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Jungck theorem for triangular maps and related results

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Abstract. We prove that a continuous triangular map G of the n-dimensional cube I^n has only fixed points and no other periodic points if and only if G has a common fixed point with every continuous triangular map F that is nontrivially compatible with G. This is an analog of Jungck theorem for maps of a real compact interval. We also discuss possible extensions of Jungck theorem, Jachymski theorem and some related results to more general spaces. In particular, the spaces with the fixed point property and the complete invariance property are considered.

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1. Introduction

Continuous selfmaps of a real compact interval for which every periodic point is a fixed point were studied by many authors (see e.g. [4], [17], [3], [9], [7]). Recently some of these results were investigated from the point of view of triangular maps (cf. [6]). The class of triangular maps for which $\operatorname{Per} G = \operatorname{Fix} G$ admits much stranger behavior than the analogous class of one–dimensional maps (see e.g. [12], [5]). Among others, the pointwise convergence of the sequence of the iterations does not characterize this class of triangular maps. Nevertheless, the concept of compatible mappings introduced in [10] allows us to describe not only the class of selfmaps of the interval (cf. [9]) but, as we are going to show, also the class of triangular maps.

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Throughout the paper all maps are assumed to be continuous even if we do not state it explicitly. If X and Y are topological spaces, $C(X,Y)$ denotes the class of continuous maps from X to Y. Let I be a real compact interval, say the unit interval $[0,1]$. Let n be a positive integer. In the n-dimensional cube I^n we will use the Euclidean metric. By a *triangular map* we mean a continuous map $G : I^n \longrightarrow I^n$ of the form

$$G(x_1, x_2, \ldots, x_n) = (g_1(x_1), g_2(x_1, x_2), \ldots, g_n(x_1, x_2, \ldots, x_n)),$$

shortly $G = (g_1, g_2, \ldots, g_n)_\triangle$. The map g_1 is called the *basis map* of G. The class of all triangular maps of I^n will be denoted by $C_\triangle(I^n, I^n)$. If $n = 1$ then $C_\triangle(I,I) = C(I,I)$. By $I(v_1, v_2, \ldots, v_k) := (v_1, v_2, \ldots, v_k) \times I^{n-k}$ we denote the *fibre* over the point $(v_1, v_2, \ldots, v_k)$ and the map $G_{v_1,v_2,\ldots,v_k} \in C_\triangle(I^{n-k}, I^{n-k})$ defined by

$$\begin{aligned} G_{v_1,v_2,\ldots,v_k}(x_{k+1}, \ldots, x_n) \quad := \quad (\quad & g_{k+1}(v_1, v_2, \ldots, v_k, x_{k+1}), \\ & \ldots, \\ & g_n(v_1, v_2, \ldots, v_k, x_{k+1}, \ldots, x_n) \ ) \end{aligned}$$

is the *fibre map* of G working in the fibre $I(v_1, v_2, \ldots, v_k)$.

Continuous interest in triangular maps is, among others, caused by the fact that they display a kind of a dualism. On one hand, they are close to one–dimensional maps in the sense that some important dynamical features extend to triangular maps. For instance, Sharkovsky's Theorem holds for them (see [11]). On the other hand, they already display other important properties which are typical for higher dimensional maps and cannot be found in the one–dimensional maps. For instance, there are triangular maps with positive topological entropy having only periodic points whose periods are (all) powers of two (see [12]). For more information on triangular maps see, e.g., [11], [12], [13], [1], [5]. A triangular map defines a discrete dynamical system that is, especially in ergodic theory, sometimes called a skew product of (one–dimensional or less dimensional) dynamical systems. As far as the authors know, in fixed point theory triangular maps have not been studied yet.

Let $\operatorname{Fix} G$ or $\operatorname{Per} G$ denote the set of all fixed points or periodic points of G, respectively. Recall (cf. [8]) that selfmaps f and g of a set X are *compatible* if they commute on the set of their *coincidence points*, i.e., on the set $\operatorname{Coin}(f,g) := \{x \in X : f(x) = g(x)\}$. If f and g are compatible and $\operatorname{Coin}(f,g)$ is nonempty, we will say that f and g are *nontrivially compatible* (cf. [9]).

Note that throughout the paper we will often prefer another but, as one can easily show, equivalent definition of nontrivial compatibility: f and g are nontrivially compatible if and only if $\operatorname{Coin}(f,g)$ is nonempty and for every $x \in \operatorname{Coin}(f,g)$, the whole trajectories of x under f and g coincide (i.e., $f^n(x) = g^n(x)$ for every $n \in \mathbb{N}$).

G. Jungck proved the following (cf. [9, Theorem 3.6]):

Theorem 1.1 (Jungck Theorem). *A map $g \in C(I,I)$ has a common fixed point with every map $f \in C(I,I)$ which is nontrivially compatible with g if and only if* $\operatorname{Per} g = \operatorname{Fix} g$.

The main aim of this paper is to show that the analog of this theorem applies to triangular maps (see Theorem 2.7).

In Section 3 we discuss possible extensions of Jungck theorem and some related results for general continuous maps to more general spaces (see Theorems 3.2, 3.3, and 3.4).

Before going to our results notice that the fact that Jungck theorem holds for triangular maps is a new illustration of their dualistic character when they are compared with selfmaps of I. In fact, in some other aspects of the fixed point theory they *differ* from interval maps: for instance, in [14] one can find an example of a triangular map G in I^2 and a compact subinterval $J \subset I$ such that $G(J^2) \supset J^2$ but G has no fixed point in J^2.

2. Jungck theorem for triangular maps

Lemma 2.1. *Let $g \in C(I,I)$. If $\operatorname{Per} g = \operatorname{Fix} g$ and $x_0 \in \operatorname{Fix} g$, then there is a map $f \in C(I,I)$ and $\alpha, \beta \in I$ such that $\alpha \le x_0 \le \beta$ and*

$$\operatorname{Coin}(f,g) = \{\alpha, x_0, \beta\} \subseteq g^{-1}(x_0).$$

Proof. We can additionally suppose that $x_0 \in \operatorname{int} I$ (if $x_0 = 0$ or $x_0 = 1$ then the proof is similar). First we define the function f on the right side of x_0. Take

$$\beta := \sup\{x \in I : g(x) = x_0\}.$$

Then $x_0 \le \beta$. Put $f(x) = x_0$ for every $x \in [\beta, 1]$. Let $f|_{[x_0,\beta]}$ be an arbitrary continuous function such that $f(x_0) = f(\beta) = x_0$ and $1 \ge f(x) > g(x)$ for all $x \in (x_0, \beta)$. To see that such a map exists it is enough to realize that $g(x) < 1$ for every $x \in [x_0, \beta]$. In fact, if $g(z) = 1$ for some $z \in [x_0, \beta]$ then $g([x_0, z]) \cap g([z, \beta]) \supseteq [x_0, z] \cup [z, \beta]$, i.e., g has a 2-horseshoe. This implies that g has a periodic point of period greater than 1 (g has even positive topological entropy [2, Proposition 4.3.2]). This contradicts the assumption on g.

Similarly we proceed on the left side of x_0 by putting $\alpha := \inf\{x \in I : g(x) = x_0\}$, $f(x) = x_0$ for every $x \in [0, \alpha]$ and taking into account that $g(x) > 0$ for all $x \in [\alpha, x_0]$. The obtained α, β and f fulfill all desired conditions. $\square$

For $n \ge m$ let $\pi_m : I^n \longrightarrow I^m$ be the projection of the space I^n onto the space I^m defined by $\pi_m(x_1, \dots, x_n) = (x_1, \dots, x_m)$. In the notation π_m we suppressed n, since it will always be clear what is the dimension of the domain of π_m.

The following extension lemma is a generalization of [13, Lemma 1]. We will use in it the notation $C_\triangle(K, I^n)$ for the set of triangular maps from K into I^n which are defined analogously as triangular maps from I^n into I^n.

Lemma 2.2. *Let $K \subseteq I^n$ be a compact set, $\Phi = (\phi_1, \phi_2, \dots, \phi_n)_\triangle \in C_\triangle(K, I^n)$. Then there is a map $F = (f_1, f_2, \dots, f_n)_\triangle \in C_\triangle(I^n, I^n)$ such that $F|_K = \Phi$.*

Moreover, we can prescribe any of the maps f_m, $m = 1, 2, \dots, n$, requiring only that $f_m \in C(I^m, I)$ be an extension of $\phi_m \in C(\pi_m(K), I)$.

Proof. Since $\Phi \in C(K, I^n)$, for every i the map ϕ_i is continuous if we consider it as a map $K \longrightarrow I$. Note that there is no sense in extending each ϕ_i to a map f_i defined on the whole I^n and putting $F = (f_1, f_2, \dots, f_n)$ because then $F \in C(I^n, I^n)$, in general, would not be *triangular*. But the map ϕ_i depends only on the first i variables and so we can consider it also as a map $\pi_i(K) \longrightarrow I$. We are going to show that, due to the compactness of K, ϕ_i is still continuous, i.e., $\phi_i \in C(\pi_i(K), I)$. For $i = n$ this is trivial since $\phi_n \in C(K, I)$. For $i = 1, 2, \dots, n-1$ this follows from the repeated use of the following

Claim 2.3. *If $M \subseteq I^k$ is a compact set and $\Psi = (\psi_1, \psi_2, \dots, \psi_k)_\triangle \in C_\triangle(M, I^k)$ then $\psi_k \in C(M, I)$, $\pi_{k-1}(M) \subseteq I^{k-1}$ is a compact set and the map Ψ^* defined as $\Psi^* = (\psi_1, \psi_2, \dots, \psi_{k-1})_\triangle$ belongs to $C_\triangle(\pi_{k-1}(M),\ I^{k-1})$.*

Proof. (of the claim). The compactness of $\pi_{k-1}(M)$ is obvious. We trivially have $\psi_k \in C(M, I)$ and so we need only prove that the map

$$\Psi^* = (\psi_1, \psi_2, \dots, \psi_{k-1})_\triangle : \pi_{k-1}(M) \longrightarrow I^{k-1}$$

is continuous. Assume, on the contrary, that Ψ^* is discontinuous at a point $z \in \pi_{k-1}(M)$. Then there is a sequence of points $z_i \in \pi_{k-1}(M)$ for which $\lim_{i\to\infty} z_i = z$ and the sequence $(\Psi^*(z_i))$ does not tend to $\Psi^*(z)$. Since I^{k-1} is compact, there is a convergent subsequence of $(\Psi^*(z_i))$. Without loss of generality we may assume that $\lim_{i\to\infty} \Psi^*(z_i) = a \neq \Psi^*(z)$. Take points $v_i \in I$ such that $(z_i, v_i) \in M$. There is a converging subsequence of (v_i). We may assume that $\lim_{i\to\infty} v_i = v$. Then $(z_i, v_i) \longrightarrow (z, v)$. Since M is closed, $(z, v) \in M$. The point $\Psi(z, v)$ belongs to the fibre $I(\Psi^*(z))$ and the sequence $(\Psi^*(z_i))$ does not converge to $\Psi^*(z)$. So $(\Psi(z_i, v_i))$ does not converge to $\Psi(z, v)$, and we have a contradiction with the continuity of Ψ. Thus Claim 2.3 is proved.

So we have proved that for $m = 1, 2, \dots, n$, the map $\phi_m : \pi_m(K) \longrightarrow I$ is continuous. By Tietze extension theorem the functions $\phi_m \in C(\pi_m(K), I)$, $1 \le m \le n$ have continuous extensions $f_m \in C(I^m, I)$, $1 \le m \le n$, respectively. Now it suffices to put $F = (f_1, f_2, \dots, f_n)_\triangle$ with arbitrary such extensions. □

Lemma 2.4. *Let $G = (g_1, g_2, \dots, g_n)_\triangle \in C_\triangle(I^n, I^n)$, $n \ge 2$ have a common fixed point with every triangular map which is nontrivially compatible with G. Then*

(i) $\operatorname{Per} g_1 = \operatorname{Fix} g_1$, *and*
(ii) *for every $a_1 \in \operatorname{Fix} g_1$, G_{a_1} has a common fixed point with every triangular map which is nontrivially compatible with G_{a_1}.*

Proof. (i) To shorten the notation we will write $y = (x_2, \dots, x_n)$. Fix a map $f \in C(I, I)$ which is nontrivially compatible with g_1. Putting

$$F(x_1, y) := (f(x_1), g_2(x_1, x_2), \dots, g_n(x_1, y)) \quad \text{for every } (x, y) \in I^n,$$

we see that

$$\operatorname{Coin}(F, G) = \operatorname{Coin}(f, g_1) \times I^{n-1}$$

is nonempty and for $(x_1, y) \in \operatorname{Coin}(F, G)$

$$\begin{aligned} F(G(x_1, y)) &= (f(g_1(x_1)), g_2(g_1(x_1), g_2(x_1, x_2)), \dots, g_n(g_1(x_1), \dots)) \\ &= (g_1(f(x_1)), g_2(f(x_1), g_2(x_1, x_2)), \dots, g_n(f(x_1), \dots)) \\ &= G(F(x_1, y)). \end{aligned}$$

Thus, by the assumption of the lemma, $\mathrm{Fix}\,F \cap \mathrm{Fix}\,G \neq \varnothing$. Let $(\widetilde{x_1}, \dots, \widetilde{x_n})$ be a common fixed point of F and G. Then $f(\widetilde{x_1}) = g_1(\widetilde{x_1}) = \widetilde{x_1}$, so $\mathrm{Fix}\,f \cap \mathrm{Fix}\,g_1 \neq \varnothing$ and by Jungck theorem we get $\mathrm{Per}\,g_1 = \mathrm{Fix}\,g_1$.

(*ii*) Let $a_1 \in \mathrm{Fix}\,g_1$ and let $\Phi \in C_\triangle(I^{n-1}, I^{n-1})$ be nontrivially compatible with G_{a_1}. We will prove that Φ and G_{a_1} have a common fixed point. Take $y_0 \in \mathrm{Coin}(G_{a_1}, \Phi)$. By Lemma 2.1 there exists a map $f_1 \in C(I, I)$ and $\alpha, \beta \in I$ such that $\alpha \leq a_1 \leq \beta$ and $\mathrm{Coin}(f_1, g_1) = \{\alpha, a_1, \beta\} \subseteq g_1^{-1}(a_1)$. We are going to define a triangular map $F \in C_\triangle(I^n, I^n)$. We start with the triangular selfmap of the compact set $\{\alpha, a_1, \beta\} \times I^{n-1}$ which sends (a_1, y) to $(a_1, \Phi(y))$ and (x_1, y) with $x_1 \in \{\alpha, \beta\} \setminus \{a_1\}$ to (a_1, y_0). Using Lemma 2.2 we can extend this map to a triangular map F whose basis map is the above mentioned map f_1. It is obvious that F is nontrivially compatible with G. Therefore $\mathrm{Fix}\,F \cap \mathrm{Fix}\,G \neq \varnothing$. From the definition of F we get that $\mathrm{Fix}\,\Phi \cap \mathrm{Fix}\,G_{a_1} \neq \varnothing$. □

In the sequel we will use the following simple properties of triangular maps: If $F \in C_\triangle(I^n, I^n)$, then

$$\pi_1(\mathrm{Fix}\,F) = \mathrm{Fix}\,f_1 \text{ and } \pi_1(\mathrm{Per}\,F) = \mathrm{Per}\,f_1.$$

Lemma 2.5. *If a map $G = (g_1, g_2, \dots, g_n)_\triangle \in C_\triangle(I^n, I^n)$, $n \geq 1$, has a common fixed point with every triangular map F which is nontrivially compatible with G, then* $\mathrm{Per}\,G = \mathrm{Fix}\,G$.

Proof. For $n = 1$ this holds by Jungck theorem. So let $n \geq 2$. Take $(a_1, a_2, \dots, a_n) \in \mathrm{Per}\,G$. By Lemma 2.4 we get $a_1 \in \mathrm{Per}\,g_1 = \mathrm{Fix}\,g_1$ which means that $g_1(a_1) = a_1$. Then $(a_2, \dots, a_n) \in \mathrm{Per}\,G_{a_1}$, where

$$G_{a_1} = (g_2(a_1, \cdot), g_3(a_1, \cdot, \cdot), \dots, g_n(a_1, \cdot, \cdot, \dots, \cdot))_\triangle.$$

Moreover, G_{a_1} has a common fixed point with every triangular map which is nontrivially compatible with G_{a_1}.

Now applying Lemma 2.4 to the map G_{a_1} we obtain that $a_2 \in \mathrm{Fix}\; g_2(a_1, \cdot)$, so $g_2(a_1, a_2) = a_2$. It means that $(a_3, \dots, a_n) \in \mathrm{Per}\,G_{a_1 a_2}$. Proceeding in this way we see that also $a_{n-1} \in \mathrm{Fix}\,g_{n-1}(a_1, a_2, \dots, a_{n-2}, \cdot)$ and $a_n \in \mathrm{Per}\,G_{a_1 a_2 \dots a_{n-1}}$. Moreover, $G_{a_1 a_2 \dots a_{n-1}}$ is an interval selfmap that has a common fixed point with every continuous map which is nontrivially compatible with $G_{a_1 a_2 \dots a_{n-1}}$. By Jungck theorem a_n belongs to $\mathrm{Fix}\,G_{a_1 a_2 \dots a_{n-1}}$, whence $g_n(a_1, a_2, \dots, a_{n-1}, a_n) = a_n$. Thus we have proved that $G(a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, a_n)$. Therefore $\mathrm{Per}\,G = \mathrm{Fix}\,G$. □

Before proving the converse statement fix some notation. When $F \in C_\triangle\,(I^n, I^n)$ and $(x_1, \dots, x_n) \in I^n$, the symbol $\omega_F(x_1, \dots, x_n)$ will denote the ω-limit set of the point $(x_1, \dots, x_n)$ under F, i.e., the set of all limit points of the trajectory $(F^k(x_1, \dots, x_n))_{k=1}^\infty$.

Further recall that if X is a Hausdorff topological space and $f, g \in C(X, X)$ are nontrivially compatible then $\mathrm{Coin}(f, g)$ is closed and, as one can easily show, for every $x \in \mathrm{Coin}(f, g)$ we have $\omega_f(x) = \omega_g(x) \subseteq \mathrm{Coin}(f, g)$.

Lemma 2.6. *Assume that $G \in C_\triangle(I^n, I^n)$, $n \geq 1$. If* $\mathrm{Per}\,G = \mathrm{Fix}\,G$, *then G has a common fixed point with every triangular map F which is nontrivially compatible with G.*

Proof. For $n = 1$ this holds by Jungck theorem. Assume that the lemma holds for some $n \geq 1$ and take any $G = (g_1, g_2, \dots, g_{n+1})_\triangle \in C_\triangle(I^{n+1}, I^{n+1})$ with $\operatorname{Per} G = \operatorname{Fix} G$ and any $F = (f_1, f_2, \dots, f_{n+1})_\triangle \in C_\triangle(I^{n+1}, I^{n+1})$ which is nontrivially compatible with G. To finish the proof we need to show that G and F have a common fixed point. To this end take a point $(a_1, a_2, \dots, a_{n+1}) \in \operatorname{Coin}(F, G)$. Since $\operatorname{Per} G = \operatorname{Fix} G$ we have also $\operatorname{Per} g_1 = \operatorname{Fix} g_1$. Therefore the sequence $(g_1^k(a_1))_{k=1}^\infty$ tends to some point $v \in \operatorname{Fix} g_1$ (see [4], [17] or [3]). Hence

$$\omega_G(a_1, a_2, \dots, a_{n+1}) = \omega_F(a_1, a_2, \dots, a_{n+1}) \subseteq I(v).$$

The set on the left side of this inclusion is nonempty and is a subset of $\operatorname{Coin}(F, G)$. So, $\operatorname{Coin}(F, G)$ contains a point of the form $(v, b_2, \dots, b_{n+1})$. Since $v \in \operatorname{Fix} g_1$, this implies that also $v \in \operatorname{Fix} f_1$. Now consider the maps G_v and F_v from $C_\triangle(I^n, I^n)$. These maps are compatible because F and G are compatible and the fibre $I(v)$ is mapped into itself both by F and G. Moreover, $(b_2, \dots, b_{n+1}) \in \operatorname{Coin}(F_v, G_v)$ and so F_v and G_v are nontrivially compatible. If we finally take into account that $\operatorname{Per} G = \operatorname{Fix} G$ and so $\operatorname{Per} G_v = \operatorname{Fix} G_v$, we can apply the induction hypothesis to the map G_v to get that there is a point

$$(c_2, \dots, c_{n+1}) \in \operatorname{Fix} G_v \cap \operatorname{Fix} F_v.$$

Then

$$(v, c_2, \dots, c_{n+1}) \in \operatorname{Fix} G \cap \operatorname{Fix} F$$

which ends the proof. □

From Lemma 2.5 and Lemma 2.6 we immediately get the following generalization of Jungck theorem:

Theorem 2.7. [Jungck theorem for triangular maps.] *For each $n \geq 1$, a map $G \in C_\triangle(I^n, I^n)$ has a common fixed point with every map $F \in C_\triangle(I^n, I^n)$ which is nontrivially compatible with G if and only if* $\operatorname{Per} G = \operatorname{Fix} G$.

This result allows us also to extend the list of conditions which characterize the triangular maps with fixed points as unique periodic points (cf. [6, Corollary 3.1]):

Corollary 2.8. *Let $G \in C_\triangle(I^n, I^n)$. Then the following conditions are equivalent:*

(i) $\operatorname{Per} G = \operatorname{Fix} G$,
(ii) $C \cap \operatorname{Fix} G \neq \varnothing$ *for any nonempty closed set $C \subseteq I^n$ such that $G(C) \subseteq C$,*
(iii) *G has a common fixed point with every map $F \in C_\triangle(I^n, I^n)$ that commutes with G on* $\operatorname{Fix} F$,
(iv) *G has a common fixed point with every map $F \in C(I^n, I^n)$ that commutes with G on* $\operatorname{Fix} F$,
(v) *G has a common fixed point with every map $F \in C_\triangle(I^n, I^n)$ which is nontrivially compatible with G.*

3. On a generalization of results related to Jachymski theorem and Jungck theorem

Looking at conditions in Corollary 2.8 it seems to be natural to ask whether some of the implications do not hold for continuous (not necessarily triangular) selfmaps of more general spaces than the n-dimensional cube. In this section we give answers to some questions of this type.

First recall that J. Jachymski proved the following (cf. [7, Proposition 1]):

Theorem 3.1. [Jachymski theorem.] *Let A be a nonempty compact and convex subset of a normed linear space and let g be a continuous selfmap of A. Then the following conditions are equivalent:*

(i) $C \cap \mathrm{Fix}\, g \neq \varnothing$ *for any nonempty closed set* $C \subseteq A$ *such that* $g(C) \subseteq C$,

(ii) g *has a common fixed point with every map* $f \in C(A, A)$ *that commutes with* g *on* $\mathrm{Fix}\, f$.

We are going to give a more general formulation of this result. To this end recall after L. E. Ward (cf. [18]) the following definition. A subset F of a topological space X is a fixed point set of X if there exists a continuous selfmap of X whose set of fixed points is exactly F. The space X has the *complete invariance property* (CIP) if each of its nonempty closed subsets is a fixed point set.

L. E. Ward proved that a convex subset of a normed linear space has the CIP (cf. [18, Corollary 1.1]; for other examples of topological spaces with the CIP see [15] or [16]). Further, by Schauder theorem the space A from Jachymski theorem has the fixed point property (FPP). So, the following is a generalization of Jachymski theorem:

Theorem 3.2. [Generalization of Jachymski theorem.] *Let X be a Hausdorff topological space with the FPP and the CIP and let g be a continuous selfmap of X. Then the following conditions are equivalent:*

(i) $C \cap \mathrm{Fix}\, g \neq \varnothing$ *for any nonempty closed set* $C \subseteq X$ *such that* $g(C) \subseteq C$,

(ii) g *has a common fixed point with every map* $f \in C(X, X)$ *that commutes with* g *on* $\mathrm{Fix}\, f$.

Moreover, $(i) \Longrightarrow (ii)$ *does not require the CIP and* $(ii) \Longrightarrow (i)$ *does not require that X be Hausdorff and have the fixed point property.*

Proof. $(i) \Longrightarrow (ii)$. Let a continuous map $f : X \longrightarrow X$ commute with g on $\mathrm{Fix}\, f$. Then $g(\mathrm{Fix}\, f) \subseteq \mathrm{Fix}\, f$. Further, the set $\mathrm{Fix}\, f$ is nonempty since X has the fixed point property and closed since X is Hausdorff. Therefore by *(i)*, $\mathrm{Fix}\, f \cap \mathrm{Fix}\, g \neq \varnothing$.

$(ii) \Longrightarrow (i)$. Let C be a nonempty closed subset of X such that $g(C) \subseteq C$. By the CIP of X, there exists a continuous map $f : X \longrightarrow X$ with $\mathrm{Fix}\, f = C$. For $x \in C$, $g(f(x)) = g(x)$ since $f(x) = x$, and $f(g(x)) = g(x)$ since $g(C) \subseteq C$ and $\mathrm{Fix}\, f = C$. Thus f and g commute on $\mathrm{Fix}\, f$. By *(ii)*, $\mathrm{Fix}\, f \cap \mathrm{Fix}\, g \neq \varnothing$, i.e., $C \cap \mathrm{Fix}\, g \neq \varnothing$. □

Theorem 3.3. *Let X be a Hausdorff topological space with the CIP and let g be a continuous selfmap of X. Then the following conditions are equivalent:*

(i) *$C \cap \operatorname{Fix} g \neq \varnothing$ for any nonempty closed set $C \subseteq X$ such that $g(C) \subseteq C$,*

(ii) *g has a common fixed point with every map $f \in C(X,X)$ which is nontrivially compatible with g.*

Moreover, (i) $\Longrightarrow$ (ii) does not require the CIP and (ii) $\Longrightarrow$ (i) does not require that X be Hausdorff.

Proof. (i) $\Longrightarrow$ (ii). Let $f \in C(X,X)$ be nontrivially compatible with g. The set $\operatorname{Coin}(f,g)$ is nonempty and, since X is Hausdorff, closed. Moreover, $g(\operatorname{Coin}(f,g)) \subseteq \operatorname{Coin}(f,g)$. Indeed, if $x \in \operatorname{Coin}(f,g)$ then $f(x) = g(x)$ and, by what was said in Introduction, $f^2(x) = g^2(x)$. Then $f(g(x)) = f^2(x) = g^2(x)$, so $g(x) \in \operatorname{Coin}(f,g)$. By *(i)*, $\operatorname{Coin}(f,g) \cap \operatorname{Fix} g \neq \varnothing$. Let $x_0 \in \operatorname{Fix} g$ be a coincidence point of f and g. Then $x_0 = g(x_0) = f(x_0)$, which means that $\operatorname{Fix} f \cap \operatorname{Fix} g \neq \varnothing$.

(ii) $\Longrightarrow$ (i). Let $C \subseteq X$ be a nonempty closed set such that $g(C) \subseteq C$. By the assumption, there is a map $h \in C(X,X)$ with $\operatorname{Fix} h = C$. Put $f := h \circ g$. Then $C \subseteq \operatorname{Coin}(f,g)$. Take any $x \in \operatorname{Coin}(f,g)$. Then $g(x) \in C$ (since otherwise we would have $f(x) = h(g(x)) \neq g(x)$) and so $g^n(x) \in C$ for all $n \in \mathbb{N}$. From this and the facts that $C \subseteq \operatorname{Coin}(f,g)$ and $f(x) = g(x)$ we get $f^n(x) = g^n(x)$ for all $n \in \mathbb{N}$. Hence f and g are nontrivially compatible. By *(ii)*, $\operatorname{Fix} f \cap \operatorname{Fix} g \neq \varnothing$. Let x_0 be a common fixed point of f and g. Then $x_0 = f(x_0) = h(g(x_0)) = h(x_0)$, so $x_0 \in C$. Therefore $C \cap \operatorname{Fix} g \neq \varnothing$. □

If X is a Hausdorff space then any periodic orbit of a continuous selfmap of X, being finite, is a closed set. Thus the condition (i) from Theorem 3.3 implies $\operatorname{Per} g = \operatorname{Fix} g$ (the converse is not true: for instance, let X be the unit disc, C its circumference and g an irrational rotation). Hence in a Hausdorff space with the CIP the condition (ii) from Theorem 3.3 implies $\operatorname{Per} g = \operatorname{Fix} g$. But it turns out that this is true even under weaker assumptions on the space than to be Hausdorff and to have the CIP.

Theorem 3.4. [Generalization of a part of Jungck theorem).] *Let X be a topological space with the property that for every nonempty finite set $A \subseteq X$ there exists a map $h \in C(X,X)$ such that $\operatorname{Fix} h = A$. If a map $g \in C(X,X)$ has a common fixed point with every map $f \in C(X,X)$ which is nontrivially compatible with g, then $\operatorname{Per} g = \operatorname{Fix} g$.*

Proof. Suppose on the contrary that g has a periodic orbit A of period greater than one. Take $h \in C(X,X)$ with $\operatorname{Fix} h = A$ and define $f := h \circ g$.

Clearly, $f \in C(X,X)$ and $A \subseteq \operatorname{Coin}(f,g)$. Take any $x \in \operatorname{Coin}(f,g)$. Then $g(x) \in A$ since otherwise we would have $f(x) = h(g(x)) \neq g(x)$. Hence $f^n(x) = g^n(x)$ for all $n \geq 1$. Thus we have proved that f is nontrivially compatible with g. But g and f have no common fixed point since if $g(x) = x$ then $x \notin A$ and so $f(x) = h(g(x)) = h(x) \neq x$. This contradiction finishes the proof. □

The example below shows that the converse implication is not true.

Example 3.5. Let X be the unit disc and g an irrational rotation of X. Then X has the property from the previous theorem and $\operatorname{Per} g = \operatorname{Fix} g$. Nevertheless, there is a map $f \in C(X,X)$ which is nontrivially compatible with g but has no common fixed point with g. To see this, take any $h \in C(X, X)$ whose fixed point set is the circumference of X and put $f := h \circ g$. □

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