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Advanced Form-Finding for Cable Dome Structures

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Abstract

A numerical method is presented for form-finding of cable domes. The topology and the types of members are the only information that requires in this form-finding process. Dummy elements are used to transform the cable dome with supports into self-stressed system without supports. The eigenvalue decomposition of the force density matrix and the Singular value decomposition of the equilibrium matrix are performed iteratively to find the feasible sets of nodal coordinates and force densities which satisfy the minimum required rank deficiencies of the force density and equilibrium matrices, respectively. Based on numerical examples it is found that the proposed method is very efficient, robust and versatile in searching self-equilibrium configurations of cable dome structures.

Keywords: Cable dome, Singular value decomposition, Form-finding, New configuration.

1. Introduction

The Cable dome structures first proposed by Geiger [6] have been developed in recent years due to their innovative forms, lightweight and deployability. They belong to a class of pre-stressed pin-jointed systems that cannot be stable without introducing prestresses to some members (Pellegrino [16]). Kawaguchi et al. [8] proposed a least-square problem of nodal displacements with the specified external forces for obtaining optimum shapes of a cable dome structure. Recently, Ohsaki and Kanno [13] investigated the form-finding of cables domes under specified stresses by nonlinear mathematical programming problem. Deng et al. [4] suggested problem of shape finding of cable-strut assemblies which could be incomplete with missing or slack cables during construction by using the iterative algorithm until equilibrium equations satisfied. More recently, optimum prestressing of domes with a single or with multiple integral prestress modes is also examined by Yuan et al. [22].

In this paper, a numerical method is presented for form-finding of cable domes. The topology and the types of members, i.e. either compression or tension are the only information that requires in this form-finding process. In other words, the initial nodal coordinates are not necessary for the present form-finding. Dummy elements are used to transform the cable dome with supports into self-stressed system without supports. The force density matrix is derived from an incidence matrix and an initial set of force densities assigned from prototypes, while the equilibrium matrix is defined by the incidence matrix and nodal coordinates. The eigenvalue decomposition of the force density matrix and the Singular value decomposition of the equilibrium matrix are performed iteratively to find the feasible sets of nodal coordinates and force densities which satisfy the minimum required rank deficiencies of the force density and equilibrium matrices, respectively.

2. Formulation of Equilibrium for cable dome structures

For a *d*-dimensional cable dome structure with b members, n free nodes and n_f fixed nodes (supports), its topology can be expressed by a connectivity matrix $\mathbf{C}_s (\in \mathrm{R}^{b\times(n+n_f)})$ as discussed in Schek [18], Motro [11], and Zhang and Ohsaki [23]. Suppose member k connects nodes i and (i < j), then the i^{th} and j^{th} elements of the k^{th} row of $\mathbf{C}_s$ are set to 1 and -1, respectively, as follows:

$$\mathbf{C}_{s(k,p)} = \begin{cases} 1 & for\ p = i \\ -1 & for\ p = j \\ 0 & otherwise \end{cases} \tag{1}$$

If the free nodes are numbered first, then to the fixed nodes, $\mathbf{C}_s$ can be divided into two parts as

$$\mathbf{C}_s = [\mathbf{C}\ \mathbf{C}_f] \tag{2}$$

where $\mathbf{C} (\in \mathrm{R}^{b\times n})$ and $\mathbf{C}_f (\in \mathrm{R}^{b\times n_f})$ describe the connectivities of the members to the free and fixed nodes, respectively. Let $\mathbf{x, y, z} (\in \mathrm{R}^{n})$ and $\mathbf{x}_f, \mathbf{y}_f, \mathbf{z}_f (\in \mathrm{R}^{n_f})$ denote the nodal coordinate vectors of the free and fixed nodes, respectively, in x, y and z directions.

The equilibrium equations in each direction of a general pin-jointed structure given by Schek [18] can be stated as

$$\mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}\mathbf{x} + \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}_f\mathbf{x}_f = \mathbf{p}_x \tag{3a}$$

$$\mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}\mathbf{y} + \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}_f\mathbf{y}_f = \mathbf{p}_y \tag{3b}$$

$$\mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}\mathbf{z} + \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}_f\mathbf{z}_f = \mathbf{p}_z \tag{3c}$$

where $\mathbf{p}_x$, $\mathbf{p}_y$ and $\mathbf{p}_z (\in \mathrm{R}^{n})$ are the vectors of external loads applied at the free nodes in x, y and z directions, respectively. The symbol, $(.)^T$, denotes the transpose of a matrix or vector. And $\mathbf{Q} (\in \mathrm{R}^{b\times b})$ is diagonal square matrix, calculated by

$$\mathbf{Q} = \mathrm{diag}(\mathbf{q}) \tag{4}$$

where $\mathbf{q} (\in \mathrm{R}^{b})$ suggested in Schek [18] is the force density vector, defined by

$$\mathbf{q} = \{q_1, q_2, \ldots, q_b\}^T \tag{5}$$

in which each component of this vector is the force f_i to length l_i ratio $q_i = f_i/l_i$ known as force density or self-stressed coefficient in Vassart and Motro [20]. Without external loading, Eq. (3) can be rewritten neglecting the self-weight of the structure as

$$\mathbf{Dx} = -\mathbf{D}_f\mathbf{x}_f \tag{6a}$$

$$\mathbf{Dy} = -\mathbf{D}_f\mathbf{y}_f \tag{6b}$$

$$\mathbf{Dz} = -\mathbf{D}_f\mathbf{z}_f \tag{6c}$$

where matrices $\mathbf{D}$ ($\in \mathrm{R}^{n\times n}$) and $\mathbf{D}_f$ ($\in \mathrm{R}^{n\times n_f}$) are, respectively, given by

$$\mathbf{D} = \mathbf{C}^T\mathbf{Q}\mathbf{C} \tag{7a}$$

$$\mathbf{D}_f = \mathbf{C}^T\mathbf{Q}\mathbf{C}_f \tag{7b}$$

or by

$$\mathbf{D} = \mathbf{C}^T\mathrm{diag}(\mathbf{q})\mathbf{C} \tag{8a}$$

$$\mathbf{D}_f = \mathbf{C}^T\mathrm{diag}(\mathbf{q})\mathbf{C}_f \tag{8b}$$

In order to solve Eq. (6) with respect to the unknown coordinates x, y and z of the free nodes, the coordinates $\mathbf{x}_f$, $\mathbf{y}_f$ and $\mathbf{z}_f$ of the fixed nodes must be given. In order to perform the advanced form-finding of the cable dome structures without given nodal coordinates of the fixed nodes, the dummy elements are used to free the fixed nodes. In other words, by using the concept of dummy elements the pre-stressed cable dome structure can be converted into free-standing self-stressed structure without supports.

It is noted that when external load and self-weight are ignored, a self-stressed system does not require any fixed node, and the self-stressed geometry is defined by the relative position of the nodes, and the system can be considered as free, forming a rigid body free in space (Vassart and Motro [20]). In this context, Eqs. (7b), (8b) vanish, and Eq. (6) becomes:

$$\mathbf{Dx} = \mathbf{0} \tag{9a}$$

$$\mathbf{Dy} = \mathbf{0} \tag{9b}$$

$$\mathbf{Dz} = \mathbf{0} \tag{9c}$$

where $\mathbf{D}$ known as force density matrix (Tibert and Pellegrino [19], Estrada et al. [5]) or stress matrix (Connelly [2, 3]).

For simplicity, Eq. (9) can be reorganized as

$$\mathbf{D}[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]=[\mathbf{0}\ \mathbf{0}\ \mathbf{0}] \tag{10}$$

However, by substituting Eq. (8) into Eq. (9) the equilibrium equations of the self-stressed structure can be expressed as

$$\mathbf{C}^T\mathrm{diag}(\mathbf{q})\mathbf{C}\mathbf{x} = \mathbf{0} \tag{11a}$$

$$\mathbf{C}^T\mathrm{diag}(\mathbf{q})\mathbf{C}\mathbf{y} = \mathbf{0} \tag{11b}$$

$$\mathbf{C}^T\mathrm{diag}(\mathbf{q})\mathbf{C}\mathbf{z} = \mathbf{0} \tag{11c}$$

Eqs. (11) can be reorganized as

$$\mathbf{A}\mathbf{q} = \mathbf{0} \tag{12}$$

where $\mathbf{A}$ ($\in \mathrm{R}^{dn\times b}$) is known as the equilibrium matrix in Motro [11], defined by

$$\mathbf{A}=\begin{pmatrix}\mathbf{C}^T diag(\mathbf{Cx})\\ \mathbf{C}^T diag(\mathbf{Cy})\\ \mathbf{C}^T diag(\mathbf{Cz})\end{pmatrix} \tag{13}$$

Eq. (10) presents the relation between force densities and nodal coordinates, while Eq. (12) shows the relation between projected lengths in x,y and z directions, respectively and force densities. Both Eqs. (10) and (12) are linear homogeneous systems of self-equilibrium equations with respect to nodal coordinates and force densities, respectively.

3. Requirement on Rank Deficiency Conditions

Let $\mathbf{q}$ be the vector of force density and $\mathbf{C}$ be the incidence matrix of a *d*-dimensional self-stressed structure in self-equilibrium. It is well known that the set of all solutions to the homogeneous system of Eq. (10) is the null space of $\mathbf{D}$. The dimension of this null space or rank deficiency of $\mathbf{D}$ is defined as

$$n_{\mathbf{D}} = n - r_{\mathbf{D}} \tag{14}$$

where $r_{\mathbf{D}} = rank(\mathbf{D})$. It is obvious that vector $\bar{\mathbf{I}}_1 = \{1,1,\cdots 1\}^T$ ($\in \mathrm{R}^{n\times 1}$), is a solution of Eq. (10) since the sum of the elements of a row or a column of $\mathbf{D}$ is always equal to zero (Tibert and Pellegrino [19]). The most important rank deficiency condition related to semi-definite matrix $\mathbf{D}$ of Eq. (10) is defined by

$$n_{\mathbf{D}} \geq d+1 \tag{15}$$

This condition forces Eq. (10) to yield at least d useful particular solutions (Meyer [10]) which exclude the above vector $\bar{\mathbf{I}}_1$ due to degenerating geometry of self-stressed structure (Tibert and Pellegrino [19], Zhang and Ohsaki [23]). These d particular solutions form a vector space basis for generating a *d*-dimensional self-stressed structure. Therefore, the minimum rank deficiency or nullity of $\mathbf{D}$ must be (d+1) for configuration of any self-stressed structure embedding into R^d, which is equivalent to the maximum rank condition of $\mathbf{D}$ proposed by Connelly [2, 3], and Motro [11] as follows:

$$max(r_{\mathbf{D}}) = n-(d+1) \tag{16}$$

Similarly, the set of all solutions to the homogeneous system of Eq. (12) lies in the null space of $\mathbf{A}$. Let $n_{\mathbf{A}}$ denote dimension of null space of the equilibrium matrix $\mathbf{A}$ which is computed by

$$n_{\mathbf{A}} = b - r_{\mathbf{A}} \tag{17}$$

where $r_{\mathbf{A}} = rank(\mathbf{A})$. The second rank deficiency condition which ensures the existence of at least one state of self-stress can be stated as

$$s = n_{\mathbf{A}} \geq 1 \tag{18}$$

where s is known as the number of independent states of self-stress, while the number of infinitesimal mechanisms is computed by $m = dn - r_{\mathbf{A}}$, as presented in Calladine [1] and Pellegrino and Calladine [15]. It is clear that Eq. (18) allows Eq. (12) to create at least one useful particular solution (Meyer [10]).

4. Form-Finding Process

The proposed form-finding procedure only needs to know the topology of structure in terms of the incidence matrix **C**, and type of each member, i.e. either cable or strut which is under tension or compression, respectively. Based on element type, the initial force density coefficients of cables (tension) are automatically assigned as +1 while those of struts (compression) as -1, respectively, as follows:

$$\mathbf{q}^0 = \{\underbrace{+1+1\cdots+1}_{cables} \quad \underbrace{-1-1\cdots-1}_{struts}\}^T \tag{19}$$

Subsequently, the force density matrix **D** is calculated from $\mathbf{q}^0$ by Eq. (8). After that, the nodal coordinates are selected from the eigenvalue decomposition of the matrix **D** which is discussed in the next section. These nodal coordinates are substituted into Eq. (12) to define force density vector **q** by the Singular value decomposition of the equilibrium matrix **A** which is also presented in the next section. The force density matrix **D** is then updated by Eq. (8). The process is iteratively calculated for searching a set of nodal coordinates [**x y z**] and force density vector **q** until the rank deficiencies of Eqs. (15) and (18) are satisfied, which forces Eqs. (10) and (12) become true. In this context, at least one state of self-stress can be created, s≥1. In this study, based on required rank deficiencies from Eqs. (15) and (18) the form-finding process is stopped as

$$n_{\mathbf{D}}^* = \mathrm{d} + 1 \tag{20a}$$

$$n_{\mathbf{A}}^* = 1 \tag{20b}$$

where $n_{\mathbf{D}}^*$ and $n_{\mathbf{A}}^*$ are minimum required rank deficiencies of the force density and equilibrium matrices, respectively.

4.1 Eigenvalue decomposition of force density matrix

The square symmetric force density matrix **D** can be factorized as follows by using the eigenvalue decomposition (Meyer [10]):

$$\mathbf{D} = \Phi \Lambda \Phi^T \tag{21}$$

where Φ $(\in \mathrm{R}^{n\times n})$ is the orthogonal matrix ($\Phi\Phi^T = I_n$, in which $I_n \in \mathrm{R}^{n\times n}$ is the unit matrix) whose i^{th} column is the eigenvector basis ϕ_i $(\in \mathrm{R}^n)$ of **D**. Λ $(\in \mathrm{R}^{n\times n})$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, i.e., $\Lambda_{ii} = \lambda_i$. The eigenvector ϕ_i of Φ corresponds to eigenvalue λ_i of Λ. The eigenvalues are in increasing order as

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \tag{22}$$

It is clear that the number of zero eigenvalues of **D** is equal to the dimension of its null space. Let p be the number of zero and negative eigenvalues of **D**. There are two cases need to be considered. The first one is $p \le n_{\mathbf{D}}^*$ and the other is $p > n_{\mathbf{D}}^*$.

Case 1, the first $n_{\mathbf{D}}^*$ orthonormal eigenvectors of Φ are directly taken as potential nodal coordinates

$$[\mathbf{x}\ \mathbf{y}\ \mathbf{z}] \in \bar{\Phi} = [\phi_1\ \phi_2 \cdots \phi_{n_{\mathbf{D}}^*}] \tag{23}$$

The algorithm then iteratively modifies the force density vector **q** as small as possible to make the first $n_{\mathbf{D}}^*$ eigenvalues of **D** become null as

$$\lambda_i = 0, (i = 1, 2, \cdots, n_{\mathbf{D}}^*) \tag{24}$$

Accordingly, **D** will finally have the required rank deficiency $n_{\mathbf{D}}^*$ without any negative eigenvalue. It implies **D** is positive semi-definite, and any self-stressed structure falling into this case is super-stable regardless of material properties and level of self-stress coefficients.

Case 2, where $p > n_{\mathbf{D}}^*$, the rank deficiency may be forced to be larger than requirement or enough but **D** may not be positive semi-definite during iteration. Additionally, the proposed form-finding procedure will evaluate the tangent stiffness matrix of the pre-stressed cable dome structure which is given in Murakami [12], Guest [7] and Zhang and Ohsaki [23]. If the tangent stiffness matrix is positive-definite, then the structure is stable when its rigid-body motions are constrained. Using this criterion, stability of any pre-stressed or self-stressed structure can be controlled by checking eigenvalues of tangent stiffness matrix of the structure (Murakami [12] and Ohsaki and Zhang [14]).

In short, the best scenario of configuration in 3-dimensional space is formed by three best candidate eigenvectors selected from the first fourth eigenvector bases which corresponding to the first fourth smallest eigenvalues, respectively. These eigenvalues will be gradually modified to be zero by the proposed iterative form-finding algorithm. In other words, the proposed form-finding procedure has repeatedly approximated equilibrium configuration such that

$$\mathbf{D}[\mathbf{x}\ \mathbf{y}\ \mathbf{z}] \approx [\mathbf{0}\ \mathbf{0}\ \mathbf{0}] \tag{25}$$

4.2 Singular value decomposition of the equilibrium matrix

The equilibrium matrix **A** is computed by substituting the set of approximated nodal coordinates [**x y z**] from Eq. (25) into Eq. (13). In order to solve linear homogeneous system (Eq. (12)) the Singular value decomposition (Meyer [10]) is carried out on the equilibrium matrix **A**:

$$\mathbf{A} = \mathbf{U}\mathbf{V}\mathbf{W}^{\mathrm{T}} \tag{26}$$

where $\mathbf{U}(\in \mathbb{R}^{dn \times dn}) = [\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_{dn}]$ and $\mathbf{W}(\in \mathbb{R}^{b \times b}) = [\mathbf{w}_1\ \mathbf{w}_2 \cdots \mathbf{w}_b]$ are the orthogonal matrices. $\mathbf{V}(\in \mathbb{R}^{dn \times b})$ is a diagonal matrix with non-negative Singular values of **A** in decreasing order as

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_b \geq 0 \tag{27}$$

As indicated in Eq. (20b), the iterative form-finding algorithm is successful in case of $n_{\mathbf{A}}^{*} = 1$. Accordingly, there are also two cases for s during the iterative form-finding procedure:

Case 1: s=0, there exists no null space of **A**. In other words, the right Singular value (σ_b) of **A** in **V** is not equal to zero. It denotes that Eq. (12) has no non-zero force density vector **q** as a solution. In this case, if the right single vector basis $(\mathbf{w}_b)$ in **W** corresponding to smallest singular value (σ_b) in **V** is used as the approximated **q**, the sign of **q** may not match with that of $\mathbf{q}^0$. Thus, all columns of **W** employed to compute a vector **q** that best matches **q** are scanned by form-finding procedure. The procedure stops sign-finding until the sign of all components of $\mathbf{w}_j (j = b, b-1, ..., 1)$ is identical to that of $\mathbf{q}^0$, i.e. $sign(\mathbf{w}_j) \equiv sign(\mathbf{q}^0)$. That vector $\mathbf{w}_j$ is directly taken as the approximated **q**. In doing so, the form-finding procedure defines the approximated **q** that matches in signs with $\mathbf{q}^0$, such that

$$\mathbf{Aq} \approx \mathbf{0} \tag{28}$$

Case 2: s=1, it is known (Pellegrino [17]) that the bases of vector spaces of force densities and mechanisms of any self-stressed structure are calculated from the null spaces of the equilibrium matrix. In this case, the matrices **U** and **W** from Eq. (26) can be expressed, respectively, as

$$\mathbf{U} = [\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_{r_{\mathbf{A}}} \mid \mathbf{m}_1 \cdots \mathbf{m}_{dn-r_{\mathbf{A}}}] \tag{29a}$$

$$\mathbf{W} = [\mathbf{w}_1\ \mathbf{w}_2 \cdots \mathbf{w}_{b-1} \mid \mathbf{q}_1] \tag{29b}$$

where the vectors $\mathrm{m} (\in \mathrm{R}^{dn})$ denote the $m (= dn - r_{\mathbf{A}})$ infinitesimal mechanisms; and the vector $\mathbf{q}_1 (\in \mathrm{R}^{b})$ matching in signs with $\mathbf{q}^0$ is indeed the single state of self-stress which satisfies the homogeneous Eq. (12).

In summary, the eigenvalue decomposition of force density matrix **D** and the Singular value decomposition of the equilibrium matrix **A** are performed iteratively to find the feasible set of nodal coordinates [**x y z**] and force density vector **q** which satisfy the minimum required rank deficiencies of the force density and equilibrium matrices as presented in Eq. (20), respectively.

Since the self-stressed structure should satisfy the self-equilibrium conditions, the vector of unbalanced forces $\varepsilon_f \ (\in \mathrm{R}^{dn})$ defined as follows can be used for evaluating the accuracy of the results:

$$\varepsilon_f = \mathbf{Aq} \tag{30}$$

The Euclidean norm of ε_f is used to define the design error ε as

$$\varepsilon = \sqrt{\varepsilon_f (\varepsilon_f)^T} \tag{31}$$

5. Numerical Examples

Numerical examples are presented for two and three-dimensional cable dome structures using Matlab Version 7.4(R2007a) (Yang et al. [21]).

5.1. Two-dimensional cable dome structure

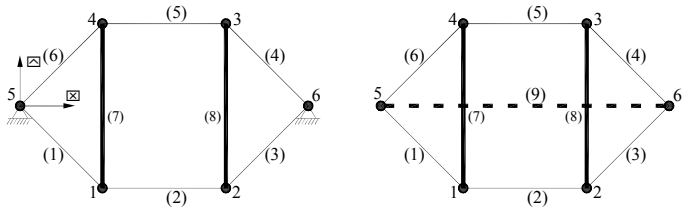

Figure 1: (a). A two-dimensional cable dome structure, (b). Its equivalent free-standing 2-D self-stressed structure with dummy element to remove the supports.

The initial topology of a two-dimensional cable dome structure (Fig. 1a) comprises two struts and six cables. The supports can be converted into free nodes by using the dummy element in order to obtain the self-stressed system. After implementation of form-finding using the method proposed in Section 4, the dummy element will be removed to transform the two nodes back to the supports. By connecting the two supports with dummy element 9, the topology of the equivalent free-standing 2-D self-stressed structure is described in Fig.1b where thin, thick and dashed lines represent the cables, struts and dummy elements, respectively.

No nodal coordinates as well as symmetry, member lengths and force density coefficients are known in advance. The only information is the incidence matrix **C** and the type of each member which is employed to automatically assign the initial force density vector by proposed form-finding procedure as

$$\mathbf{q}^0 = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}^T = \{1, 1, 1, 1, 1, 1, -1, -1, -1\}^T \tag{32}$$

The obtained force density vector normalized with respect to the force density coefficient of the cable 1 is as follows:

$$\mathbf{q} = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}^T = \{1.0000, 1.2808, 1.0000, 1.0000, 1.2808, 1.0000, -0.5001, -0.5001, -0.7192\}^T \tag{33}$$

The associated stable configuration of the structure after neglecting the dummy element 9 is plotted in Fig. 2.

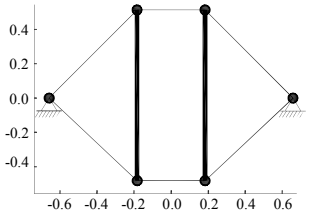

Figure 2: The obtained geometry of the two-dimensional cable dome structure.

The form-finding procedure for converges in only one iteration with the design error (ε) defined in Eq. (37) less than 10^{-15}. The structure obtained has only one self-stress state ($s=1$) and one infinitesimal mechanism ($m=1$) when their rigid-body motions are constrained indicating they are statically indeterminate and kinematically indeterminate (Pellegrino and Calladine [15]). The force density matrices **D** is positive semi-definite, and the structure is certainly super stable regardless of materials and prestress levels (Connelly [2, 3]). In other words, the introduction of single prestress stiffens the infinitesimal mechanism to make the structures stable in all but three directions. Consequently, the proposed form-finding procedure with limited information about the incidence matrix and element prototype is indeed capable of finding a self-equilibrium stable cable dome structure by imposing the two necessary rank deficiency conditions.

5.2 Three-dimensional cable dome structure

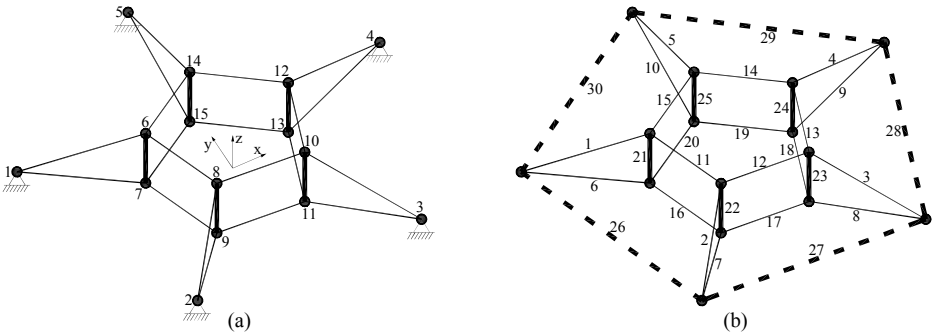

Figure 3: (a). A three-dimensional five-strut cable dome structure, (b). Its equivalent free-standing 3-D self-stressed structure with dummy elements to remove the supports.

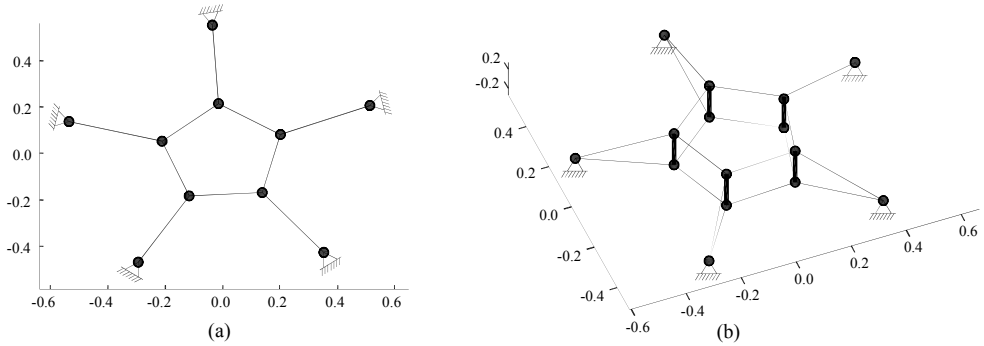

Figure 4: The obtained geometry of the three-dimensional five-strut cable dome structure, (a). Top view, (b). Perspective view.

Consider the three-dimensional cable dome with five struts and twenty cables shown in Fig. 3a. Its equivalent model is depicted in Fig. 3b. Similar to above example, the input information is the incidence matrix **C** and the type of each member which is used to automatically assign the initial force density vector by proposed form-finding procedure as

$$\mathbf{q}^0 = \{q_1 \sim q_{20} = 1, q_{21} \sim q_{30} = -1\}^T \tag{34}$$

The calculated force density vector after normalizing with respect to the force density coefficient of the cable 1 is:

$$\mathbf{q} = \{q_1 \sim q_{10} = 1.0000, q_{11} \sim q_{20} = 1.1206, q_{21} \sim q_{25} = -0.5000, q_{26} \sim q_{30} = -0.8794\}^T \tag{35}$$

The associated stable configuration of the structure after neglecting the dummy elements $(26:\ 30)$ is plotted in Fig. 4. The design error (ε) is about 10^{-14}. The obtained structure possesses one self-stress state $(s=1)$ and ten infinitesimal mechanisms $(m=10)$ excluding their six body-rigid motions. In this problem, the force density matrix **D** is negative semi-definite indicating the structure is not super stable. Accordingly, tangent stiffness of the structures has been investigated and found to be positive. It confirms that the structure is mechanically stable (Murakami [12], Ohsaki and Zhang [14]).

6. Concluding Remarks

The advanced form-finding procedure for cable dome structures has been proposed. The force density matrix is derived from the incidence matrix and initial set of force densities formed by the vector of type of member forces. The elements of this vector consist of unitary entries +1 and -1 for members in tension and compression, respectively. The equilibrium matrix is defined by the incidence matrix and nodal coordinates. The eigenvalue decomposition of the force density matrix and the Singular value decomposition of the equilibrium matrix are performed iteratively to find the range of feasible sets of nodal coordinates and force densities. In the numerical examples, a very good convergence of the proposed method has been shown for two-dimensional and three-dimensional cable dome

structures. The proposed algorithm is strongly capable of searching novel configurations with limited information of topology and the member's type. As a natural extension of this research, form-finding with more complicated constraints awaits further attention.

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