity point if $d(x, Tx) = d(A_i, A_{i+1})$.

Definition 2.2 ([1]). Let $A_1, A_2, \dots, A_p$ be nonempty subsets of a metric space X. A p-cyclic map T on $\cup_{i=1}^{p} A_i$ is a p-cyclic contraction mapping if for some $k \in (0, 1)$,

$$d(Tx, Ty) \leq k d(x, y) + (1-k) d(A_i, A_{i+1}) \tag{2.1}$$

for all $x \in A_i, y \in A_{i+1}, i = 1, 2, \dots, p$.

Remark 2.3. Note that Definition 2.2 implies that T satisfies $d(Tx, Ty) \leq d(x, y)$, for all $x \in A_i, y \in A_{i+1}$, moreover, the inequality (2.1) can be written as $d(Tx, Ty) - d(A_i, A_{i+1}) \leq k[d(x, y) - d(A_i, A_{i+1})]$ for all $x \in A_i, y \in A_{i+1}$.

Definition 2.4 ([7]). Let $A_1, A_2, \dots, A_p$ be nonempty subsets of a metric space X. Then a p-cyclic mapping $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is called a p-cyclic nonexpansive mapping if $d(Tx, Ty) \leq d(x, y)$ for all $x \in A_i, y \in A_{i+1}, i = 1, 2, \dots, p$.

The nonexpansive condition ensures the equality of distance between consecutive sets.

Lemma 2.5 ([7]). *Let (X,d) be a metric space and let $A_1, A_2, \dots, A_p$ be nonempty subsets of X. If $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ is a p-cyclic nonexpansive mapping then $d(A_i, A_{i+1}) = d(A_{i+1}, A_{i+2}) = \dots = d(A_1, A_2), i = 1, 2, \dots, p-1$.*

Lemma 2.6 ([1]). *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space . Let $\{x_n\}$ and $\{z_n\}$ be a sequences in A , and let $\{y_n\}$ be a sequence in B satisfying*

(i) $\|z_n - y_n\| \to d(A,B)$,
(ii) *for every $\epsilon > 0$, there exists $N_0 \in N$, such that for all $m > n > N_0, \|x_m - y_n\| \leq d(A,B) + \epsilon$.*

Then, for every $\in > 0$, there exists $N_1 \in N$, such that for all $m > n > N_1, \|x_m - z_n\| \leq \epsilon$.

Lemma 2.7 ([1]). *Let A be a nonempty closed convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let$\{x_n\}$ and$\{z_n\}$ be a sequences in A and let $\{y_n\}$ be a sequence in B satisfying*

(i) $\|x_n - y_n\| \to d(A,B)$,
(ii) $\|z_n - y_n\| \to d(A,B)$.

Then $\|x_n - z_n\|$ converges to zero.

Theorem 2.8 ([7]). *Let $A_1, A_2, \dots, A_p$ be nonempty subsets of a metric space X and let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic mapping. If for some $x \in A_i$, the sequence $\{T^{pn}x\} \in A_i$ contains a convergent subsequence $\{T^{pn_j}x\}$ converging to $\xi \in A_i$, then ξ is a best proximity point in A_i.*

Definition 2.9. Let $A_1, A_2, \dots, A_p$ be nonempty subsets of a metric space X. A p-cyclic mapping T on $\bigcup_{i=1}^p A_i$ is said to be a p-cyclic contractive map if $d(Tx, Ty) < d(x,y)$, for all $x \in A_i, y \in A_{i+1}$ satisfying $d(x,y) > d(A_i, A_{i+1})$, for all $i = 1, \dots, p$.

Definition 2.10. The nonempty subsets $A_1, A_2, \dots, A_p$ of a metric space X are said to satisfy cyclical proximal property if there exists $x_i \in A_i$ for all $1 \leq i \leq p$ such that $x_i = x_{i+p}$ for all $i = 1, \dots, p$ whenever $\|x_i - x_{i+1}\| = d(A_i, A_{i+1})$.

3. Main Results

The following lemma shows that any p-cyclic contractive mapping is also p-cyclic non-expansive.

Lemma 3.1. *Let $A_1, A_2, \dots, A_p$ be nonempty closed and convex subsets of a uniformly convex Banach space X. Let $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be such that*

(i) $T(A_i) \subset A_{i+1},\ i = 1, 2, \dots, p$, *where* $A_{p+i} = A_i$,
(ii) $\|Tx - Ty\| < \|x - y\|$, *for all* $x \in A_i,\ y \in A_{i+1}$ *and* $\|x - y\| \neq d(A_i, A_{i+1})$.

Then $\|Tx - Ty\| \leq \|x - y\|$, for all $x \in A_i,\ y \in A_{i+1}$.

Proof. It is easy to observe that $d(A_i, A_{i+1}) = d(A_{i+1}, A_{i+2})$, for all $i = 1, \ldots, p-1$. We shall prove that $\|Tx - Ty\| = d(A_i, A_{i+1})$, whenever $\|x - y\| = d(A_i, A_{i+1})$. Assume that $\|x - y\| = d(A_i, A_{i+1})$, then it is possible to choose sequences $\{x_n\} \in A_i$ and $\{y_n\} \in A_{i+1}$ such that $\|x_n - y_n\| > d(A_i, A_{i+1})$ and $\|x_n - y_n\| \to d(A_i, A_{i+1})$ with $x_n \neq x, y_n \neq y$. Since $d(A_i, A_{i+1}) \leq \|Tx_n - Ty\| < \|x_n - y\|$, $\|Tx_n - Ty\| \to d(A_i, A_{i+1})$. Similar argument asserts that $\|Ty_n - Tx\| \to d(A_i, A_{i+1})$. Since $\|P_{A_{i+1}}Ty - Ty\| \leq \|Tx_n - Ty\|$, $Tx_n \to P_{A_{i+1}}Ty$ and $Ty_n \to P_{A_{i+2}}Tx$. As $\|Tx_n - Ty_n\| \to d(A_i, A_{i+1})$, we have $\|P_{A_{i+1}}Ty - P_{A_{i+2}}Tx\| = d(A_i, A_{i+1})$. By uniqueness of the proximal point, $Ty = P_{A_{i+2}}Tx$, $Tx = P_{A_{i+1}}Ty$. Hence the lemma. □

It is necessary to ensure the non-expansive condition as it may not be explicitly given in the contractive condition for example Theorem 3.4, whereas the conditions used in Theorem 3.6 directly imply the non-expansive condition.

Theorem 3.2. *Let $A_1, A_2, \ldots, A_p$ be nonempty closed and convex subsets of a strictly convex Banach space X satisfying cyclical proximal property. Further, assume one of the subsets is compact. Let $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a p-cyclic mapping such that $\|Tx - Ty\| < \|x - y\|$ for all $x \in A_i,\ y \in A_{i+1}$ and $\|x - y\| \neq d(A_i, A_{i+1})$, then for each i, $1 \leq i \leq p$, there exists a unique best proximity point such that, for any $x_0 \in A_{i_0}$ (with respect to A_{i+1}), the sequence $\{x_{pn}\}$ converges to the best proximity point.*

Proof. Assume A_i is compact. Define $\phi : A_{i_0} \to \mathbb{R}^+$ by $\phi(y) = d(y, Ty)$ for all $y \in A_{i_0}$. From the Lemma 2.7 it is easy to observe that T is continuous on A_{i_0}. In general, T^m is continuous on any $A_i, i = 1, \ldots, p$, where m is positive integer. So ϕ is continuous and hence there exists $y_0 \in A_{i_0}$, such that $d(y_0, Ty_0) = \phi(y_0) = \inf_{y \in A_{i_0}} d(y, Ty)$. Suppose $d(y_0, Ty_0) > d(A_i, A_{i+1})$, then $d(T^p y_0, T^{p+1} y_0) < d(y_0, Ty_0)$ which is a contradiction. Hence $d(y_0, Ty_0) = d(A_i, A_{i+1})$. Assume that $x_0 \in A_{i_0}$, and $\{x_{pn}\} \in A_{i_0}$, for all $n = 1, 2, \ldots$.

Suppose for some n, $x_{pn} = y_0$, then $x_{pn+1} = Tx_{pn} = Ty_0$, Assume $x_{pn} \neq y_0$ for any n. Since $\|T^n y_0 - T^{n+1} y_0\| = d(A_i, A_{i+1})$ and $T^p y_0 = y_0$, by cyclical proximal property.

$$d(x_{pn}, P_{A_{i+1}}(y_0)) = d(T^p x_{pn-p}, T^{p+1} y_0) \leq d(x_{pn-p}, Ty_0) = d(x_{p(n-1)}, P_{A_{i+1}}(y_0)).$$

Therefore $d(x_{pn}, P_{A_{i+1}}(y_0))$ is a decreasing sequences converging to some $r \geq 0$. Since A_i is compact, it follows that the sequence $\{x_{pn}\}$ has a subsequence $\{x_{pn_k}\}$ converging to some $z \in A_i$. If $d(z, P_{A_{i+1}}(y_0)) \leq d(A_i, A_{i+1})$, then there is nothing to prove. Assume that $d(z, P_{A_{i+1}}(y_0)) > d(A_i, A_{i+1})$, then

$$\begin{aligned}
d(z, P_{A_{i+1}}(y_0)) &= \lim_{n\to\infty} d(x_{pn}, P_{A_{i+1}}(y_0)) = \lim_{n\to\infty} d(T^p x_{pn}, P_{A_{i+1}}(y_0)) \\
&= \lim_{k\to\infty} d(T^p x_{pn_k}, P_{A_{i+1}}(y_0)) = d(T^p z, T^{p+1} y_0) \\
&\qquad\qquad (\text{Since } T^p \text{ is continuous on } A_{i_0}) \\
&< d(z, Ty_0) = d(z, P_{A_{i+1}}(y_0)),
\end{aligned}$$

which is a contradiction. Therefore $z = y_0$. Since any convergent subsequence of $\{x_{pn}\}$ converges to y_0, $\{x_{pn}\}$ itself converges to y_0 which is the best proximity point.

For uniqueness, suppose there exists $z \in A_i$ with $z \neq y_0$ such that $\|z - Tz\| = d(A_i, A_{i+1})$, by cyclical proximal property $T^p y_0 = y_0, T^p z = z$. If $\|y_0 - Tz\| - d(A_i, A_{i+1}) > 0$ then

$$\begin{aligned}\|Ty_0 - T^2 z\| - d(A_i, A_{i+1}) &< \|y_0 - Tz\| - d(A_i, A_{i+1}) \\ &= \|T^p y_0 - T^{p+1} z\| - d(A_i, A_{i+1}) \\ &\leq \|Ty_0 - T^2 z\| - d(A_i, A_{i+1}).\end{aligned}$$

which is a contradiction. □

Example 3.3. Let $A_1 = \{(0,0,x) \in \mathbb{R}^3 / x \geq 1\}$, $A_2 = \{(0,1,x) \in \mathbb{R}^3 / x \geq 1\}$, $A_3 = \{(1,1,x) \in \mathbb{R}^3 / x \geq 1\}$, and $A_4 = \{(1,0,x) \in \mathbb{R}^3 / x \geq 1\}$ be subsets in the space $\mathbb{R}^3$ with euclidean norm. Clearly A_1, A_2, A_3 and A_4 satisfy cyclical proximal property. Define T on $\cup_{i=1}^4 A_i$ as

$$\begin{aligned}T(0,0,x) &= \Big(0,1,x+\frac{1}{x}\Big), \text{ for } (0,0,x) \in A_1, \\ T(0,1,x) &= \Big(1,1,x+\frac{1}{x}\Big), \text{ for } (0,1,x) \in A_2 \\ T(1,1,x) &= \Big(1,0,x+\frac{1}{x}\Big), \text{ for } (1,1,x) \in A_3, \\ T(1,0,x) &= \Big(0,0,x+\frac{1}{x}\Big), \text{ for } (1,0,x) \in A_4.\end{aligned}$$

For any $(0,0,x) \in A_1$, and $(0,1,y) \in A_2$. If $\|(0,0,x) - (0,1,y)\| > d(A_1, A_2) = 1$, then $x \neq y$. Also

$$\begin{aligned}\|T(0,0,x) - T(0,1,y)\| &= \|\Big(0,1,x+\frac{1}{x}\Big) - \Big(1,1,y+\frac{1}{y}\Big)\| < (1 + (x-y)^2)^{\frac{1}{2}} \\ &= \|(0,0,x) - (0,1,y)\|\end{aligned}$$

Hence T is a cyclic contractive map. Also for any $(0,0,x) \in A_1$,

$$\begin{aligned}\|(0,0,x) - T(0,0,x)\| &= \|(0,0,x) - \Big(0,1,x+\frac{1}{x}\Big)\| \\ &= \Big(1 + \Big(\frac{1}{x}\Big)^2\Big)^{\frac{1}{2}} > 1 = d(A_1, A_2).\end{aligned}$$

Here T does not admit any best proximity point as none of the sets are compact.

Next we consider two of the famous extensions of Banach contraction theorem due to Boyd-Wong and Gregathy.

Theorem 3.4. *Let $A_1, A_2, \ldots, A_p$ be nonempty closed subsets of a complete metric space (X, d). Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic mapping . Suppose $d(Tx, Ty) \leq \psi(d(x,y) - d(A_i, A_{i+1})) + d(A_i, A_{i+1})$ for all $x \in A_i,\ y \in A_{i+1}$, where $\psi : [0,\infty) \to [0,\infty)$ is upper semi-continuous from the right and satisfies $0 \leq \psi(t) < t$ for all $t > 0$. Then*

(i) $d(T^{pn}x, T^{pn+1}y) \to d(A_i, A_{i+1})$ *as* $n \to \infty$
(ii) $d(T^{p(n+1)}x, T^{pn+1}y) \to d(A_i, A_{i+1})$ *as* $n \to \infty$

Note: The contractive condition here does not directly guarantee the non-expansive condition and hence the importance of Lemma 3.1.

Proof. (i) Choose $x_0 \in A_i$, set $s_n = d(T^{pn}x_0, T^{pn+1}x_0) - d(A_i, A_{i+1})$. Given $\psi(t) < t$ for all $t > 0$, from the Lemma 3.1, it follows that

$$d(T^{p(n+1)}x_0, T^{p(n+1)+1}x_0) \leq d(T^{pn}x_0, T^{pn+1}x_0).$$

Therefore $\{s_n\}$ is a decreasing sequence and hence converges . Let r be the limit of s_n. Then $r \geq 0$.

Suppose $r > 0$. Then

$$\begin{aligned} d(T^{p(n+1)}x_0, T^{p(n+1)+1}x_0) - d(A_i, A_{i+1}) &\leq d(T^{p(n+1)-1}x_0, T^{p(n+1)}x_0) \\ &\leq d(T^{p(n+1)-2}x_0, T^{p(n+1)-1}x_0) \\ &\leq \dots \\ &\leq d(T^{pn+1}x_0, T^{pn+2}x_0) \\ &\leq \psi(d(T^{pn}x_0, T^{pn+1}x_0) \\ &\qquad -d(A_i, A_{i+1})). \end{aligned}$$

Taking lim sup on both sides,

$$\begin{aligned} &\limsup d(T^{p(n+1)}x_0, T^{p(n+1)+1}x_0) - d(A_i, A_{i+1}) \\ &\qquad \leq \limsup \psi(d(T^{pn}x_0, T^{pn+1}x_0) - d(A_i, A_{i+1})). \end{aligned}$$

We obtain $r \leq \psi(r)$, which is a contradiction. Hence $d(T^{pn}x_0, T^{pn+1}x_0) \to d(A_i, A_{i+1})$ as $n \to \infty$. Similar argument shows that $d(T^{p(n+1)}x, T^{pn+1}y) \to d(A_i, A_{i+1})$ as $n \to \infty$. ☐

Theorem 3.5. *Let $A_1, A_2, \dots, A_p$ be nonempty closed and convex subsets of a uniformly convex Banach space X. Let $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a p-cyclic mapping such that $d(Tx, Ty) \leq \psi(d(x, y) - d(A_i, A_{i+1})) + d(A_i, A_{i+1})$ for all $x \in A_i$, $y \in A_{i+1}$, where $\psi : [0, \infty) \to [0, \infty)$ is upper semi-continuous from the right and satisfies $0 \leq \psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. Then for each $i, 1 \leq i \leq p$, there exists a unique best proximity point such that, for any $x_0 \in A_i$, $\{T^{pn}x_0\}$ converges to the best proximity point.*

Proof. Choose $x_0 \in A_i$. Suppose $d(A_i, A_{i+1}) = 0$, then T has a unique fixed point $x \in \cap_{i=1}^p A_i$, see in [8]. Assume that $d(A_i, A_{i+1}) \neq 0$, then by Theorem 3.4 it follows that $\|T^{pn}x_0 - T^{pn+1}x_0\| \to d(A_i, A_{i+1})$ and $\|T^{p(n+1)}x_0 - T^{pn+1}x_0\| \to d(A_i, A_{i+1})$. By Lemma 2.7, it follows that $\|T^{pn}x_0 - T^{p(n+1)}x_0\| \to 0$. Similarly $\|T^{pn+1}x_0 - T^{p(n+1)+1}x_0\| \to 0$. To complete the proof, we have to show that for every $\epsilon > 0$, there exists N_0, such that for all $m > n \geq N_0$, $\|T^{pm}x_0 - T^{pn+1}x_0\| \leq d(A_i, A_{i+1}) + \epsilon$. Suppose not, then there exists $\epsilon > 0$, such that for all $k \in N$ there exists $m_k > n_k \geq k$ for which $\|T^{pm_k}x_0 - T^{pn_k+1}x_0\| \geq d(A_i, A_{i+1}) + \epsilon$. This m_k can be chosen such that it is the least integer greater than n_k to satisfy the above inequality and $\|T^{p(m_k-1)}x_0 -$

$T^{pn_k+1}x_0\| < d(A_i,A_{i+1})+\epsilon$. Consequently $\|T^{pn}x_0 - T^{pn+1}x_0\| \to d(A_i,A_{i+1})$ and $\|T^{p(n+1)}x_0 - T^{pn+1}x_0\| \to d(A_i,A_{i+1})$. By Lemma2.7, it follows that $\|T^{pn}x_0 - T^{p(n+1)}x_0\| \to 0$. Similarly $\|T^{pn+1}x_0 - T^{p(n+1)+1}x_0\| \to 0$.

$$\begin{aligned} d(A_i,A_{i+1})+\epsilon &\leq \|T^{pm_k}x_0 - T^{pn_k+1}x_0\| \\ &\leq \|T^{pm_k}x_0 - T^{p(m_k-1)}x_0\| + \|T^{p(m_k-1)}x_0 - T^{pn_k+1}x_0\| \\ &\leq \|T^{pm_k}x_0 - T^{p(m_k-1)}x_0\| + d(A_i,A_{i+1})+\epsilon. \end{aligned}$$

This implies that $lim_{k\to\infty}\|T^{pm_k}x_0 - T^{pn_k+1}x_0\| = d(A_i,A_{i+1})+\epsilon$. Since

$$\|T^{p(m_k+1)}x_0 - T^{p(n_k+1)+1}x_0\| \leq \|T^{pm_k+1}x_0 - T^{pn_k+2}x_0\|,$$

$$\begin{aligned} \|T^{pm_k}x_0 - T^{pn_k+1}x_0\| &\leq \|T^{pm_k}x_0 - T^{p(m_k+1)}x_0\| + \|T^{p(m_k+1)}x_0 \\ &\quad -T^{p(n_k+1)+1}x_0\| + \|T^{p(n_k+1)+1}x_0 - T^{pn_k+1}x_0\| \\ &\leq \|T^{pm_k}x_0 - T^{p(m_k+1)}x_0\| \\ &\quad +\|T^{pm_k+1}x_0 - T^{pn_k+2}x_0\| \\ &\quad +\|T^{p(n_k+1)+1}x_0 - T^{pn_k+1}x_0\| \\ &\leq \|T^{pm_k}x_0 - T^{p(m_k+1)}x_0\| \\ &\quad +\psi(\|T^{pm_k}x_0 - T^{pn_k+1}x_0\| \\ &\quad -d(A_i,A_{i+1})) + d(A_i,A_{i+1}) \\ &\quad +\|T^{p(n_k+1)+1}x_0 - T^{pn_k+1}x_0\|, \end{aligned}$$

which yields that

$$\begin{aligned} &\|T^{pm_k}x_0 - T^{pn_k+1}x_0\| - d(A_i,A_{i+1}) \\ &\qquad \leq \|T^{pm_k}x_0 - T^{p(m_k+1)}x_0\| + \psi(\|T^{pm_k}x_0 - T^{pn_k+1}x_0\| \\ &\qquad\quad -d(A_i,A_{i+1})) + \|T^{p(n_k+1)+1}x_0 - T^{pn_k+1}x_0\|. \end{aligned}$$

Therefore $\limsup_k \|T^{pm_k}x_0 - T^{pn_k+1}x_0\| - d(A_i,A_{i+1}) \leq \limsup_k \psi(\|T^{pm_k}x_0 - T^{pn_k+1}x_0\| - d(A_i,A_{i+1}))$, as $\|T^{pm_k}x_0 - T^{p(m_k+1)}x_0\| \to 0$ and $\|T^{p(n_k+1)+1}x_0 - T^{pn_k+1}x_0\| \to 0$. Hence $\epsilon \leq \psi(\epsilon)$, a contradiction. By Lemma 2.6, $\{T^{pn}x_0\}$ is a cauchy sequence and converges to $x \in A_i$. From Theorem 2.8, it follows that $\|x - Tx\| = d(A_i,A_{i+1})$.
To see that $T^p x = x$, we note that

$$\begin{aligned} \|x - T^{p+1}x\| &= \lim_{n\to\infty} \|T^{pn}x_0 - T^{p+1}x\| \\ &\leq \lim_{n\to\infty} \|T^{p(n-1)}x_0 - Tx\| \\ &= \|x - Tx\| = d(A_i,A_{i+1}). \end{aligned}$$

Since A_{i+1} is convex set and X is uniformly convex Banach space, $Tx = T^{p+1}x$. Consequently

$$\|T^p x - Tx\| = \|T^p x - T^{p+1}x\| \leq \|x - Tx\| = d(A_i,A_{i+1}).$$

Hence $T^p x = x$. Uniqueness follows as in Theorem 3.2. □

The following result on Geraghty contractive condition can be proved in a similar fashion.

Theorem 3.6. *Let $A_1, A_2, \ldots, A_p$ be nonempty closed and convex subsets of a uniformly convex Banach space X and let $\mathbb{S} = \{\alpha : \mathbb{R}^+ \to [0,1) : \alpha(t_n) \to 1 \Rightarrow t_n \to 0\}$. Let $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a p-cyclic mapping such that $\|Tx - Ty\| \leq \alpha(\|x-y\|)(\|x-y\|) + (1-\alpha(\|x-y\|))d(A_i, A_{i+1})$ for all $x \in A_i$, $y \in A_{i+1}$, where $\alpha \in \mathbb{S}$. Then for each $i, 1 \leq i \leq p$, there exists a unique best proximity point such that, for any $x_0 \in A_i$, $\{T^{pn}x_0\}$ converges to the best proximity point.*

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References

[1] A. Anthony Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl. **323**, no.2 (2006), 1001–1006.
[2] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458–464.
[3] C. Vetro, *Best proximity points: Convergence and existence theorem for p-cyclic mappings*, Nonlinear Anal. **73** (2010), 2283–2291.
[4] C. Di Bari, T. Suzuki and C. Vetro, *Best proximity points for cyclic Meir-Keeler contraction*, Nonlinear Anal. **69** (2008) 3790–3794.
[5] M. Edelstein, *On fixed and periodic points under contractive mapping*, J. London Math. Soc. **37** (1962), 74–79.
[6] M. Geraghty, *On contractive mapping*, Proc. Amer. Math. Soc. **40** (1973), 604–608.
[7] S. Karpagam and S. Agrawal, *Best proximity point theorems for p-cyclic Meir-Keeler contractions*, Fixed Point Theory Appl. **2009** (2009), Article ID 197308.
[8] W. A. Kirk, P. S. Srinivasan and P. Veeramani, *Fixed points for mapping satisfying cyclical contractive condition*, Fixed Point Theory **4** (2003), 79–89.

