Proceedings of the International Association for Shell and Spatial Structures (IASS) Symposium 2009, Valencia
Evolution and Trends in Design, Analysis and Construction of Shell and Spatial Structures
28 September - 2 October 2009, Universidad Politecnica de Valencia, Spain
Alberto DOMINGO and Carlos LAZARO (eds.)

# Wrinkling analysis of orthotropic membranes via semidefinite programming

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## Abstract

In this paper we propose a numerical algorithm for computing the equilibrium configuration of an orthotropic wrinkling membrane in the small deformation. We consider the minimization problem of total potential energy of orthotropic membrane structures, which is regarded as the infinite dimensional optimization problem. By introducing a tensor representing the amount of wrinkle, we reformulate this problem into an infinite-dimensional SDP problem. Throughout the numerical examples it is shown that our method can find wrinkling states, as well as the equilibrium configurations, of orthotropic membranes without any difficulty.

**Keywords:** Orthotropic membrane, Wrinkling analysis, Minimization of total potential energy, Semidefinite programming, Interior-point method.

## 1. Introduction

In this paper we propose an optimization-based algorithm for finding the wrinkling state, as well as the equilibrium configuration, of an orthotropic membrane structure in the small deformation. We assume that a membrane consists of completely *no-compression material*, which cannot transmit compressive forces but otherwise behaves as the linear elastic material. Due to this property membrane structures may have wrinkles at the equilibrium state.

It should be emphasized that the constitutive law of membranes depends on the wrinkling states, i.e. on the existence and directions of wrinkles. This is considered as a major difficulty of equilibrium analysis of membrane structures, because a trial-and-error processes is usually required in the conventional methods in order to find the wrinkling states which does not conflict with the unknown displacements and stress states. For orthotropic membranes the principal directions of stress do not coincide with the principal directions of strain in general, and hence it is difficult to find the compatible wrinkling states by using the conventional

displacement method combined with trial-and-error processes. Therefore, it is desirable to develop a method based on a formulation irrespective of stress states. Lu *et al.* [4] proposed a numerical method for wrinkling membranes under the assumption of no-compression material. Pipkin [5] and Atai and Steigmann [2] formulated the strain energy function for membrane as a quasi-convexification of that for plate.

For planar isotropic membranes, Kanno and Ohsaki [3] proposed a formulation based on the *semidefinite program* (SDP), which is independent of the wrinkling state. SDP is a class of nonlinear convex optimization problems, which can be solved effectively by using the primal-dual interior-point method. In this paper, we show that the method in [3] can be extended to orthotropic membranes.

We first consider the minimization problem of total potential energy of orthotropic membrane structures. Then we reformulate this problem into an infinite-dimensional SDP problem by introducing an auxiliary variables representing the amount of wrinkle. It is shown from the optimality condition of this problem that the constitutive law of no-compression material is realized at the optimal solution.

By applying the conventional displacement-based finite-element discretizaion procedure, we achieve a finite-dimensional SDP problem. Then the equilibrium configuration is obtained as an optimal solution of the presented SDP problem by using the primal-dual interior-point method.

We present numerical examples to demonstrate that our method can find wrinkling states, as well as the equilibrium configurations, of orthotropic membrane structures without any difficulty. It should be emphasized that our approach does not require any knowledge of wrinkling patterns *a priori*, and can converge to the solution without any difficulty even if the equilibrium state is unstable.

## 2. Constitutive law

In this section we introduce the constitutive law of orthotropic no-compression material.

Consider an orthotropic membrane structure in the two-dimensional space, which occupies a bounded connected domain $\Omega \subset \mathbb{R}^2$ with a sufficiently smooth boundary $\Gamma := \partial\Omega$. We denote by $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ the displacement field, which is also assumed to be smooth enough.

Let $\mathscr{S}^n$ denote the set of $n \times n$ real symmetric matrices. we write $A \succeq O$ and $A \succeq B$ if the matrices $A$ and $(A-B)$ are positive semidefinite, respectively. The linear strain tensor $E \in \mathscr{S}^2$ is defined by

$$E = \frac{1}{2}\left\{(\boldsymbol{u}\otimes\nabla) + (\boldsymbol{u}\otimes\nabla)^\top\right\}. \tag{1}$$

For simplicity, we write

$$\mathscr{E}\cdot\boldsymbol{u} = \frac{1}{2}\left\{(\boldsymbol{u}\otimes\nabla) + (\boldsymbol{u}\otimes\nabla)^\top\right\}, \tag{2}$$

where $\mathscr{E} : \mathbb{R}^2 \to \mathscr{S}^2$ can be regarded as the linear mapping.

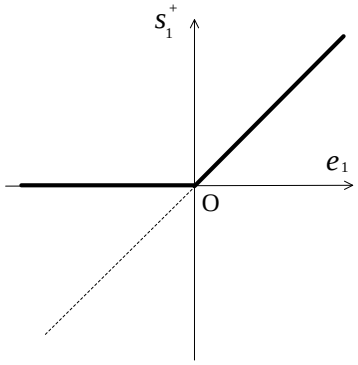



Figure 1: Constitutive law of no-compression isotropic material in the case of $s_2^+ = 0$.

Let $Q \in \mathscr{S}^2$ denote the stress tensor corresponding to $E$. Consider a linear elastic material satisfying

$$Q = \mathsf{C} : E,$$

where $\mathsf{C}$ is the elasticity tensor. The elastic stored energy function is given by

$$\tilde{w}(E) = \int Q : \mathrm{d}E = \frac{1}{2} E : \mathsf{C} : E. \tag{3}$$

For no-compression material we denote by $S^+ \in \mathscr{S}^2$ the stress tensor compatible to $E$. Let $e_i$ and $s_i^+$ $(i = 1,2)$ denote the principal strains and stresses, respectively. The relation between $s_1^+$ and $e_1$ is illustrated in Figure 1 for isotropic no-compression material, where we assume $s_2^+ = 0$ for simplicity. Note that the strain $E$ consists of two amounts, i.e. the amount of wrinkling which does not cause any stress and the one which contributes the elastic stored energy. We denote by $Z - E$ the one corresponding to the wrinkling, where $Z \in \mathscr{S}^2$. The principal values of $Z$ and $Z - E$ are denoted by $z_i$ and $\lambda_i$, respectively $(i = 1,2)$.

Now, we consider the constitutive law of no-compression material. Recall that the no-tension material cannot transmit compression stresses, which implies that the principal stresses should be nonnegative, i.e., $s_i^+ \geq 0$ $(i = 1,2)$. Consequently, we have $S^+ \succeq O$. Moreover, since $Z - E$ corresponds to the amount of wrinkle, we also have $Z - E \succeq O$. The internal work done by the wrinkle $Z - E$ is equal to be zero, which yields $S^+ : (Z - E) = 0$. It follows from the duality theory of the semidefinite program [1, Lemma 3] that $S^+$ and $Z - E$ share a common system of eigenvectors. If infinitesimal compression stresses are applied, then the material immediately losses the stiffness in the corresponding principal directions as a consequence

of appearance of wrinkles. In other words, $s_i^+ = 0$ if $\lambda_i > 0$.On the other hand, if the principal stress $s_i^+$ is positive, then no wrinkling appears in the corresponding principal direction of the stress, that is, $\lambda_i = 0$ if $s_i^+ > 0$. If there exists no wrinkling, then the no-compression material obeys Hooke's law, which is represented by $S^+ = \mathsf{C} : Z$, where $\mathsf{C}$ is the elasticity tensor.

Summarizing above discussions, the constitutive law of no-compression material is given as follows.

$$S^+ = \mathsf{C} : Z, \tag{4}$$

$$S^+ \text{ and } Z - E \text{ commute}, \begin{cases} \lambda_i = 0, & \text{if } s_i^+ > 0 \\ s_i^+ = 0, & \text{if } \lambda_i > 0 \end{cases} \tag{5}$$

$$s_i^+ \geq 0, \quad \lambda_i \geq 0 \quad (i = 1, 2). \tag{6}$$

We give some remarks regarding principal directions of $S^+$, $Z - E$, $Z$, and $E$. As mentioned in (5), for orthotropic membranes, the principal directions of $S^+$ coincide with the principal directions of $Z - E$, i.e., they share a common system of eigenvectors. In addition, in the case of isotropic membrane, $Z$ and $E$ also share the same system of eigenvectors. In contrast, for orthotropic membrane, eigenvectors of $Z$ are different from those of $E$ in general.

For the implementation, it is convenient to represent (4) by using vectors instead of tensors. Consider the $L$-$T$ coordinate system, where the $L$- and $T$-axes are parallel to the elastic principal directions of orthotropic membrane, respectively. Define the components of $S^+$ and $Z$ with respect to the $L$-$T$ coordinate system as

$$S^+ = \begin{bmatrix} S_L^+ & S_{LT}^+ \\ S_{TL}^+ & S_T^+ \end{bmatrix}, \quad Z = \begin{bmatrix} Z_L & Z_{LT}/2 \\ Z_{TL}/2 & Z_T \end{bmatrix}. \tag{7}$$

Since $S^+$ and $Z$ are symmetric tensors, $S_{LT}^+ = S_{TL}^+$ and $Z_{LT} = Z_{TL}$ hold. Hence, we consider the three-dimensional vectors $[S_L^+ \ S_T^+ \ S_{LT}^+]^\top$ and $[Z_L \ Z_T \ Z_{LT}]^\top$ instead of $S^+$ and $Z$. Define $A \in \mathscr{S}^3$ by

$$A = \begin{bmatrix} E_L/(1-\nu_L\nu_T) & \nu_T E_L/(1-\nu_L\nu_T) & 0 \\ \nu_L E_T/(1-\nu_L\nu_T) & E_T/(1-\nu_L\nu_T) & 0 \\ 0 & 0 & G_{LT} \end{bmatrix}, \tag{8}$$

where $E_L$ and $E_T$ are Young's moduli in the $L$- and $T$-directions, respectively, $\nu_L$ and $\nu_T$ are Poisson's ratios, and $G_{LT}$ is shear modulus. Then (4) is rewritten as

$$\begin{bmatrix} S_L^+ \\ S_T^+ \\ S_{LT}^+ \end{bmatrix} = A \begin{bmatrix} Z_L \\ Z_T \\ Z_{LT} \end{bmatrix}. \tag{9}$$

We next consider the constitutive law with respect to the $x$-$y$ coordinate system. Figure 2 illustrates the relationship between the $L$-$T$ and $x$-$y$ coordinate systems. Note that the orientation of the principal directions of elasticity is specified by $\theta$. We write the components of

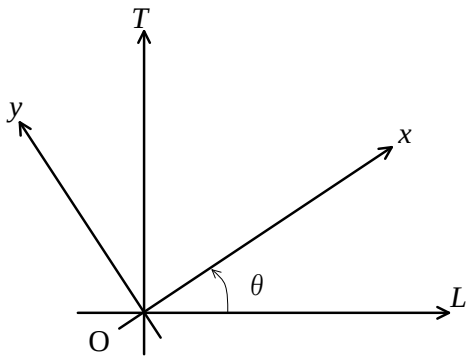



Figure 2: $L$-$T$ and $x$-$y$ coordinates.

$S^+$ and $Z$ with respect to the $x$-$y$ coordinate system as

$$S^+ = \begin{bmatrix} S_x^+ & S_{xy}^+ \\ S_{yx}^+ & S_y^+ \end{bmatrix}, \quad Z = \begin{bmatrix} Z_x & Z_{xy}/2 \\ Z_{yx}/2 & Z_y \end{bmatrix}.$$

Let $T(\theta)$ be the transformation matrix defined by

$$T(\theta) = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}.$$

By using vectors $[S_x^+ \; S_y^+ \; S_{xy}^+]^\top$ and $[Z_x \; Z_y \; Z_{xy}]^\top$, (9) is rewritten as

$$\begin{bmatrix} S_x^+ \\ S_y^+ \\ S_{xy}^+ \end{bmatrix} = A' \begin{bmatrix} Z_x \\ Z_y \\ Z_{xy} \end{bmatrix}, \tag{10}$$

where $A'$ is the matrix defined by $A' := T(\theta)AT(\theta)^\top$.

## 3. Minimization of total potential energy

In this section, we formulate the minimization problem of total potential energy (TPE) for orthotropic membranes. An equilibrium configuration is obtained as an optimal solution of the proposed TPE minimization problem.

For the given $E$ we denote by $S^+(E)$ the quantity $S^+$ satisfying (4)–(6). The strain energy function for a membrane material is written as

$$w^+(E) = \int S^+(E) : \mathrm{d}E. \tag{11}$$

Let $\boldsymbol{f}$ denote the body force per unit volume. The Dirichlet and Neumann boundary conditions are imposed on $\Gamma_u$ and $\Gamma_t$, respectively, where $\Gamma_u \cup \Gamma_t$ is a partition of $\Gamma$. We denote by $\boldsymbol{t}$ the traction at $\Gamma_t$ per unit length. Suppose that the boundary conditions $\boldsymbol{u} = \underline{\boldsymbol{u}}$ and $\boldsymbol{t} = \underline{\boldsymbol{t}}$ are given on $\Gamma_u$ and $\Gamma_t$, respectively. Then the external work is written as

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \mathrm{d}\Omega + \int_{\Gamma_t} \underline{\boldsymbol{t}} \cdot \boldsymbol{u} \mathrm{d}\Gamma. \tag{12}$$

From (2), (11), and (12), the minimization problem of total potential energy for a membrane structure is formulated as

$$(\Pi^+) : \begin{cases} \min & \displaystyle\int_\Omega (w^+(E) - \boldsymbol{f} \cdot \boldsymbol{u}) \mathrm{d}\Omega - \int_{\Gamma_t} \underline{\boldsymbol{t}} \cdot \boldsymbol{u} \mathrm{d}\Gamma \\ \text{s.t.} & E = \mathscr{E} \cdot \boldsymbol{u} \qquad (\text{in } \Omega), \\ & \boldsymbol{u} = \underline{\boldsymbol{u}} \qquad (\text{on } \Gamma_u = \Gamma \backslash \Gamma_t), \end{cases} \tag{13}$$

where $\boldsymbol{u}$ and $E$ are the variables. Note that at the optimal solution, $(\boldsymbol{u}^*, E^*)$, of the problem $(\Pi^+)$, $\boldsymbol{u}^*$ and $E^*$ coincide with the displacement and strain corresponding to the equilibrium state, respectively. In the next section we consider a tractable reformulation of $(\Pi^+)$.

## 4. SDP formulation

In section 3 we have shown that the equilibrium configuration of a membrane can be obtained as the optimal solution of $(\Pi^+)$. However, it is difficult to solve $(\Pi^+)$ directly, because the function $w^+$ depends on the wrinkling states. This motivates us to reformulate $(\Pi^+)$ into a tractable form, which is presented as the problem $(\mathrm{P}^+)$ below.

We now establish the relation between the strain energy functions for the membrane and the reference linearly elastic body as follows.

**Lemma 4.1.** *Let $\tilde{w}$ and $w^+$ be given by* (3) *and* (11)*, respectively. Then*

$$w^+(E) = \min_{Y \in \mathscr{S}^n} \{\tilde{w}(Y) \mid Y \succeq E\} \tag{14}$$

*holds for any $E \in \mathscr{S}^n$.*

**Lemma 4.2.** *$E$, $S^+$, and $Z$ satisfy* (4)–(6) *if and only if they satisfy* (4) *and*

$$Z - E \succeq O, \quad S^+ \succeq O, \quad S^+ : (Z - E) = 0.$$

Lemmas 4.1 and 4.2 play important roles to our reformulation of $(\Pi^+)$.

It follows from Lemma 4.1 that $(\Pi^+)$ is equivalently rewritten as

$$\begin{cases} \min & \displaystyle\int_\Omega \left( \min_{Y \in \mathscr{S}^n} \{\tilde{w}(Y) \mid Y \succeq E\} - \boldsymbol{f}\cdot\boldsymbol{u} \right) \mathrm{d}\Omega - \int_{\Gamma_t} \underline{\boldsymbol{t}}\cdot\boldsymbol{u}\,\mathrm{d}\Gamma \\ \text{s.t.} & E = \mathscr{E}\cdot\boldsymbol{u} \qquad (\text{in } \Omega), \\ & \boldsymbol{u} = \underline{\boldsymbol{u}} \qquad (\text{on } \Gamma_u). \end{cases} \tag{15}$$

Observe that the problem (15) includes a minimization problem in its objective function. However, in (15) $Y$ is subjected only to the constraint condition $Y \succeq E$ in the inner minimization problem. Hence, (15) is equivalently rewritten as

$$(\mathrm{P}^+): \begin{cases} \min & \displaystyle\int_\Omega (\tilde{w}(Y) - \boldsymbol{f}\cdot\boldsymbol{u})\mathrm{d}\Omega - \int_{\Gamma_t} \underline{\boldsymbol{t}}\cdot\boldsymbol{u}\,\mathrm{d}\Gamma \\ \text{s.t.} & Y \succeq E, \quad E = \mathscr{E}\cdot\boldsymbol{u} \qquad (\text{in } \Omega), \\ & \boldsymbol{u} = \underline{\boldsymbol{u}} \qquad (\text{on } \Gamma_u). \end{cases} \tag{16}$$

Note that the problem $(\mathrm{P}^+)$ is an SDP problem in the variables $\boldsymbol{u}$, $Y$, and $E$.

Applying the conventional displacement-based finite-element discretization procedure, we obtain the discritized version of $(\mathrm{P}^+)$ with the finite number of variables. It should be emphasized that $(\mathrm{P}^+)$ includes $\tilde{w}$, instead of $w^+$, which does not depend on the wrinkling state. Hence, no knowledge of wrinkling patterns is required in advance, and the optimal solution of $(\mathrm{P}^+)$ can be found without any trial-and-error procedure.

Let $(\boldsymbol{u}^*, Y^*, E^*)$ denote the optimal solution of $(\mathrm{P}^+)$. From the construction of $(\mathrm{P}^+)$ it follows that $(\boldsymbol{u}^*, E^*)$ is an optimal solution of $(\Pi^+)$. Hence, $\boldsymbol{u}^*$ corresponds to the displacement at the equilibrium state. In addition, we can show that $Y^*$ corresponds to $Z$ satisfying (4)–(6) as follows.

Consider the optimization problem in the variable $Y$

$$\begin{cases} \min & \tilde{w}(Y) = \dfrac{1}{2} Y : \mathsf{C} : Y \\ \text{s.t.} & Y - E \succeq O, \end{cases} \tag{17}$$

which appears in Lemma 4.1. Since $Y^*$ is optimal in $(\mathrm{P}^+)$, it is also an optimizer of the problem (17). Observe that (17) is a minimization problem of a convex function over the convex constraint condition. Hence, $Y^*$ is an optimal solution of (17) if and only if there exists a Lagrange multiplier $W \in \mathscr{S}^n$ satisfying the KKT conditions, which are written as

$$W - \mathsf{C} : Y^* = O, \tag{18}$$

$$Y^* - E^* \succeq O, \quad W \succeq O, \quad W : (Y^* - E^*) = 0. \tag{19}$$

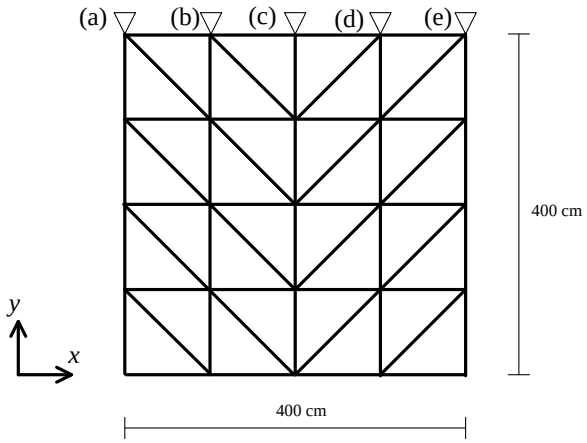


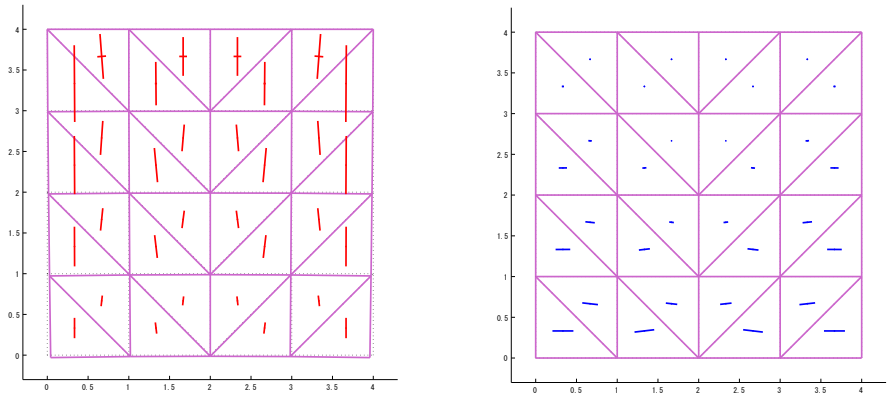

Figure 3: The membrane structure.

By putting $W = S^+$, it follows from Lemma 4.2 that (18) and (19) are equivalent to the constitutive law of orthotropic membrane, (4)–(6). Hence, the optimal solution $Y^*$ corresponds to $Z$ in (4)–(6). The stress $S^+$ at the equilibrium state can be computed by (4).

## 5. Numerical examples

Equilibrium configurations are computed for structures consisting of orthotropic membranes by solving the SDP problem $(\mathrm{P}^+)$. The SDPs are solved by using SeDuMi Ver. 1.1 (Pólik [6], Strum [7]). Computation has been carried out on Inter Core2 Duo CPU T7300 (2.00 GHz with 1.99 GB memory) with MATLAB Ver. 7.5.0.342.

Consider a planar membrane structure in the plane stress as shown in Figure 3. The structure is discretized into three-node triangular isoparametric finite elements, where the number of elements is 32. Nodes (a)–(e) are pin-supported. Uniform loads of 1.0 kN are applied to all nodes in the negative direction of the $y$-axis. Young's modulus $E_L$ in the $L$-direction and Poisson's ratio $\nu_L$ in (8) are 0.025 GPa and 0.2, respectively. Poisson's ratio in the $T$-direction, $\nu_T$, satisfies $\nu_T = (E_T/E_L)\nu_L$. Note that an isotropic membrane is defined by $G_{LT} = E_L/2(1+\nu_L)$ and $E_T/E_L = 1.0$. For an orthotropic membrane, we also give $G_{LT}$ as $G_{LT} = E_L/2(1+\nu_L)$, and compute the equilibrium configurations for various values of $E_T/E_L$ and $\theta$, where $\theta$ is defined in Figure 2.



(a) deformed configuration and principal stresses (displacement amplified 40 times)

(b) wrinkle strains

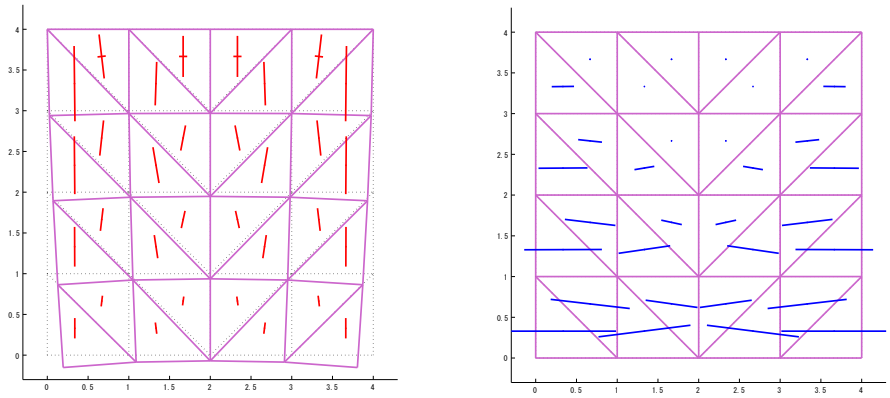
Figure 4: The equilibrium configuration and stress states of the isotropic membrane ($E_T/E_L = 1.0,\ \theta = 0$).

### 5.1. Isotropic membrane

Consider an isotropic membrane defined by $E_T/E_L = 1.0$ and $\theta = 0$. The equilibrium configuration obtained by solving $(\mathrm{P}^+)$ is illustrated in Figure 4 (a). The CPU time required to solve the SDP problem is 0.60 sec. The principal values and directions of stresses on each element are also illustrated, where the solid segments are parallel to the principal axes of stresses, and the length of each segment is proportional to the modulus of corresponding principal value. Figure 4 (b) illustrates the principal directions and values of $Z - E$ of each element, where the direction of each segment is orthogonal to the wrinkling pattern, and the length of each segment is proportional to the amount of wrinkle.

### 5.2. Orthotropic membrane

We next consider two examples of orthotropic membranes. Figure 5 illustrates the case in which $E_T/E_L = 0.2$ and $\theta = 0$, i.e. principal directions of elasticity are parallel to the *x*- and *y*-axes, and the stiffness in the direction of the *y*-axis is one fifth of that in the direction of the *x*-axis. The CPU time required to solve the SDP problem is 0.56 sec. Figure 5 (a) illustrates the obtained configuration and the stress states, while Figure 5 (b) illustrates the principal values and directions of $Z - E$ in a manner similar to Figure 4. It is observed from Figure 5 that the decrease of the stiffness in the direction of the *y*-axis causes larger deformation compared with the isotropic membrane. We can also observe the increase of



(a) deformed configuration and principal stresses (displacement amplified 40 times)

(b) wrinkle strains

Figure 5: The equilibrium configuration and stress states of the orthotropic membrane ($E_T/E_L = 0.2,\ \theta = 0$).

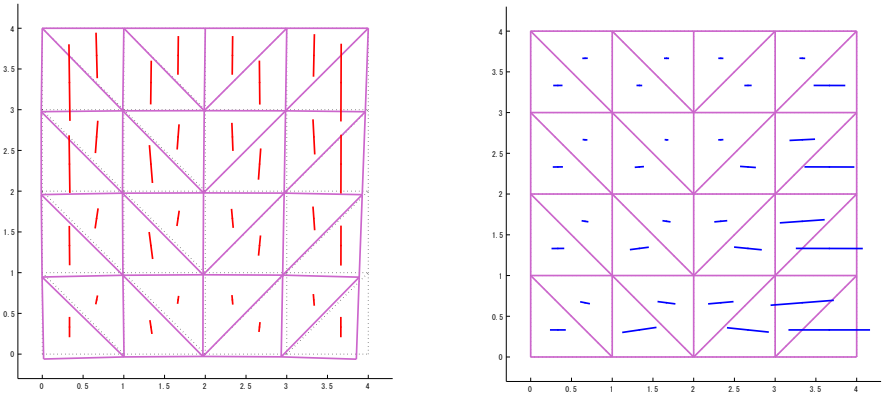
amount of wrinkle, which is caused by Poisson's effect.

Figure 6 illustrates the case in which $E_T/E_L = 0.2\ \theta = \pi/4$. The CPU time required to solve the SDP problem is 0.52 sec. The principal directions of elasticity are illustrated in Figure 7, where the thick and thin lines depict the strong and weak directions, respectively. It is observed from Figure 6 that the equilibrium configuration and the distribution of wrinkle are not symmetric, although we consider the symmetric boundary and loading conditions. It is shown from the numerical examples above that the proposed method can find the equilibrium configurations of orthotropic membranes without any difficulty, even if many elements are in the wrinkling states.

## 6. Conclusions

An approach based on semidefinite programming (SDP) problem has been proposed for finding the equilibrium configuration of structures consisting of orthotropic no-compression membrane under the assumption of small deformations. We have shown that the minimization problem of total potential energy for orthotropic membrane structure can be reformulated as a SDP problem. The equilibrium configuration can be obtained as an optimal solution of the presented SDP problem by using the primal-dual interior-point method.

The presented SDP formulation requires any knowledge on the wrinkling states in advance.



(a) deformed configuration and principal stresses (displacement amplified 40 times)

(b) wrinkle strains

Figure 6: The equilibrium configuration and stress states of the orthotropic membrane ($E_T/E_L = 0.2,\ \theta = \pi/4$).

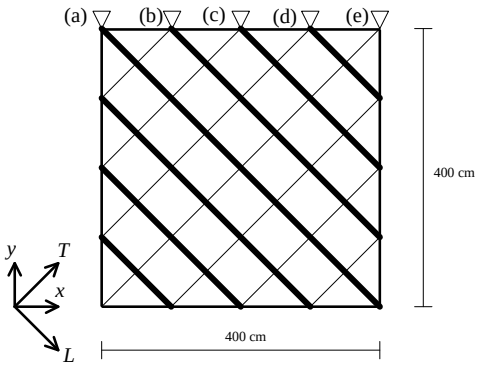



Figure 7: The elastic principal directions of the orthotropic membrane structure in Figure 6.

Therefore the method does not involve any processes of trial-and-error even if the structure has a complicated wrinkling pattern. It is guaranteed that the number of arithmetic operations required by this method is bounded by a polynomial of the size of problem. Numerical examples have shown the effectiveness of this method for the cases in which many wrinkles appear at equilibrium configurations.

Besides these advantages, there exist several well-developed software packages which solve SDPs efficiently. Hence, it is sufficient to prepare the codes which provides the data matrices and vectors of the corresponding SDP problem. Since our finite-element discretization is based on the usual displacement-based finite element methods, we can easily utilize the conventional software of the finite element method in order to compute the data matrices and vectors of the SDP problem to be solved.

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