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A note on weakly pseudocompact locales

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Abstract

We revisit weak pseudocompactness in pointfree topology, and show that a locale is weakly pseudocompact if and only if it is G_δ-dense in some compactification. This localic approach (in contrast with the earlier frame-theoretic one) enables us to show that finite localic products of locales whose non-void G_δ-sublocales are spatial inherit weak pseudocompactness from the factors. We also show that if a locale is weakly pseudocompact and its G_δ-sublocales are complemented then it is Baire.

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1. Introduction and motivation

Generalizing pseudocompactness in completely regular Hausdorff spaces, García-Ferreira and García-Maynez [11] define a completely regular Hausdorff space to be weakly pseudocompact in case it is G_δ-dense in some compactification. This generalizes pseudocompactness since pseudocompact spaces (throughout, by "space" we mean a completely regular Hausdorff space) are precisely the spaces X that are G_δ-dense in βX.

The reader will recall that a subspace A of a topological space X is G_δ-dense in X if it meets every nonempty G_δ-subspace of X. Ball and Walters-Wayland showed in [2] that this concept is expressible frame-theoretically in the following way. A subspace A of X is G_δ-dense if and only if the frame homomorphism

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$\mathfrak{O}X \to \mathfrak{O}A$ induced by the subspace inclusion $A \hookrightarrow X$ is coz-codense. It was on the basis of this that in [10] Walters-Wayland and I defined a completely regular frame L to be weakly pseudocompact if there is a compactification $h\colon M \to L$ of L such that the homomorphism h is coz-codense. Assuming the Boolean Ultrafilter Theorem, one quickly observes that this definition is "conservative", which is to say a space is weakly pseudocompact precisely when the frame of its open sets is weakly pseudocompact.

Now a natural question to ask is whether weak pseudocompactness in the pointfree setting is equivalent to a condition lifted verbatim from the spatial definition, subject to replacing "subspace" with "sublocale". That is, is L weakly pseudocompact if and only if it is G_δ-dense in some compactification? Of course G_δ-denseness in locales is to be taken (exactly as in spaces) to mean having non-void intersection with every non-void sublocale which is a countable intersection of open sublocales. Before we leap to conclusions and say that should clearly be so in view of the result of Ball and Walters-Wayland mentioned above, we should reflect on the fact that their result is about subspaces, and spaces can have more sublocales than subspaces.

It is the main aim of this note to answer the question in the affirmative. This is how we go about doing so. We first show that a sublocale of a compact regular locale is G_δ-dense precisely when it has non-void intersection with every non-void zero-set sublocale (Lemma 3.2). This then shows that a sublocale is G_δ-dense if and only if the associated frame surjection from the ambient locale to the sublocale is coz-codense. Thence it is easy to deduce that a locale is weakly pseudocompact if and only if it is G_δ-dense in some compactification (Proposition 3.4).

This done, we then prove the main result in the paper (Theorem 3.7). We carry out the proof in **Frm**, and thus deal with coproducts of frames. In Remark 3.8 we explain why this result does not follow from the result that the topological product of weakly pseudocompact spaces is weakly pseudocompact.

In [11], the authors show that every weakly pseudocompact space is Baire. It will be recalled that a space is Baire if the intersection of countably many dense open sets is dense. In locales the Baire property cannot be defined by decreeing that countable intersections of open dense sublocales be dense, because all intersections of dense sublocales are dense. This led Isbell [14] to define Baire locales in terms of first and second category sublocales. We will recall the definition at the right place. In Proposition 4.3 we show that a weakly pseudocompact locale is Baire if its G_δ-sublocales are complemented. This rather extraneous condition seems to be necessitated by the fact that the sublocale lattice is not necessarily Boolean.

2. Preliminaries

2.1. **Frames, very briefly.** We shall use the terms "frame" and "locale" interchangeably. Our reference for frames and locales are [16] and [17]. Our notation follows these texts, to a large extent. One little deviation is that we write $\mathfrak{O}X$ for the frame of open sets of a space X. We denote by h_* the right adjoint of a frame homomorphism h. The completely below relation is denoted by $\prec\!\!\prec$. All frames in this paper are completely regular.

By a *point* of a frame L we mean an element p such that $p < 1$ and $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. We write $\mathrm{Pt}(L)$ for the set of points of L. A frame is *spatial* if it as enough points, in the sense that every element is the meet of points above it. Compact regular frames have enough points, modulo the Boolean Ultrafilter Theorem, which we will assume throughout. An element of a frame is *dense* if it has nonzero meet with every nonzero element. This is precisely when its pseudocomplement is 0.

An element a of L is a *cozero element* if it is expressible as $a = \bigvee a_n$, where $a_n \prec\!\!\prec a_{n+1}$ for every $n \in \mathbb{N}$. The set of all cozero elements of L is denoted by $\mathrm{Coz}\, L$. It is a σ-frame, and it generates L if and only if L is completely regular. A frame homomorphism $h\colon L \to M$ is *coz-faithful* if it is one-one on $\mathrm{Coz}\, L$. This is the case precisely when it is *coz-codense*, meaning that for any $c \in \mathrm{Coz}\, L$, $h(c) = 1$ implies $c = 1$. A frame L is *normal* if whenever $a \vee b = 1$ in L, there exist $u, v \in L$ such that

$$u \wedge v = 0 \quad \text{and} \quad a \vee u = 1 = b \vee v.$$

The compact completely regular coreflection of L is denoted by βL. Let $\mathfrak{L}(\mathbb{R})$ denote the *frame of reals* (see [17, Chapter XIV]). As in spaces, a frame L is said to be *pseudocompact* if every frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$ is *bounded*, which is to say $f(p, q) = 1$ for some $p, q \in \mathbb{Q}$. There several characterizations of pseudocompact frames (see [3, 5]). A pertinent one here is that L is pseudocompact if and only if the homomorphism $\beta L \to L$ is coz-codense [19] .

2.2. **Coproducts of frames.** We shall have occasion to deal only with binary coproducts of frames. We write $L \oplus M$ for the coproduct of L and M. The actual construction of coproducts is described in detail in Chapter IV of [17]. We recall only the following. The elements $a \oplus b$ – which are called *basic elements* – generate $L \oplus M$ by joins. Thus, each $U \in L \oplus M$ is expressible as

$$U = \bigvee \{a \oplus b \mid (a, b) \in U\}.$$

The top element of $L \oplus M$ is $1_L \oplus 1_M$. If $0 \neq a \oplus b \leq u \oplus v$, then $a \leq u$ and $b \leq v$. The coproduct injections

$$L \xrightarrow{\ i\ } L \oplus M \xleftarrow{\ j\ } M$$

map $a \in L$ to $a \oplus 1$, and $b \in M$ to $1 \oplus b$. Hence, $i_*(a \oplus 1) = a$ and $j_*(1 \oplus b) = b$, for every $a \in L$ and every $b \in M$. If $h_\alpha\colon L_\alpha \to M_\alpha$ is a frame homomorphism for $\alpha = 1, 2$, then there is an induced frame homomorphism $h_1 \oplus h_2\colon L_1 \oplus L_2 \to M_1 \oplus M_2$ such that

$$(h_1 \oplus h_2)\Big(\bigvee_k (a_k \oplus b_k)\Big) = \bigvee_k \big(h_1(a_k) \oplus h_2(b_k)\big).$$

Its right adjoint maps as follows (see [6, Lemma 2]):

$$(h_1 \oplus h_2)_*(U) = \bigvee\{(h_1)_*(a) \oplus (h_2)_*(b) \mid a \oplus b \leq U\}.$$

2.3. **Sublocales.** We denote the lattice of all sublocales of a locale L by $\mathcal{S}(L)$. All joins and meets of sublocales will be computed in this lattice. Recall that the meet is simply set-theoretic intersection. The join is not set-theoretic union, in general. We shall not need to know how joins are calculated in $\mathcal{S}(L)$. The smallest sublocale of L is denoted by $\mathsf{O} = \{1\}$, and is called the *void* sublocale. We say a sublocale S *meets* a sublocale T if $S \cap T \neq \mathsf{O}$. Otherwise, S *misses* T.

The lattice $\mathcal{S}(L)$ is a *coframe*, which is to say for any $S \in \mathcal{S}(L)$ and any family $\{T_\alpha\}$ of sublocales, the distributive law below holds:

$$S \vee \bigwedge_\alpha T_\alpha = \bigwedge_\alpha (S \vee T_\alpha).$$

Our notation for open and closed sublocales is that of [17]. Thus, for any $a \in L$, $\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ denote the open and closed sublocales of L determined by a, respectively. We shall freely use properties of these sublocales, such as

$$\mathfrak{o}(0) = \mathfrak{c}(1) = \mathsf{O} \quad \text{and} \quad \mathfrak{o}(1) = \mathfrak{c}(0) = L,$$

and

$$\mathfrak{c}(a) \subseteq \mathfrak{o}(b) \quad \Longleftrightarrow \quad a \vee b = 1.$$

To say a sublocale of L is *complemented* means that it has a complement in $\mathcal{S}(L)$. The sublocales $\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complements of each other. A complemented sublocale is *linear*, which means that if S is a complemented sublocale of L, then

$$S \wedge \bigvee_\alpha T_\alpha = \bigvee_\alpha (S \wedge T_\alpha),$$

for all families $\{T_\alpha\}$ of sublocales of L.

The closure of a sublocale S will be denoted by $\overline{S}$. We remind the reader that

$$\overline{S} = \uparrow\Big(\bigwedge S\Big).$$

A sublocale is *dense* if its closure is the whole locale. The smallest dense sublocale of L will be denoted by $\mathfrak{B}L$. A sublocale S of L is *nowhere dense* if $S \wedge \mathfrak{B}L = \mathsf{O}$. The complement of a complemented nowhere dense sublocale is dense. The closure of a nowhere dense sublocale is nowhere dense (see [18, p. 278]). If $h\colon M \to L$ is a compactification of L, we shall identify L with the sublocale $h_*[L]$ of M, and, by abuse of language, say L is a sublocale of M.

3. A localic approach to weak pseudocompactness

Extending terminology from spaces, we say a sublocale of a frame L is a *G_δ-sublocale* if it is a countable intersection of open sublocales. Isbell [13] calls such sublocales "O_δ-sublocales" for reasons he explains. In particular, open sublocales are G_δ-sublocales, and every *zero-set sublocale*, that is, a sublocale of the form $\mathfrak{c}(c)$ for some $c \in \operatorname{Coz} L$, is a G_δ-sublocale [12, Remark 2.6]. We should caution the reader that the calculation in [12] is carried out in $\mathcal{S}(L)^{\mathrm{op}}$, so the joins there are intersections in $\mathcal{S}(L)$.

As in spaces, we say a sublocale of L is *G_δ-dense* in L if it meets every non-void G_δ-sublocale of L. As mentioned earlier, it is for this reason that the following definition was formulated in [10].

Definition 3.1. A frame L is *weakly pseudocompact* if there is a compactification $\gamma\colon \gamma L \to L$ of L such that the homomorphism γ is coz-codense.

The equivalence of G_δ-density of a subspace $A \subseteq X$ to coz-codensity of the homomorphism $\mathfrak{O}X \to \mathfrak{O}A$ makes it clear that this definition is conservative. Less clear is that this frame-theoretic definition agrees with the one that would be adopted verbatim from the spatial one as described in the Introduction. To show that they agree, we need a lemma which, although not generalizing the result of Ball and Walters-Wayland, brings to the fore what cannot be deduced from their result. Let us expatiate a little on this. The result we are alluding to does not generalize that of Ball and Walters-Wayland in that we do not state it for sublocales of any locale, but rather for sublocales of a compact regular locale. On the other hand, it does not follow from that of Ball and Walters-Wayland because even a compact Hausdorff space (when viewed as a locale) can have far more sublocales than it has subspaces.

Recall that if $S \in \mathcal{S}(L)$, then the corresponding frame surjection is

$$\nu_S\colon L \to S \quad \text{defined by} \quad \nu_S(a) = \bigwedge\{s \in S \mid s \geq a\}.$$

Observe that, for any $a \in L$ and $S \in \mathcal{S}(L)$,

$$S \subseteq \mathfrak{o}(a) \quad \Longleftrightarrow \quad \nu_S(a) = 1.$$

Lemma 3.2. *The following are equivalent for a sublocale S of a compact regular locale L.*

(1) *S is G_δ-dense in L.*
(2) *S meets every non-void zero-set sublocale of L.*
(3) *$\nu_S\colon L \to S$ is coz-codense.*

Proof. (1) ⇒ (2): This follows from the fact that every zero-set sublocale is a G_δ-sublocale.

(2) ⇒ (1): We show first that every non-void G_δ-sublocale of L is above some non-void zero-set sublocale. So consider a set $\{a_n \mid n \in \mathbb{N}\} \subseteq L$ such that the G_δ-sublocale $G = \bigcap\limits_{n=1}^{\infty} \mathfrak{o}(a_n)$ is non-void. Since L is compact regular, [17, Corollary VII 7.3] tells us that G is spatial. Since points of a sublocale are precisely the points of the locale belonging to the sublocale, there is a point $p \in \operatorname{Pt}(L)$ such that $p \in G$. Consequently, $p \in \mathfrak{o}(a_n)$ for each n, so that $\mathfrak{c}(p) = \{p, 1\} \subseteq \mathfrak{o}(a_n)$ and hence $p \vee a_n = 1$. Since compact regular locales are normal, for each n there exists $c_n \in \operatorname{Coz} L$ such that $c_n \leq p$ and $c_n \vee a_n = 1$ (see [1, Corollary 8.3.2]). Put $c = \bigvee c_n$, so that $c \in \operatorname{Coz} L$, $c \leq p < 1$, and

$c \vee a_n = 1$ for every n. Consequently, $\mathfrak{c}(c) \subseteq \mathfrak{o}(a_n)$ for every n, and so

$$\mathsf{O} \neq \mathfrak{c}(c) \subseteq \bigcap_{n=1}^{\infty} \mathfrak{o}(a_n).$$

Thus, if S meets every non-void zero-set sublocale, then it meets every non-void G_δ-sublocale, which shows that (2) implies (1).

(2) $\Leftrightarrow$ (3): This follows from the fact that, for any $c \in \operatorname{Coz} L$, $S \cap \mathfrak{c}(c) = \mathsf{O}$ if and only if $S \subseteq \mathfrak{o}(c)$ if and only if $\nu_S(c) = 1$. □

Remark 3.3. We can actually add another equivalent condition. As in spaces, define the *δ-closure* of a sublocale S of L to be the sublocale

$$\overline{S}^{\delta} = \bigcap \{\mathfrak{o}(a) \mid a \in \operatorname{Coz} L \text{ and } S \subseteq \mathfrak{o}(a)\}.$$

Then S is G_δ-dense in L if and only if $\overline{S}^{\delta} = L$.

In view of Lemma 3.2 we immediately have the following characterization.

Proposition 3.4. *A locale is weakly pseudocompact if and only if it is G_δ-dense in some compactification.*

As an illustration of the usefulness of this characterization, we shall prove that certain binary products of weakly pseudocompact locales are weakly pseudocompact. In spaces, every product of weakly pseudocompact spaces is weakly pseudocompact [11]. The proof relies, among other things, on the fact that the lattice of subspaces of a space is a Boolean algebra, so that, for instance, if U is an open subspace of a space X, then for any $x \in X$ we have $x \in U$ or $x \in X \smallsetminus U$. In locales this latter result of course fails. Indeed, if $0 < a \in L$ is not dense, then $0 \notin \mathfrak{o}(a)$ and $0 \notin \mathfrak{c}(a)$, even though $\mathfrak{c}(a)$ is the complement of $\mathfrak{o}(a)$. Points in a locale behave differently though, as shown in [15, Lemma 2.1]. We recall this result, but paraphrase it as follows.

Lemma 3.5. *For any $a \in L$,* $\operatorname{Pt}(L) \subseteq \mathfrak{o}(a) \cup \mathfrak{c}(a)$.

Let us recall another result; one that describes points in a coproduct in terms of points in the summands. It is taken from [8, Lemma 4.2] where it is proved for T_1-locales, and hence is valid for completely regular locales.

Lemma 3.6. $\operatorname{Pt}(L \oplus M) = \{(p \oplus 1) \vee (1 \oplus q) \mid p \in \operatorname{Pt}(L)$ *and* $q \in \operatorname{Pt}(M)\}$.

In the result that follows we shall place a condition on the locales L and M that all their non-void G_δ-sublocales be spatial. This is strictly weaker than requiring all sublocales to be spatial. Indeed, if a locale is compact regular, then all its non-void G_δ-sublocales are spatial [17, Corollary VII 7.3], but of course not all its sublocales need be spatial. As shown in [15], requiring all sublocales of L to be spatial is equivalent to requiring that $\mathcal{S}(L)^{\mathrm{op}}$ be spatial.

Theorem 3.7. *Let L and M be frames all of whose non-void G_δ-sublocales are spatial. If L and M are weakly pseudocompact, then $L \oplus M$ is weakly pseudocompact.*

Proof. Let γL and γM be compactifications of L and M, respectively, such that L is G_δ-dense in γL, and M is G_δ-dense in γM. Let $\gamma_L\colon \gamma L \to L$ and $\gamma_M\colon \gamma M \to M$ be the corresponding frame surjections. For brevity, write ρ_L and ρ_M for their right adjoints, so that $\rho_L\colon L \to \gamma L$ is the inclusion map $L \hookrightarrow \gamma L$, and similarly for ρ_M. By the result of Banaschewski and Vermeulen recalled in the Preliminaries, for any $U \in L \oplus M$,

$$\begin{aligned}(\gamma_L \oplus \gamma_M)_*(U) &= \bigvee\{\rho_L(a) \oplus \rho_M(b) \mid a \oplus b \le U\}\\ &= \bigvee\{a \oplus b \mid a \oplus b \le U\}\\ &= U,\end{aligned}$$

whence $(\gamma_L \oplus \gamma_M)_*[L \oplus M] = L \oplus M$, showing that $L \oplus M$ is a sublocale of the compact frame $\gamma L \oplus \gamma M$. It is a dense sublocale because L and M are dense sublocales of γL and γM, respectively. Thus, $\gamma L \oplus \gamma M$ is a compactification of $L \oplus M$.

Now consider a non-void G_δ-sublocale $\mathcal{G}$ of $\gamma L \oplus \gamma M$. Since the elements $a \oplus b$, for $a \in \gamma L$ and $b \in \gamma M$, generate $\gamma L \oplus \gamma M$, and $\gamma L \oplus \gamma M$ is spatial, we may assume, without loss of generality, that $\mathcal{G}$ is of the form $\mathcal{G} = \bigcap\limits_{n=1}^{\infty} \mathfrak{o}(a_n \oplus b_n) \neq \mathsf{O}$ for some sequences (a_n) and (b_n) of non-zero elements of γL and γM, respectively. By [17, Corollary VII 7.3], $\mathcal{G}$ is spatial, and hence contains a point of $\gamma L \oplus \gamma M$. Thus, by Lemma 3.6, there exist $p \in \mathrm{Pt}(\gamma L)$ and $q \in \mathrm{Pt}(\gamma M)$ such that

$$(p \oplus 1) \vee (1 \oplus q) \in \bigcap_{n=1}^{\infty} \mathfrak{o}(a_n \oplus b_n). \tag{†}$$

We claim that $p \in \bigcap_n \mathfrak{o}(a_n)$. If not, there in an index m such that $p \notin \mathfrak{o}(a_m)$. Then, by Lemma 3.5, $p \in \mathfrak{c}(a_m)$, which implies $a_m \le p$, whence

$$a_m \oplus b_m \le p \oplus 1 \le (p \oplus 1) \vee (1 \oplus q),$$

and therefore, by (†), $(p\oplus 1)\vee(1\oplus q) \in \mathfrak{c}(a_m \oplus b_m) \cap \mathfrak{o}(a_m \oplus b_m)$; which is false. Thus, $\bigcap_n \mathfrak{o}(a_n) \neq \mathsf{O}$. Similarly, $\bigcap_n \mathfrak{o}(b_n) \neq \mathsf{O}$. Since L and M are G_δ-dense in γL and γM, respectively,

$$L \cap \bigcap_{n=1}^{\infty} \mathfrak{o}(a_n) \neq \mathsf{O} \quad \text{and} \quad M \cap \bigcap_{n=1}^{\infty} \mathfrak{o}(b_n) \neq \mathsf{O}.$$

Since $L \cap \mathfrak{o}(a_n)$ is an open sublocale of L, and since

$$L \cap \bigcap_{n=1}^{\infty} \mathfrak{o}(a_n) = \bigcap_{n=1}^{\infty} \Big(L \cap \mathfrak{o}(a_n)\Big),$$

it follows that $L \cap \bigcap_n \mathfrak{o}(a_n)$ is a (non-void) G_δ-sublocale of L. Since non-void G_δ-sublocales of L are spatial, by hypothesis, there is an $s \in \mathrm{Pt}(\gamma L)$ such that $s \in L \cap \bigcap_n \mathfrak{o}(a_n)$. Similarly, there is a $t \in \mathrm{Pt}(\gamma M)$ such that $t \in M \cap \bigcap_n \mathfrak{o}(b_n)$. We claim that the point $(s\oplus 1)\vee(1\oplus t)$ of $\gamma L \oplus \gamma M$ belongs to $(L \oplus M) \cap \mathcal{G}$. To show that $(s \oplus 1) \vee (1 \oplus t) \in L \oplus M$, we need to exercise care because joins in

$L \oplus M$ are calculated differently from joins in $\gamma L \oplus \gamma M$. So let us denote binary join in $L \oplus M$ by $\sqcup$. Now, $(s \oplus 1) \sqcup (1 \oplus t)$ is a point in $L \oplus M$ by Lemma 3.6, and is therefore a point in $\gamma L \oplus \gamma M$. But $(s \oplus 1) \vee (1 \oplus t) \leq (s \oplus 1) \sqcup (1 \oplus t)$; so the two are equal by maximality of points in regular frames. Suppose, by way of contradiction, that $(s \oplus 1) \vee (1 \oplus t) \notin \mathcal{G}$. Then there is an index k such that $(s \oplus 1) \vee (1 \oplus t) \notin \mathfrak{o}(a_k \oplus b_k)$. Then, by Lemma 3.5, $(s \oplus 1) \vee (1 \oplus t) \in \mathfrak{c}(a_k \oplus b_k)$, which implies

$$(a_k \oplus 1) \wedge (1 \oplus b_k) = (a_k \oplus b_k) \leq (s \oplus 1) \vee (1 \oplus t),$$

and therefore

$$a_k \oplus 1 \leq (s \oplus 1) \vee (1 \oplus t) \quad \text{or} \quad 1 \oplus b_k \leq (s \oplus 1) \vee (1 \oplus t)$$

because $(s \oplus 1) \vee (1 \oplus t)$ is a point in $\gamma L \oplus \gamma M$. Suppose the former, and consider the coproduct injection $i \colon \gamma L \to \gamma L \oplus \gamma M$. Then

$$a_k = i_*(a_k \oplus 1) \leq i_*((s \oplus 1) \vee (1 \oplus t)) = s;$$

the latter equality holding because s and $i_*((s \oplus 1) \vee (1 \oplus t))$ are points of γL with

$$s = i_*(s \oplus 1) \leq i_*((s \oplus 1) \vee (1 \oplus t)).$$

Consequently, $s \in \mathfrak{c}(a_k)$, which is false because $s \in \mathfrak{o}(a_k)$. A similar contradiction is arrived at if we assume that $1 \oplus b_k \leq (s \oplus 1) \vee (1 \oplus t)$. Thus, $(s \oplus 1) \vee (1 \oplus t) \in \mathcal{G}$; and so $L \oplus M$ is G_δ-dense in $\gamma L \oplus \gamma M$, and is therefore weakly pseudocompact. □

Remark 3.8. This result is of course about the localic product of some weakly pseudocompact spaces; so one may wonder if it does not follow immediately from the spatial result. Let us explain via an analogous situation. If X and Y are topological spaces for which the (topological) product $X \times Y$ is pseudocompact, then the frame $\mathfrak{O}X \oplus \mathfrak{O}Y$ is pseudocompact, even if $\mathfrak{O}X \oplus \mathfrak{O}Y$ is not isomorphic to $\mathfrak{O}(X \times Y)$. The reason is that $\mathfrak{O}(X \times Y)$ is isomorphic to a dense sublocale of $\mathfrak{O}X \oplus \mathfrak{O}Y$ [17, Proposition IV 5.4.1], and a locale which has dense pseudocompact sublocale is pseudocompact [9, Lemma 4.3]. Now, a locale which has a dense weakly pseudocompact sublocale is not necessarily weakly pseudocompact. Indeed, let L be a weakly pseudocompact locale which is not pseudocompact. Then λL, the Lindelöf reflection of L in **Loc**, is not weakly pseudocompact, for if it were then by [10, Corollary 2.8] it would be compact, which would make L pseudocompact. So, with reference to the previous theorem, knowing that $X \times Y$ is weakly pseudocompact does not tell us anything about the weak pseudocompactness, or otherwise, of $\mathfrak{O}X \oplus \mathfrak{O}Y$.

4. Weak pseudocompactness and Baireness

In [14], Isbell defines Baire locales as follows. A sublocale S of a locale L is of *first category* if there are countably many nowhere dense sublocales K_n, $n \in \mathbb{N}$, such that $S \leq \bigvee_n K_n$. Otherwise, it is of *second category*. A *Baire* locale is one in which every non-void open sublocale is of second category. Observe that since open sublocales are complemented, an open sublocale is of first category if and only if it can be written as a join of countably many nowhere dense sublocales. This agrees with the characterization of Baire spaces as precisely those spaces in which every countable union of closed sets with empty interior has empty interior.

We mentioned in the Preliminaries that the closure of a nowhere dense sublocale is nowhere dense [18, p. 278]). Plewe's proof in [18] is in terms of sublocales. For the sake of completeness let us give a proof, but a different one carried out entirely in **Frm**. Recall from [7] that a quotient $\eta\colon L \to N$ of L is called *nowhere dense* if for every nonzero $a \in L$ there exists a nonzero $b \leq a$ in L such that $\eta(b) = 0$. This agrees with the localic notion in the sense that $\eta\colon L \to N$ is nowhere dense if and only if the sublocale $\eta_*[N]$ is nowhere dense in L. It is shown in [7, Lemma 3.2] that $\eta\colon L \to N$ is nowhere dense if and only if $\eta_*(0)$ is a dense element in L.

Lemma 4.1. *The closure of a nowhere dense quotient is nowhere dense.*

Proof. Let $\eta\colon L \to N$ be a nowhere dense quotient of L. The closure of this quotient is $\varphi\colon L \to {\uparrow}\eta_*(0)$, where φ is the map $x \mapsto x \vee \eta_*(0)$. The right adjoint of φ is the inclusion map ${\uparrow}\eta_*(0) \hookrightarrow L$. Since the zero of ${\uparrow}\eta_*(0)$ is $\eta_*(0)$, we have $\varphi_*(0_{{\uparrow}\eta_*(0)}) = \eta_*(0)$, which is a dense element in L. Therefore $\varphi\colon L \to {\uparrow}\eta_*(0)$ is nowhere dense. □

For use below, we also need to know that if $N \subseteq L$ are sublocales of M with N nowhere dense in L, and L dense in M, then N is nowhere dense in M. To prove this, recall first that if $g\colon A \to B$ is a dense onto frame homomorphism, then $g_*(b)$ is dense in A whenever b is dense in B. For, if $g_*(b) \wedge a = 0$ for any $a \in A$, then $0 = g\big(g_*(b) \wedge a\big) = b \wedge g(a)$, implying $g(a) = 0$ by the density of b, and hence $a = 0$ by the density of g.

Lemma 4.2. *Suppose that in the composite $M \xrightarrow{h} L \xrightarrow{\eta} N$ of quotient maps h is dense and η is nowhere dense. Then $\eta h\colon M \to N$ is nowhere dense.*

Proof. Since η is nowhere dense, $\eta_*(0)$ is dense, and hence $h_*(\eta_*(0))$ is dense as h is dense. But $h_*(\eta_*(0)) = (\eta h)_*(0)$, so the result follows. □

Now, in light of Lemma 4.1, if U is a first category sublocale of L, then there are countably many *closed* nowhere dense sublocales K_n with $U \leq \bigvee_n K_n$. Recall that if L is a sublocale of M, then the closed sublocales of L are precisely the sublocales $\mathfrak{c}_L(a) = L \cap \mathfrak{c}(a)$, for $a \in M$.

Proposition 4.3. *A weakly pseudocompact locale in which G_δ-sublocales are complemented is Baire.*

Proof. Let L be such a locale, and let M be a compactification of L such that L is G_δ-dense in M. Let U be a first category open sublocale of L. We aim to show that $U = \mathsf{O}$. Find countably many elements $(a_n)_{n\in\mathbb{N}}$ in M such that each $\mathfrak{c}_L(a_n)$ is nowhere dense in L, and $U \leq \bigvee_{n=1}^{\infty} \mathfrak{c}_L(a_n)$. We claim that, for any n, $\mathfrak{c}(a_n)$ is nowhere dense in M. If not, then there exists $x \in M$ such that $1 \neq$

$x^{**} \in \mathfrak{c}(a_n)$. Since L is dense in M, $\mathfrak{B}M \subseteq L$, hence $x^{**} \in L \cap \mathfrak{c}(a_n) = \mathfrak{c}_L(a_n)$, contradicting the fact that $\mathfrak{c}(a_n) \cap \mathfrak{B}M = \mathsf{O}$ since $\mathfrak{c}_L(a_n)$ is nowhere dense in M by Lemma 4.2. Consequently, $\mathfrak{o}(a_n)$ is a dense sublocale of M because complements of complemented nowhere dense sublocales are dense. Now pick an open sublocale $\mathfrak{U}$ of M such that $U = L \cap \mathfrak{U}$, and observe that $\mathfrak{U} \cap \bigcap\limits_{n=1}^{\infty} \mathfrak{o}(a_n)$ is a G_δ-sublocale of M with

$$\begin{aligned} L \cap \Big(\mathfrak{U} \cap \bigcap_{n=1}^{\infty} \mathfrak{o}(a_n)\Big) &= (L \cap \mathfrak{U}) \cap \Big(L \cap \bigcap_{n=1}^{\infty} \mathfrak{o}(a_n)\Big) \\ &= U \cap \Big(L \cap \bigcap_{n=1}^{\infty} \mathfrak{o}(a_n)\Big) \\ &= U \wedge \bigwedge_{n=1}^{\infty} \mathfrak{o}_L(a_n) \\ &\leq \Big(\bigvee_{k=1}^{\infty} \mathfrak{c}_L(a_k)\Big) \wedge \Big(\bigwedge_{n=1}^{\infty} \mathfrak{o}_L(a_n)\Big) \\ &= \bigvee_{k=1}^{\infty} \Big(\mathfrak{c}_L(a_k) \wedge \bigwedge_{n=1}^{\infty} \mathfrak{o}_L(a_n)\Big) \quad \text{since } \bigwedge_{n=1}^{\infty} \mathfrak{o}_L(a_n) \text{ is complemented} \\ &\leq \bigvee_{k=1}^{\infty} (\mathfrak{c}_L(a_k) \wedge \mathfrak{o}_L(a_k)) \\ &= \mathsf{O}. \end{aligned}$$

Since L is G_δ-dense in M, it follows that $\mathfrak{U} \cap \bigcap_n \mathfrak{o}(a_n) = \mathsf{O}$. But now $\bigcap_n \mathfrak{o}(a_n)$ is a dense sublocale of M missing the open sublocale $\mathfrak{U}$, so we must have $\mathfrak{U} = \mathsf{O}$, which then implies $U = \mathsf{O}$. Therefore L is Baire. □

Remark 4.4. The calculation in the foregoing proof, starting with the equality

$$L \cap \Big(\mathfrak{U} \cap \bigcap_{n=1}^{\infty} \mathfrak{o}(a_n)\Big) = (L \cap \mathfrak{U}) \cap \Big(L \cap \bigcap_{n=1}^{\infty} \mathfrak{o}(a_n)\Big),$$

can be used to show that if L is a G_δ-dense sublocale of a Baire locale, and G_δ-sublocales of L are complemented, then L is also Baire.

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