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Finite products of limits of direct systems induced by maps

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Abstract

Let Z, H be spaces. In previous work, we introduced the direct (inclusion) system induced by the set of maps between the spaces Z and H. Its direct limit is a subset of $Z \times H$, but its topology is different from the relative topology. We found that many of the spaces constructed from this method are pseudo-compact and Tychonoff. We are going to show herein that these spaces are typically not sequentially compact and we will explore conditions under which a finite product of them will be pseudo-compact.

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1. Introduction

In [2] we introduced the concept of the direct system $\mathbf{X}$ induced by the set of maps $\mathcal{F} = C(Z, H)$ between spaces Z and H, and in [3] we extended these ideas to include systems induced by a perhaps proper nonempty subset $\mathcal{F}$ of $C(Z, H)$. This system $\mathbf{X}$ consists of all the unions of the graphs of finite nonempty subsets of $\mathcal{F}$ and the inclusion maps induced when one finite subset is contained in another. Such an entity is an "inclusion" direct system. Its direct limit $X_\delta = \operatorname{dirlim} \mathbf{X}$, as a set, is the union of the graphs of the elements

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of $\mathcal{F}$ and hence $X_\delta \subset Z \times H$ as a set. But the topology of X_δ might be larger than that induced by $Z \times H$. In [2] we were concerned mainly (Theorems 5.1, 6.1(2)) with determining when X_δ is pseudo-compact. By a space being pseudo-compact, we mean that every map of it to $\mathbb{R}$ has bounded image (see [1] where additional, unnecessary requirements are placed on such a space). In [3] we considered pseudo-compactness and other properties such as regularity, complete regularity, and normality. There we proved the existence of a large, but abstract class of pseudo-compact spaces. We are going to show herein that these spaces are typically not sequentially compact but that finite products of them are pseudo-compact. Our main result is Theorem 2.13 which shows that finite products of these spaces are pseudo-compact. Whether infinite products are pseudo-compact is an open question.

2. Products of Pseudo-compact Spaces

In Section 4 of [4], there is a discussion about products of pseudo-compact spaces. For example it is pointed out that even the product of two pseudo-compact spaces need not be pseudo-compact. A useful fact from this reference is its Theorem 4.1.

Theorem 2.1. *Let $\{X_a \,|\, a \in A\}$ be a collection of spaces. If $X = \prod\{X_a \,|\, a \in A\}$ is pseudo-compact, then for every subset B of A, $\prod\{X_a \,|\, a \in B\}$ is pseudo-compact. If A is infinite and X is not pseudo-compact, then there is a countably infinite subset J of A such that $\prod\{X_a \,|\, a \in J\}$ is not pseudo-compact.*

Theorem 4.4 of [4] asserts the following.

Proposition 2.2. *Every product of pseudo-compact spaces of which all but one are sequentially compact is pseudo-compact.*

For example, let Ω denote the first uncountable ordinal. The first uncountable ordinal space $[0, \Omega)$ is not compact, but it is pseudo-compact and sequentially compact, so any product of copies of this space is pseudo-compact. Yet, sequential compactness is not a necessary condition for such an outcome. Let ω be the first infinite ordinal. Then we have the next result.

Example 2.3. Let X denote the space $[0, \Omega] \times [0, \omega] \setminus \{(\Omega, \omega)\}$. Then X is not sequentially compact, but X^ω is pseudo-compact and hence X is pseudo-compact.

Proof. Clearly X is not sequentially compact. It is not difficult to see that for each nonempty open subset U of X^ω, there exists $w = (w_1, w_2, \dots) \in U$ such that for each i the first coordinate of w_i does not equal Ω. If X^ω is not pseudo-compact, then we may choose a map $f : X^\omega \to \mathbb{R}$ and a set $\{x_i \,|\, i \in \mathbb{N}\}$ in X^ω such that $f(x_i) = i$ for each i. Let $V_i = (i - \frac{1}{3}, i + \frac{1}{3})$ and $W_i = f^{-1}(V_i)$.

For each i, choose $y_i = (y_{i,1}, y_{i,2}, \dots) \in W_i$ so that for each j, the first coordinate $z_{i,j}$ of $y_{i,j}$ does not equal Ω. The countable collection of these $z_{i,j}$ has an upper bound $\alpha < \Omega$. It follows that $\{y_i \,|\, i \in \mathbb{N}\} \subset ([0, \alpha] \times [0, \omega])^\omega$, a compact subset of X^ω. This shows that the unbounded set $\{f(y_i) \,|\, i \in \mathbb{N}\}$ is contained in a compact subset of $\mathbb{R}$, a contradiction. □

In [3] we proved the existence of a large class of pseudo-compact spaces, which are the direct limits of certain direct systems. Let us review the main ideas surrounding the construction of such direct systems. For the remainder of this section, let Z and H be nonempty Hausdorff spaces, and $C(Z,H)$ denote the set of maps of Z to H. Fix a nonempty subset $\mathcal{F}$ of $C(Z,H)$ and let A be the collection of nonempty finite subsets of $\mathcal{F}$ ordered by inclusion, which we denote $\preceq$. Thus, $(A,\preceq)$ is a directed set. Recall that whenever $f\in C(Z,H)$, then $G_f\subset Z\times H$, the graph of f, is closed in $Z\times H$.

Definition 2.4. For each $a\in A$, let $X_a=\bigcup\{G_f\,|\,f\in a\}\subset Z\times H$. Whenever $a\preceq b$, let $p_a^b: X_a\to X_b$ denote the inclusion map.

Lemma 2.5. *The system* $\mathbf{X}=(X_a,p_a^b,(A,\preceq))$ *is a direct system of closed inclusion maps and Hausdorff spaces* X_a. *Let* $X_\delta=\operatorname{dirlim}\mathbf{X}$. *Then, as a set,* $X_\delta=\bigcup\{X_a\,|\,a\in A\}\subset Z\times H$, *but* X_δ *has the weak topology determined by* $\{X_a\,|\,a\in A\}$. *The set inclusion* $\iota_{X_\delta}: X_\delta\hookrightarrow Z\times H$ *is a map in conjunction with the respective topologies of* X_δ *and the product topology of* $Z\times H$. ☐

Definition 2.6. Let $\mathcal{F}$ be a nonempty subset of $C(Z,H)$, A be the collection of nonempty finite subsets of $\mathcal{F}$, and $\mathbf{X}=(X_a,p_a^b,(A,\preceq))$ be as in Lemma 2.5. Then we call $\mathbf{X}$ the inclusion direct system **induced** by $\mathcal{F}$.

Such systems as in Definition 2.6 frequently have direct limits X_δ that are pseudo-compact. We state Theorem 4.1 of [3].

Theorem 2.7. *Let* Z, H *be nonempty spaces,* $\mathbf{X}=(X_a,p_a^b,(A,\preceq))$ *be the inclusion direct system induced by a nonempty subset* $\mathcal{F}$ *of* $C(Z,H)$, *and* $X_\delta=\operatorname{dirlim}\mathbf{X}$. *Suppose that:*

(1) *both* Z *and* H *are sequentially compact,*
(2) H *is a sequential extensor modulo* $\mathcal{F}$ *for* Z,
(3) Z *and* H *are Hausdorff spaces,*
(4) Z *is a perfect space,*
(5) Z *is a first countable space, and*
(6) *either* $\mathcal{F}$ *is dense for* $C(Z,H)$, *or* H *is first countable.*

Then X_δ *is pseudo-compact.*

There are two notions in this theorem which should be defined. Before those, we also need the next idea.

Definition 2.8. Let $M=\{\frac{1}{i}\,|\,i\in\mathbb{N}\}\cup\{0\}\subset\mathbb{R}$. We shall refer to M as the **convergent sequence**. Any space homeomorphic to the convergent sequence will be called a convergent sequence.

Definition 2.9. Let $\mathcal{F}\subset C(Z,H)$. We will say that H is a **sequential extensor modulo** $\mathcal{F}$ for Z if for each convergent sequence $M=\{z_i\,|\,i\in\mathbb{N}\}\cup\{z\}$ in Z, sequence (f_i) in $\mathcal{F}$ such that the sequence $(f_i(z_i))$ converges to an element $w\in H$, and sequence (U_i) of neighborhoods U_i of $(z_i,f_i(z_i))$ in $Z\times H$, there exist $f\in\mathcal{F}$ and a subsequence (z_{i_j}) of (z_i) such that $(z_{i_j},f(z_{i_j}))\in U_{i_j}$ for all j.

Definition 2.10. Let $\mathbf{X} = (X_a, p_a^b, (A, \preceq))$ be the inclusion direct system induced by a nonempty subset $\mathcal{F}$ of $C(Z, H)$ and $X_\delta = \operatorname{dirlim} \mathbf{X}$. Suppose that for each sequence $(z_i)_{i\in\mathbb{N}}$ in Z converging to an element $z \in Z$, $h \in C(Z, H)$ with $(z, h(z)) \in X_\delta$, and neighborhood U of $(z, h(z))$ in X_δ, there exist $g \in C(Z, H)$ and $i \in \mathbb{N}$ such that,

(1) $g \in \mathcal{F}$,
(2) $(z_i, g(z_i)) \in U$, and
(3) $g(z_i) = h(z)$.

Then we shall say that $\mathcal{F}$ is **dense** for $C(Z, H)$.

To see how easily such conditions as in Theorem 2.7 can be made to occur, let $Z = [0, 1] = H$. Then Example 7.3 of [3] along with an application of Lemma 7.2 of [3] yield the following.

Example 2.11. Let $\mathcal{F} = C(Z, H)$, $\mathbf{X} = (X_a, p_a^b, (A, \preceq))$ be the inclusion direct system induced by $\mathcal{F}$, and $X_\delta = \operatorname{dirlim} \mathbf{X}$. Then X_δ is a completely regular, pseudo-compact Hausdorff space, but X_δ is neither compact, normal, nor sequentially compact.

We are going to show in Theorem 2.13 that finite products of spaces X_δ such as those in Theorem 2.7 are pseudo-compact. We need a lemma in support of the proof of that theorem.

Lemma 2.12. *Suppose that $\mathbf{X} = (X_a, p_a^b, (A, \preceq))$ satisfies the hypothesis of* Theorem 2.7 *and $X_\delta = \operatorname{dirlim} \mathbf{X}$. Then for each sequence (U_i) of nonempty open subsets of X_δ, there exist:*

(1) $f \in \mathcal{F}$,
(2) $z \in Z$,
(3) *a subsequence* (U_{i_j}) *of* (U_i), *and*
(4) *for all* $j \in \mathbb{N}$, *a point* $z_j \in Z$

such that for each j, $(z_j, f(z_j)) \in U_{i_j}$, (z_j) converges to $z \in Z$, and $(f(z_j))$ converges to $f(z)$ in H. Hence, $(z_j, f(z_j))$ converges to $(z, f(z))$ in $G_f \subset X_\delta$.

Proof. Let (U_i) be a sequence of nonempty open subsets of X_δ. For each $i \in \mathbb{N}$, choose $a_i \in A$, $f_i \in a_i$, and $z_i \in Z$ such that $x_i = (z_i, f_i(z_i)) \in U_i$. Using (1) of 2.7 and passing to a subsequence, if necessary, we may as well assume that the sequence $(f_i(z_i))_{i\in\mathbb{N}}$ in H converges to some element $w \in H$. Applying (1) of 2.7 again, we may also assume that the sequence $(z_i)_{i\in\mathbb{N}}$ in Z converges to an element $z \in Z$.

There are two cases to consider, either $\{z_i \mid i \in \mathbb{N}\}$ is a finite set or it is an infinite set. Let us show that if it is finite, then we can replace it with an infinite set satisfying the above conditions, independently of which of the two parts of 2.7(6) is in play. Using (3) of 2.7 and passing to a subsequence if necessary, we may assume that for each i, $z_i = z$. Making use of (3)–(5) of 2.7, choose a sequence $(z_i^*)_{i\in\mathbb{N}}$ in Z converging to z so that for $i \neq j$, $z_i^* \neq z_j^*$, and for each i, $z_i^* \neq z$.

Our plan is to show that (6) of 2.7 allows us to find a sequence $i_1 < i_2 < \cdots$ in $\mathbb{N}$ along with a sequence (g_j) so that for each j, $b_j = \{g_j\} \in A$, $x_j^* = (z_{i_j}^*, g_j(z_{i_j}^*)) \in U_j$, and the sequence $(g_j(z_{i_j}^*))$ converges in H to w. If that can be done, we will define $Q^* = \{x_j^* \,|\, j \in \mathbb{N}\}$. Then for each $j \in \mathbb{N}$: $b_j \in A$, $g_j \in b_j$, $z_{i_j}^* \in Z$, and $x_j^* = (z_{i_j}^*, g_j(z_{i_j}^*)) \in U_j$. The sequence $(z_{i_j}^*)_{j\in\mathbb{N}}$ converges to z. The reader will now observe that we may replace (z_j, x_j, a_j, f_j) by $(z_{i_j}^*, x_j^*, b_j, g_j)$, $j = 1, 2, \ldots$, to satisfy the conditions of the first paragraph of this proof. But this time $\{z_{i_j}^* \,|\, j \in \mathbb{N}\}$ is an infinite set.

We now show that whichever part of 2.7(6) is in operation, we can produce such a collection of elements $(z_{i_j}^*, x_j^*, b_j, g_j)$, $j = 1, 2, \ldots$, as just described. If, (Case 1), $\mathcal{F}$ is dense for $C(Z, H)$, then define $V_i = U_i$ for all i. If, (Case 2), $\mathcal{F}$ is not dense for $C(Z, H)$, and hence H is first countable, we define the sets V_i differently. Let $\{Q_i \,|\, i \in \mathbb{N}\}$ be a local base for the neighborhood system of w in H with $Q_{i+1} \subset Q_i$ for all i. Since $(f_i(z))$ converges to w, we may assume that $f_i(z) \in Q_i$ for each i. Define $V_i = (Z \times Q_i) \cap U_i$, noting that this equals $(Z \times Q_i) \cap X_\delta \cap U_i$. Since $\iota_{X_\delta} : X_\delta \to Z \times H$ is a map, then $\iota_{X_\delta}^{-1}(Z \times Q_i) \cap U_i = (Z \times Q_i) \cap X_\delta \cap U_i$ is open in X_δ for each i. Thus V_i is an open neighborhood of $x_i = (z_i, f_i(z_i)) = (z, f_i(z))$ in X_δ in either case of 2.7(6).

Under the assumption of (Case 1), apply Definition 2.10 with $(z_i)_{i\in\mathbb{N}}$ replaced by $(z_i^*)_{i\in\mathbb{N}}$, $h = f_1$, $(z, h(z)) = (z, f_1(z))$, and $U = V_1$. Using (1)–(3) of that definition, we obtain $i_1 \in \mathbb{N}$ and $g_1 \in \mathcal{F}$ such that $x_1^* = (z_{i_1}^*, g_1(z_{i_1}^*)) \in V_1 = U_1$ and $g_1(z_{i_1}^*) = f_1(z)$. If (Case 2) prevails, put $g_1 = f_1 \in \mathcal{F}$; we shall show, as in (Case 1), that there exists $i_1 \in \mathbb{N}$ so that $x_1^* = (z_{i_1}^*, g_1(z_{i_1}^*)) \in V_1$. Since (z_i^*) converges to $z \in Z$, then $(g_1(z_i^*))$ converges to $g_1(z)$ in H. Note that G_{g_1} is a closed subspace of both $Z \times H$ and X_δ. So the sequence $(z_i^*, g_1(z_i^*))$ converges in G_{g_1} to $(z, g_1(z))$. Since $V_1 \cap G_{g_1}$ is a neighborhood of $(z, g_1(z)) = (z, f_1(z))$ in G_{g_1}, then for some i_1, $(z_{i_1}^*, g_1(z_{i_1}^*)) \in V_1$. So in either case, there exist g_1 and i_1 so that $\{g_1\} \in A$ and $(z_{i_1}^*, g_1(z_{i_1}^*)) \in V_1$, but in the first case we also have that $(z_{i_1}^*, g_1(z_{i_1}^*)) = (z_{i_1}^*, f_1(z))$.

Replace the sequence $(z_i^*)_{i\in\mathbb{N}}$ by $(z_i^*)_{i>i_1}$. Using the latter, apply the same technique we just employed, this time with (f_2, x_2) replacing (f_1, x_1), to find $i_2 > i_1$ and $g_2 \in \mathcal{F}$ such that $x_2^* = (z_{i_2}^*, g_2(z_{i_2}^*)) \in V_2 \subset U_2$ with the additional property that $g_2(z_{i_2}^*) = f_2(z)$ in (Case 1). Such a process is to be done recursively so that we find a sequence $i_1 < i_2 < \cdots$ in $\mathbb{N}$, a sequence $(g_i)_{i\in\mathbb{N}}$ in $\mathcal{F}$, $b_i = \{g_i\} \in A$, and a subsequence $(z_{i_j}^*)_{j\in\mathbb{N}}$ of $(z_i^*)_{i\in\mathbb{N}}$ such that for each $j \in \mathbb{N}$, $x_j^* = (z_{i_j}^*, g_j(z_{i_j}^*)) \in V_j \subset U_j$, and, in (Case 1), $g_j(z_{i_j}^*) = f_j(z)$. In the latter case, the sequence $(g_j(z_{i_j}^*))_{j\in\mathbb{N}}$ equals $(f_j(z))_{j\in\mathbb{N}} = (f_j(z_j))_{j\in\mathbb{N}}$, so it converges to $w \in H$. In (Case 2), $x_j^* = (z_{i_j}^*, g_j(z_{i_j}^*)) \in V_j \subset Z \times Q_j$. This shows that $g_j(z_{i_j}^*) \in Q_j$, and hence the sequence $(g_j(z_{i_j}^*))_{j\in\mathbb{N}}$ converges to $w \in H$. We have proved that under the assumption 2.7(6), a sequence $(z_{i_j}^*, x_j^*, b_j, g_j)$, $j = 1, 2, \ldots$, can be produced with $\{z_{i_j}^* \,|\, j \in \mathbb{N}\}$ an infinite set as requested above.

We assume therefore that $\{z_i \,|\, i \in \mathbb{N}\}$ is an infinite set. By passing to a subsequence of $(z_i)_{i\in\mathbb{N}}$ if necessary we may stipulate that for $i \neq j$, $z_i \neq z_j$ and that for each i, $z_i \neq z$. By 2.7(3), Z is Hausdorff, so $B = \{z_i \,|\, i \in \mathbb{N}\} \cup \{z\}$ is a convergent sequence in the sense of Definition 2.8. Define $\lambda : B \to H$ by $\lambda(z_i) = f_i(z_i)$, and $\lambda(z) = w$. Then λ is a map. Applying 2.7(2) get a map $f : Z \to H$ such that $c = \{f\} \in A$ along with a subsequence (z_{i_j}) of (z_i) such that $(z_{i_j}, f(z_{i_j})) \in U_{i_j}$ for all j. Since A is a collection of subsets of $\mathcal{F}$, then $f \in \mathcal{F}$. □

Now we can prove our main result about the preservation pseudo-compactness in any finite product of the above types of spaces.

Theorem 2.13. *Let $n \in \mathbb{N}$ and $X_{\delta_1}, \ldots, X_{\delta_n}$ be spaces like those obtained in* Theorem 2.7. *Then $X = X_{\delta_1} \times \cdots \times X_{\delta_n}$ is pseudo-compact.*

Proof. For simplicity we will prove this in case $n = 2$; the reader will easily fill in the details needed for the general case. Let $(\mathcal{F}_k, Z_k, H_k)$ correspond to $(\mathcal{F}, Z, H)$ in Lemma 2.12 for $k \in \{1, 2\}$.

Suppose that X is not pseudo-compact. Then there exist a subset $\{x_i \,|\, i \in \mathbb{N}\}$ of X and a map $\omega : X \to \mathbb{R}$ such that $\omega(x_i) = i$ for each i. Put $W_i = (i - \frac{1}{3}, i + \frac{1}{3})$ and let $Q_i = \omega^{-1}(W_i)$. For each i, find open sets U_i^k of X_{δ_k}, $k \in \{1, 2\}$, such that $x_i \in U_i^1 \times U_i^2 \subset Q_i$.

First apply Lemma 2.12 to the nonempty open subsets U_i^1 of X_{δ_1}. There exist $f_1 \in \mathcal{F}_1$, $z_1 \in Z_1$, a subsequence $(U_{i_j}^1)$ of (U_i^1), and for all $j \in \mathbb{N}$, a point $z_j^1 \in Z_1$ such that for each j, $(z_j^1, f_1(z_j^1)) \in U_{i_j}^1$, (z_j^1) converges to $z_1 \in Z_1$, and $(f_1(z_j^1))$ converges to $f_1(z_1)$ in H_1. Hence, $(z_j^1, f_1(z_j^1))$ converges to $(z_1, f_1(z_1))$ in $G_{f_1} \subset X_{\delta_1}$.

By passing to a subsequence we may as well assume that $U_{i_j}^1 = U_j^1$ for each j. Now apply the same procedure to the nonempty open sets U_i^2 of X_{δ_2}. We get $f_2 \in \mathcal{F}_2$, $z_2 \in Z_2$, a subsequence $(U_{i_j}^2)$ of (U_i^2), and for all $j \in \mathbb{N}$, a point $z_j^2 \in Z_2$ such that for each j, $(z_j^2, f_2(z_j^2)) \in U_{i_j}^2$, (z_j^2) converges to $z_2 \in Z_2$, and $(f_2(z_j^2))$ converges to $f_2(z_2)$ in H_2. Hence, $(z_j^2, f_2(z_j^2))$ converges to $(z_2, f_2(z_2))$ in $G_{f_2} \subset X_{\delta_2}$.

By passing to subsequences we may assume the following. There are sequences (z_i^k) in Z_k, $k \in \{1, 2\}$, such that $(f_k(z_i^k))$ converges to $f_k(z_k)$ in G_{f_k} with $(r_i^1, r_i^2) \in U_i^1 \times U_i^2 \subset Q_i$ for each i where we define $r_i^k = (z_i^k, f_k(z_i^k))$. Let $D_k = \{r_i^k \,|\, i \in \mathbb{N}\} \cup \{(z_k, f_k(z_k))\}$. Then each $D_k \subset G_{f_k} \subset X_{\delta_k}$ is compact. Now, $\omega(r_i^1, r_i^2) \in \omega(Q_i) \subset W_i$ for each i. So $\omega(\{(r_i^1, r_i^2) \,|\, i \in \mathbb{N}\}) \subset \omega(D_1 \times D_2)$, a compact subset of $\mathbb{R}$, which is impossible. □

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