

Directed GF-spaces

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ABSTRACT. In this paper we introduce the concept of directed fractal structure, which is a generalization of the concept of fractal structure (introduced by the authors). We study the relation with transitive quasiuniformities and inverse limits of posets. We define the concept of GF-compactification and apply it to prove that the Stone-Čech compactification can be obtained as the GF-compactification of the directed fractal structure associated to the Pervin quasi-uniformity.

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1. INTRODUCTION

Looking for a topological generalization of the notion of self-similar sets (commonly known as "fractals"), the authors introduced in [2] the notion of GF-space, where the way of looking at self-similar sets like fixed points of iterated function systems was replaced by looking at them as a family of coverings recursively defined. The success of this point of view in dealing with many problems of General Topology (and not only those of self-similar sets) has motivated the authors to go one step further in their generalization.

In section 2 of this paper we give away the countability of the family of coverings and we introduce the notion of directed GF-space (in the same line of thought that gives nets from sequences or inverse spectra from inverse sequences). The categorical issues (relation with quasiuniformities or inverse limits) follow closely those of the countable case, but now the range of topological spaces we can consider grows. The most startling consequence of all the work done in sections 2, 3 and 4 is the relation between Pervin quasi-uniformity

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and Stone-Čech compactification in sections 5 and 6. We also describe the directed fractal structures induced by the Pervin and the finest transitive quasi-uniformities, as well as the directed fractal structure which yields the Stone-Čech compactification of any Tychonoff space.

2. DIRECTED GF-SPACES

2.1. Introduction. In this paper, every topological space will be T_0 .

Now, we recall some definitions and introduce some notations that will be useful in this paper.

Let Γ be a covering. Recall that $\text{St}(x, \Gamma) = \bigcup\{A \in \Gamma : x \in A\}$.

A (base \mathcal{B} of a) quasi-uniformity \mathcal{U} on a set X is a (base \mathcal{B} of a) filter \mathcal{U} of binary relations (called entourages) on X such that (a) each element of \mathcal{U} contains the diagonal Δ_X of $X \times X$ and (b) for any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ satisfying $V \circ V \subseteq U$. A base \mathcal{B} of a quasi-uniformity is called transitive if $B \circ B = B$ for all $B \in \mathcal{B}$. The theory of quasiuniform spaces is covered in [3].

If \mathcal{U} is a quasi-uniformity on X , then so is $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$, where

$$U^{-1} = \{(y, x) : (x, y) \in U\}.$$

The generated uniformity on X is denoted by \mathcal{U}^* . A base is given by the entourages $U^* = U \cap U^{-1}$. The topology $\tau(\mathcal{U})$ induced by the quasi-uniformity \mathcal{U} is that in which the sets $U(x) = \{y \in X : (x, y) \in U\}$, where $U \in \mathcal{U}$, form a neighbourhood base for each $x \in X$. There is also the topology $\tau(\mathcal{U}^{-1})$ induced by the inverse quasi-uniformity.

A quasi-uniformity \mathcal{U} is said to be half complete if each \mathcal{U}^* -Cauchy net is \mathcal{U} -convergent.

A relation \leq on a set G is called a partial order on G if it is a transitive antisymmetric reflexive relation on G . If \leq is a partial order on a set G , then (G, \leq) is called a partially ordered set.

(G, \leq, τ) will be called a poset (partially ordered set) or T_0 -Alexandroff space if (G, \leq) is a partially ordered set and τ is that in which the sets

$$[g, \rightarrow[= \{h \in G : g \leq h\}$$

form a neighborhood base for each $g \in G$ (we say that the topology τ is induced by \leq). Note that then $\overline{\{g\}} =]\leftarrow, g]$ for all $g \in G$.

Let us remark that a map $f : G \rightarrow H$ between two posets G and H is continuous if and only if it is order preserving, i.e.

$$g_1 \leq g_2 \text{ implies } f(g_1) \leq f(g_2).$$

Let Γ be a covering of X . Γ is said to be locally finite if for all $x \in X$ there exists a neighborhood of x which meets only a finite number of elements of Γ .

2.2. Directed GF-spaces.

Definition 2.1. Let Γ_1 and Γ_2 be coverings of a set X . We denote by $\Gamma_1 \prec \prec \Gamma_2$ if Γ_1 is a refinement of Γ_2 (that is, $\Gamma_1 \prec \Gamma_2$) and for each $B \in \Gamma_2$ it holds

$$B = \bigcup\{A \in \Gamma_1 : A \subseteq B\}.$$

A base of directed fractal structure over a set X is a family of coverings

$$\mathbf{\Gamma} = \{\Gamma_i : i \in I\}$$

such that for each $i, j \in I$ there exists $k \in I$ such that $\Gamma_k \prec \Gamma_i$ and $\Gamma_k \prec \Gamma_j$.

A base of directed fractal structure over a set X is said to be a directed fractal structure if given a covering Δ with $\Gamma \prec \Delta$ for any $\Gamma \in \mathbf{\Gamma}$ it holds that $\Delta \in \mathbf{\Gamma}$.

If $\mathbf{\Gamma}$ is a base of directed fractal structure over a set X then it is clear that the family of coverings

$$\{\Gamma : \text{there exists } \Gamma' \in \mathbf{\Gamma} \text{ with } \Gamma' \prec \Gamma\}$$

is a directed fractal structure.

If $\mathbf{\Gamma}$ is a directed fractal structure over X , we will say that $(X, \mathbf{\Gamma})$ is a directed GF-space. If there is no confusion about $\mathbf{\Gamma}$, we will say that X is a directed GF-space. Whenever the index set I is \mathbb{N} , the set of natural numbers with its usual order, we drop the word "directed" in this definition (this notion was introduced in [2]).

If $\mathbf{\Gamma}$ is a directed fractal structure over a set X , then it induces a transitive base of quasi-uniformity as follows.

We define $\mathcal{U}_{\mathbf{\Gamma}}$ as the quasi-uniformity of base $\mathcal{B} = \{U_{\Gamma} : \Gamma \in \mathbf{\Gamma}\}$, where

$$U_{\Gamma} = \{(x, y) \in X \times X : y \in X \setminus \bigcup\{A \in \Gamma : x \notin A\}\}.$$

Then it is easy to check that \mathcal{B} is a transitive base of quasi-uniformity and hence $\mathcal{U}_{\mathbf{\Gamma}}$ is a transitive quasi-uniformity. We will use the notations U_{Γ}^{-1} instead of $(U_{\Gamma})^{-1}$ and U_{Γ}^* instead of $(U_{\Gamma})^*$ in order to avoid using unneeded parentheses (the terms $(U^{-1})_{\Gamma}$ and $(U^*)_{\Gamma}$ has no special meaning here).

The topology induced by a directed fractal structure $\mathbf{\Gamma}$ on a set X is defined as the topology induced by the quasi-uniformity $\mathcal{U}_{\mathbf{\Gamma}}$.

If $\mathbf{\Gamma}$ is a directed fractal structure over X , and $\{\text{St}(x, \Gamma) : \Gamma \in \mathbf{\Gamma}\}$ is a neighborhood base of x for all $x \in X$, we will call $(X, \mathbf{\Gamma})$ a starbase directed GF-space and $\mathbf{\Gamma}$ a starbase directed fractal structure.

Let $\mathbf{\Gamma}$ be a (base of) directed fractal structure over X . We say that $\mathbf{\Gamma}$ is finite if Γ is a finite covering for each $\Gamma \in \mathbf{\Gamma}$. Note that a directed fractal structure is finite if any base of the directed fractal structure is finite.

A directed fractal structure can be induced in subspaces as follows. If $A \subseteq X$, and $(X, \mathbf{\Gamma})$ is a directed GF-space, then the induced directed fractal structure over A is denoted by $\mathbf{\Gamma}_A$ and is defined by $\mathbf{\Gamma}_A = \{\Gamma_A : \Gamma \in \mathbf{\Gamma}\}$, where

$$(\Gamma_A) = \{B \cap A : B \in \Gamma\}.$$

Definition 2.2. A map f between two directed GF-spaces $(X, \mathbf{\Gamma})$ and $(Y, \mathbf{\Delta})$ is said to be a GF-map if for each $\Delta \in \mathbf{\Delta}$ it follows that $f^{-1}(\Delta) \in \mathbf{\Gamma}$, where

$$f^{-1}(\Delta) = \{f^{-1}(A) : A \in \Delta\}.$$

Next, we are going to summarize some basic properties about directed fractal structures.

Proposition 2.3.

- (1) If $\mathbf{\Gamma}$ is a directed fractal structure over X , then Γ is closure preserving for each $\Gamma \in \mathbf{\Gamma}$. Moreover A is closed for every $A \in \Gamma$ and for every $\Gamma \in \mathbf{\Gamma}$.
- (2) If $\mathbf{\Gamma}$ is a directed fractal structure over X , then $U_{\Gamma}^{-1}(x) = \bigcap \{A \in \Gamma : x \in A\}$.

Proof.

- (1) The proof is as in [1] (where it is done for fractal structures).
- (2) The proof is as in [2].

□

3. RELATIONS OF DIRECTED GF-SPACES WITH TRANSITIVE QUASI-UNIFORMITIES AND INVERSE LIMITS OF POSETS

Now, we are going to relate the concepts of directed fractal structure, transitive base of quasi-uniformity and inverse limit of posets.

3.1. Directed fractal structure \rightarrow Transitive quasi-uniformity. We recall (see section 2.2) that the transitive quasi-uniformity \mathcal{U}_{Γ} induced by a directed fractal structure $\mathbf{\Gamma}$ is the quasi-uniformity with transitive base $\{U_{\Gamma} : \Gamma \in \mathbf{\Gamma}\}$, where $U_{\Gamma} = \{(x, y) \in X \times X : y \in X \setminus \bigcup \{A \in \Gamma : x \notin A\}\}$.

Proposition 3.1. *Let $(X, \mathbf{\Gamma})$ and $(Y, \mathbf{\Delta})$ be directed GF-spaces and let \mathcal{U}_{Γ} and \mathcal{U}_{Δ} be the transitive quasi-uniformities induced by $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ respectively. Let $f : (X, \mathbf{\Gamma}) \rightarrow (Y, \mathbf{\Delta})$ be a GF-map. Then $f : (X, \mathcal{U}_{\Gamma}) \rightarrow (Y, \mathcal{U}_{\Delta})$ is quasi-uniformly continuous.*

Proof. Let $\Delta \in \mathbf{\Delta}$, since f is a GF-map, there exists $\Gamma \in \mathbf{\Gamma}$ such that $\Gamma \prec \prec f^{-1}(\Delta)$. Let us prove that $f \times f(U_{\Gamma}) \subseteq U_{\Delta}$.

Let $x \in X$ and $y \in U_{\Gamma}^{-1}(x)$, and $B \in \mathbf{\Delta}$ be such that $f(x) \in B$, since $\Gamma \prec \prec f^{-1}(\Delta)$ there exists $A \in \mathbf{\Gamma}$ such that $x \in A$ and $f(A) \subseteq B$. Since

$$y \in U_{\Gamma}^{-1}(x) = \bigcap \{C \in \mathbf{\Gamma} : x \in C\},$$

then $y \in A$ and hence $f(y) \in f(A) \subseteq B$. Then

$$f(y) \in \bigcap \{B \in \mathbf{\Delta} : f(x) \in B\} = U_{\Delta}^{-1}(f(x)),$$

and therefore $f(U_{\Gamma}(x)) \subseteq U_{\Delta}(f(x))$, so f is quasi-uniformly continuous. □

3.2. Transitive base of quasi-uniformity \rightarrow Directed fractal structure. If \mathcal{B} is a transitive base for a quasi-uniformity \mathcal{U} , then we define $\mathbf{\Gamma}_{\mathcal{U}}$ as the directed fractal structure for which $\{\Gamma_V : V \in \mathcal{B}\}$ is a base, where

$$\Gamma_V = \{V^{-1}(x) : x \in X\} \text{ for each } V \in \mathcal{B}.$$

$\mathbf{\Gamma}_{\mathcal{U}}$ is called the directed fractal structure induced by the transitive quasi-uniformity \mathcal{U} .

We note that $\{\Gamma_V : V \in \mathcal{V}\}$ is in fact a base of directed fractal structure and that Γ_U does not depend on the transitive base \mathcal{B} , but on the quasi-uniformity \mathcal{U} .

Proposition 3.2. *Let (X, \mathcal{U}) and (Y, \mathcal{V}) be quasi-uniform spaces and let Γ_U and Γ_V be the directed fractal structures induced by \mathcal{U} and \mathcal{V} respectively. Let $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ be a quasi-uniformly continuous map. Then $f : (X, \Gamma_U) \rightarrow (Y, \Gamma_V)$ is a GF-map.*

Proof. Let $V \in \mathcal{V}$ be a transitive entourage and let us prove that $f^{-1}(\Gamma_V) \in \Gamma_U$. Since f is quasi-uniformly continuous, there exists $U \in \mathcal{U}$ a transitive entourage such that $f \times f(U) \subseteq V$. Let us prove that $\Gamma_U \prec\prec f^{-1}(\Gamma_V)$. Let $x \in X$ and $V^{-1}(y) \in \Gamma_V$ such that $f(x) \in V^{-1}(y)$. Let us prove that $f(U^{-1}(x)) \subseteq V^{-1}(y)$ (note that $x \in U^{-1}(x)$ and $U^{-1}(x) \in \Gamma_U$). Indeed, let $z \in U^{-1}(x)$, then $x \in U(z)$ and $f(x) \in V(f(z))$, and hence

$$f(z) \in V^{-1}(f(x)) \subseteq V^{-1} \circ V^{-1}(y) = V^{-1}(y).$$

Therefore $f(U^{-1}(x)) \subseteq V^{-1}(y)$. On the other hand, it is clear that $U^{-1}(x) \subseteq f^{-1}(V^{-1}(f(x)))$, and hence $\Gamma_U \prec\prec f^{-1}(\Gamma_V)$, so $f^{-1}(\Gamma_V) \in \Gamma_U$ and f is a GF-map. \square

3.3. Transitive base of quasi-uniformity \rightarrow Inverse limit of posets. Let X be a topological space and \mathcal{B} be a transitive base of quasi-uniformity for X . For each $U \in \mathcal{B}$, we define $G_U = \{U^*(x) : x \in X\}$. We note that G_U is a partition of X for each $U \in \mathcal{U}$.

The following statements can be proved like the suitable ones in [2].

We define $\rho_U : X \rightarrow G_U$ by $\rho_U(x) = U^*(x)$, with the order $\rho_U(x) \leq_U \rho_U(y)$ if and only if $y \in U(x)$. For $U, V \in \mathcal{B}$ with $V \subseteq U$, we define $\phi_{VU} : G_V \rightarrow G_U$ by $\phi_{VU}(\rho_V(x)) = \rho_U(x)$.

Then ρ_U and ϕ_{VU} are continuous mappings for each $U \in \mathcal{B}$ and each $V \in \mathcal{B}$ with $V \subseteq U$, and $\rho = (\rho_U)_{U \in \mathcal{B}}$ is an embedding from X into $\varprojlim \{G_U : U \in \mathcal{B}\}$ (which is the inverse limit of a directed system of posets).

We call $\varprojlim \{G_U : U \in \mathcal{B}\}$ (resp. G_U) the inverse limit (resp. poset) induced by the transitive base \mathcal{B} (resp. entourage U).

Remark 3.3. In what follows, we will identify the space X with $\rho(X)$. Then, for example, when we deal with the restriction to X of a quasi-uniformity on $\varprojlim \{G_U : U \in \mathcal{B}\}$ we mean the restriction to $\rho(X)$.

3.4. Inverse limit of posets \rightarrow Transitive base of quasi-uniformity. Let $X = \varprojlim \{G_i : i \in I\}$ be the inverse limit of posets and define

$$U_{G_i}(g) = \{h = (h_j)_{j \in I} \in \varprojlim \{G_j : j \in I\} : g_i \leq_i h_i\},$$

where $g = (g_j)_{j \in I}$. Then $\mathcal{B}_{\varprojlim G_i} = \{U_{G_i} : i \in I\}$ is a transitive base of quasi-uniformity for X , where

$$U_{G_i} = \{(g, h) \in \varprojlim G_i : h \in U_{G_i}(g)\}.$$

We say that the transitive quasi-uniformity $\mathcal{U}_{\varprojlim G_i}$ generated by the base $\mathcal{B}_{\varprojlim G_i}$ is the transitive quasi-uniformity induced by the inverse limit $\varprojlim G_i$.

Proposition 3.4. *Let $X = \varprojlim_{i \in I} G_i$ and $Y = \varprojlim_{j \in J} H_j$, and let $\mathcal{U}_{\varprojlim G_i}$ and $\mathcal{U}_{\varprojlim H_j}$ be the transitive quasi-uniformities induced by X and Y respectively. Let $f = (f_j)_{j \in J} : X \rightarrow Y$, where for each $j \in J$ there exists $i(j) \in I$ such that $f_j : G_{i(j)} \rightarrow H_j$ is order preserving, the set $\{i(j) : j \in J\}$ is cofinal in I and $\phi_{j_2 j_1} \circ f_{j_2} = f_{j_1} \circ \phi_{i(j_2), i(j_1)}$ for each $j_2 \geq j_1$, where $\phi_{j_2 j_1} : H_{j_2} \rightarrow H_{j_1}$ and $\phi_{i_2 i_1} : G_{i_2} \rightarrow G_{i_1}$ (for $j_2 \geq j_1$ and $i_2 \geq i_1$) are the bonding maps of X and Y respectively. Then $f : (X, \mathcal{U}_{\varprojlim G_i}) \rightarrow (Y, \mathcal{U}_{\varprojlim H_j})$ is quasi-uniformly continuous.*

Proof. Let $j \in J$, and let $i(j)$ be as in the statement of the proposition. Let us see that $f(U_{G_{i(j)}}) \subseteq U_{H_j}$. Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ with $y \in U_{G_{i(j)}}(x)$, then $x_{i(j)} \leq_{i(j)} y_{i(j)}$, and since f_j is order preserving, it follows that $f_j(x_{i(j)}) \leq_j f_j(y_{i(j)})$. Since $f(x) = (f_j(x_{i(j)}))_{j \in J}$ and $f(y) = (f_j(y_{i(j)}))_{j \in J}$ it follows that $f(y) \in U_{H_j}(f(x))$. Therefore f is quasi-uniformly continuous. \square

3.5. Other relations. The following relations follow in an easy way:

Proposition 3.5. *With the previous notation.*

- (1) $V = U_{\Gamma_V}$ for each transitive entourage V .
- (2) $V = U_{G_V}|_{X \times X}$ for each transitive entourage V (see Remark 3.3).
- (3) $H = G_{U_H}$ for each poset H .
- (4) $\Gamma_{U_\Gamma} \prec \prec \Gamma$ for each covering Γ .

Corollary 3.6. *With the previous notation.*

- (1) $\mathcal{V} = \mathcal{U}_{\Gamma_V}$ and $\mathcal{V} = \mathcal{U}_{\varprojlim\{G_V : V \in \mathcal{V}\}}|_{X \times X}$.
- (2) $\varprojlim\{G_i : i \in I\} = \varprojlim\{G_{U_{G_i}} : i \in I\}$.
- (3) $\Gamma \subseteq \Gamma_{\mathcal{U}_\Gamma}$.

Notation. Let Γ be a directed fractal structure over X and let \mathcal{U}_Γ be the transitive quasi-uniformity induced by Γ . In order to simplify a little the notation, we will write G_Γ (resp. $\rho_\Gamma, \phi_{\Gamma_1 \Gamma_2}$) instead of G_{U_Γ} (resp. $\rho_{U_\Gamma}, \phi_{U_{\Gamma_1} U_{\Gamma_2}}$) for each $\Gamma \in \mathbf{\Gamma}$.

Definition 3.7. *With the previous notation. Let $\mathbf{\Gamma}$ be a directed fractal structure over X . Then $X \subseteq \varprojlim\{G_\Gamma : \Gamma \in \mathbf{\Gamma}\}$ (see Remark 3.3). We can consider in $\varprojlim\{G_\Gamma : \Gamma \in \mathbf{\Gamma}\}$ the directed fractal structure $G(\mathbf{\Gamma}) = \mathbf{\Gamma}_{U_{\varprojlim\{G_\Gamma : \Gamma \in \mathbf{\Gamma}\}}}$.*

We define $G(X)$ as the set of closed points of $\varprojlim\{G_\Gamma : \Gamma \in \mathbf{\Gamma}\}$, and we will consider the directed GF-space $(G(X), G(\mathbf{\Gamma})_{G(X)})$.

We will use the symbol $(X, G(\mathbf{\Gamma}))$ (resp. $(G(X), G(\mathbf{\Gamma}))$) instead of $(X, G(\mathbf{\Gamma})_X)$ (resp. $(G(X), G(\mathbf{\Gamma})_{G(X)})$).

Proposition 3.8. *Let $(X, \mathbf{\Gamma})$ be a directed GF-space, and let \mathcal{U}_Γ (resp. $\mathcal{U}_{G(\mathbf{\Gamma})_X}$) be the transitive quasi-uniformity induced by Γ (resp. $G(\mathbf{\Gamma})$) on X . Then $\Gamma \subseteq G(\mathbf{\Gamma})_X = \Gamma_{\mathcal{U}_\Gamma}$, $\mathcal{U}_{G(\mathbf{\Gamma})_X} = \mathcal{U}_\Gamma$ and $G(G(\mathbf{\Gamma})) = G(\mathbf{\Gamma})$.*

Proof. Let us prove that $\Gamma \subseteq G(\Gamma)_X = \Gamma_{\mathcal{U}_\Gamma}$. It can be proved that $G(\Gamma)_X = (\Gamma_{\bigcup_{\Gamma \in \Gamma} \{G_\Gamma, \Gamma \in \Gamma\}})_X = \Gamma_{(\bigcup_{\Gamma \in \Gamma} \{G_\Gamma, \Gamma \in \Gamma\})_{X \times X}}$, and by Corollary 3.6 it follows that $G(\Gamma)_X = \Gamma_{\mathcal{U}_\Gamma}$. It also follows from Corollary 3.6 that $\Gamma \subseteq \Gamma_{\mathcal{U}_\Gamma}$.

Let us prove that $\mathcal{U}_{G(\Gamma)_X} = \mathcal{U}_\Gamma$. Since $G(\Gamma)_X = \Gamma_{\mathcal{U}_\Gamma}$, we have by Corollary 3.6 that $\mathcal{U}_{(G(\Gamma)_X)} = \mathcal{U}_{\Gamma_{\mathcal{U}_\Gamma}} = \mathcal{U}_\Gamma$.

The equality $G(G(\Gamma)) = G(\Gamma)$ also follows from Corollary 3.6. \square

From the preceding proposition it follows that Γ and $G(\Gamma)$ induce the same quasi-uniformity and topology on X .

Hereafter we will refer to quasi-uniformity properties of directed fractal structure (for example we will say that a directed fractal structure is half complete), and this will mean that the transitive quasi-uniformity induced by the directed fractal structure has that property.

Proposition 3.9. *Let Γ be a directed fractal structure over a topological space X , let \mathcal{U}_Γ be the transitive quasi-uniformity induced by Γ , let $G(\Gamma)$ be the directed fractal structure induced by \mathcal{U}_Γ , and let G_Γ be the posets induced by U_Γ , for each $U_\Gamma \in \mathcal{U}_\Gamma$.*

- (1) *If Γ is finite then $G(\Gamma)$ is finite.*
- (2) *$G(\Gamma)$ is finite if and only if \mathcal{U}_Γ is totally bounded if and only if G_Γ is finite for every $\Gamma \in \Gamma$.*
- (3) *If Γ is starbase then $G(\Gamma)$ is starbase.*
- (4) *$G(\Gamma)$ is starbase if and only if \mathcal{U}_Γ is locally symmetric.*

Proof. First, note that by Proposition 3.8 it follows that $G(\Gamma) = \Gamma_{\mathcal{U}_\Gamma} = \{\Gamma_{U_\Gamma} : \Gamma \in \Gamma\}$ on X and we recall that $\Gamma_{U_\Gamma} = \{U_\Gamma^{-1}(x) : x \in X\}$ whenever $\Gamma \in \Gamma$.

(1) Since $U_\Gamma^{-1}(x) = \bigcap \{A \in \Gamma : x \in A\}$, it follows that $G(\Gamma)$ is finite if Γ is.

(2) First, let us prove that \mathcal{U}_Δ is totally bounded if Δ is a finite directed fractal structure.

Since $U_\Gamma^*(x) = \bigcap \{A \in \Gamma : x \in A\} \setminus (\bigcup \{B \in \Gamma : x \notin B\})$ for every $x \in X$ and $\Gamma \in \Gamma$, then it follows that if Γ is finite then $\{U_\Gamma^*(x) : x \in X\}$ is finite and hence \mathcal{U}_Γ is totally bounded.

Applying this result to $G(\Gamma)$, we obtain that $\mathcal{U}_{G(\Gamma)}$ is totally bounded. Finally, we note that $\mathcal{U}_{G(\Gamma)} = \mathcal{U}_\Gamma$ by Proposition 3.8.

Conversely, if \mathcal{U}_Γ is totally bounded, since $U_\Gamma^{-1}(x) = \bigcup \{U_\Gamma^*(y) : y \in U_\Gamma^{-1}(x)\}$, then it follows that $\Gamma_{U_\Gamma} = \{U_\Gamma^{-1}(x) : x \in X\}$ is finite and hence $G(\Gamma)$ is a finite directed fractal structure over X .

Finally, since $G_\Gamma = \{U_\Gamma^*(x) : x \in X\}$ then it follows that \mathcal{U}_Γ is totally bounded if and only if G_Γ is finite.

(3) This is obvious, since $\Gamma \subseteq \Gamma_{\mathcal{U}_\Gamma} = G(\Gamma)$ by Corollary 3.6.

(4) Let us prove that $U_\Gamma^{-1} \circ U_\Gamma(x) = \text{St}(x, \Gamma_{U_\Gamma})$ for each $x \in X$ and $\Gamma \in \Gamma$.

$y \in U_\Gamma^{-1} \circ U_\Gamma(x)$ if and only if there exists $z \in X$ such that $x, y \in U_\Gamma^{-1}(z)$ if and only if $y \in \text{St}(x, \Gamma_{U_\Gamma})$ (recall that $\Gamma_{U_\Gamma} = \{U_\Gamma^{-1}(z) : z \in X\}$).

Therefore \mathcal{U}_Γ is locally symmetric if and only if $G(\Gamma)$ is starbase. \square

4. THE CATEGORY OF DIRECTED GF-SPACES

Let us describe the categories of directed GF-spaces and transitive quasi-uniform spaces and the relation between them.

Directed GF-spaces. An object is a space joint with a directed fractal structure. A morphism is a GF-map.

Transitive quasi-uniform spaces. An object is a space joint with a transitive quasi-uniformity. A morphism is a quasi-uniformly continuous map.

Functors. Let Γ be a directed fractal structure over a space X and \mathcal{U} be a transitive quasi-uniformity for X .

Denote by \mathcal{U}_Γ the transitive quasi-uniformity induced by Γ on X and denote by $\Gamma_\mathcal{U}$ the directed fractal structure induced by \mathcal{U} on X .

We define the functor QU from the category of directed GF-spaces to the category of transitive quasi-uniform spaces as follows. $QU(X, \Gamma) = (X, \mathcal{U}_\Gamma)$ and $QU(f) = f$. Note that it is well defined by section 3.1.

We define the functor FS from the category of transitive quasi-uniform spaces to the category of directed GF-spaces as follows. $FS(X, \mathcal{U}) = (X, \Gamma_\mathcal{U})$ and $FS(f) = f$. Note that it is well defined by section 3.2.

By Corollary 3.6, it follows that $QU \circ FS$ is the identity functor in the category of transitive quasi-uniform spaces.

5. COMPACTIFICATION

Lemma 5.1. *Let (X, \mathcal{U}) be a quasi-uniform space and let $\varprojlim\{G_U : U \in \mathcal{U}\}$ be the inverse limit induced by \mathcal{U} (with notations of section 3). Then \mathcal{U} is half complete if and only if for each $(g_U)_{U \in \mathcal{U}} \in \varprojlim\{G_U : U \in \mathcal{U}\}$ there exists $x \in X$ with $\rho_U(x) \leq_U g_U$ whenever $U \in \mathcal{U}$.*

Proof. The proof is easy once we note that $(x_U)_{U \in \mathcal{U}}$ is a \mathcal{U}^* -Cauchy net if and only if $(\rho_U(x_U))_{U \in \mathcal{U}} \in \varprojlim\{G_U : U \in \mathcal{U}\}$ and $\rho_U(x) \leq_U \rho_U(x_U)$ whenever $U \in \mathcal{U}$ if and only if (x_U) converges to x in (X, \mathcal{U}) . \square

5.1. GF-compactification. In this subsection, the GF-compactification is introduced. (We want to recall Remark 3.3).

5.1.1. Let Γ be a directed fractal structure over X such that the transitive quasi-uniformity \mathcal{U}_Γ induced by Γ is point symmetric (in fact, we only need that each \mathcal{U}_Γ^* -Cauchy net which is \mathcal{U}_Γ^{-1} -convergent to x is also \mathcal{U}_Γ -convergent to x). Then $X \subseteq G(X)$.

Proof. Let $x \in X$ and suppose that there exists $g = (\rho_\Gamma(x_\Gamma)) \in \varprojlim\{G_\Gamma : \Gamma \in \Gamma\}$ such that $(\rho_\Gamma(x_\Gamma)) < \rho(x)$, that is, there exists $\Delta \in \Gamma$ such that $\rho_\Delta(x_\Delta) <_\Delta \rho_\Delta(x)$. Then it is clear that $(x_\Gamma)_{\Gamma \in \Gamma}$ is a \mathcal{U}_Γ^* -Cauchy net, since $U_{\Gamma_2}^*(x_{\Gamma_2}) \subseteq U_{\Gamma_1}^*(x_{\Gamma_1})$ for all $\Gamma_2 \prec \Gamma_1$ (since $(\rho_\Gamma(x_\Gamma)) \in \varprojlim\{G_\Gamma : \Gamma \in \Gamma\}$) and $(x_\Gamma)_{\Gamma \in \Gamma}$ is a net which U^{-1} -converges to x , so by hypothesis we have that $(x_\Gamma)_{\Gamma \in \Gamma}$ \mathcal{U}_Γ -converges to x . Hence for all $\Gamma \in \Gamma$ there exists $\Lambda \in \Gamma$ with $\Lambda \prec \Gamma$

and such that $x_\Pi \in U_\Gamma(x)$ for each $\Pi \in \mathbf{\Gamma}$ with $\Lambda \prec\prec \Pi$, whence $\rho_\Gamma(x) \leq_\Gamma \rho_\Gamma(x_\Pi) = \rho_\Gamma(x_\Gamma)$ for each $\Gamma \in \mathbf{\Gamma}$, what contradicts that $\rho_\Delta(x_\Delta) <_\Delta \rho_\Delta(x)$. The contradiction shows that $\{\rho(x)\}$ is closed, or equivalently, $\rho(x) \in G(X)$. \square

5.1.2. Let $\mathbf{\Gamma}$ be a finite directed fractal structure over X . Then $G(\mathbf{\Gamma})$ is a finite half complete directed fractal structure over $G(X)$.

Proof. Analogously to the proof of Proposition 3.9 it can be proved that $G(\mathbf{\Gamma})$ is finite. Let us prove that it is half complete.

Let $(g_\Gamma) \in \varprojlim\{G_\Gamma : \Gamma \in \mathbf{\Gamma}\}$ and let

$$\mathcal{F} = \{(h_\Gamma) \in \varprojlim\{G_\Gamma : \Gamma \in \mathbf{\Gamma}\} : (h_\Gamma) \leq_\Gamma (g_\Gamma)\}.$$

Since $(g_\Gamma) \in \mathcal{F}$, it is nonempty, and if $(h_\Gamma^l) \in \varprojlim\{G_\Gamma : \Gamma \in \mathbf{\Gamma}\}$ is a decreasing chain, then if for each $\Gamma \in \mathbf{\Gamma}$ we define $h_\Gamma = \min\{h_\Gamma^l : l \in L\}$ (note that the minimum exists, since Γ is finite and $\{h_\Gamma^l : l \in L\}$ is a chain) we have that (h_Γ) is a bound for the chain. Then by Zorn's lemma, \mathcal{F} has a minimal element (h_Γ) . Hence $(h_\Gamma) \in G(X)$ and $(h_\Gamma) \leq (g_\Gamma)$. Therefore $G(X)$ is half complete by Lemma 5.1. \square

5.1.3. Let $\mathbf{\Gamma}$ be a finite directed fractal structure over X such that U_Γ , the transitive quasi-uniformity induced by $\mathbf{\Gamma}$, is point symmetric. Then $(G(X), G(\mathbf{\Gamma}))$ is a T_1 compactification of X .

We will denote by $G(X, \mathbf{\Gamma})$ the directed GF-space $(G(X), G(\mathbf{\Gamma}))$.

Proof. By a previous item, X is a subset of $G(X)$, and if $g = (\rho_\Gamma(x_\Gamma)) \in G(X)$, then (x_Γ) converges to g in $G(X)$ (since $g_\Gamma = \rho_\Gamma(x_\Gamma) = \rho_\Gamma(x_\Delta)$ for all $\Delta \in \mathbf{\Gamma}$ with $\Delta \prec\prec \Gamma$, and hence $x_\Delta \in U_{G_\Gamma}(g)$ for all $\Delta \geq \Gamma$). Therefore X is dense in $G(X)$.

On the other hand, $G(X)$ is T_1 , since it is the subset of closed points of $\varprojlim\{G_\Gamma : \Gamma \in \mathbf{\Gamma}\}$. $G(X)$ is compact because it is a half complete (by the previous item) totally bounded (by proposition 3.9) quasi-uniform space. \square

5.2. **Wallman compactification.** Next, we are going to show that the GF-compactification is a Wallman type compactification.

Let $\mathbf{\Gamma}$ be a finite directed fractal structure over X , such that the induced quasi-uniformity is point symmetric (for example, if $\mathbf{\Gamma}$ is starbase). We define

$$\mathcal{L} = \mathcal{L}(\mathbf{\Gamma}) = \left\{ \bigcup_{k \in K} \bigcap_{l \in L} A_{kl} : A_{kl} \in \Gamma_{kl} \in \mathbf{\Gamma}; K, L \text{ finite sets} \right\}.$$

It is clear that \mathcal{L} is a lattice. On the other hand, since $\{U_\Gamma(x) : x \in X; \Gamma \in \mathbf{\Gamma}\}$ is an open base of X , then $\{X \setminus U_\Gamma(x) : x \in X; \Gamma \in \mathbf{\Gamma}\}$ is a closed base of X , and since $X \setminus U_\Gamma(x) = \bigcup\{A \in \Gamma : x \notin A\} \in \mathcal{L}$, then \mathcal{L} is a closed base of X . Since X is T_0 , \mathcal{L} is a β -lattice. Let us show that it is an α -lattice. Let $x \in X$ and let $L \in \mathcal{L}$ such that $x \notin L$. Since L is closed, and since X is point symmetric, then there exists $\Gamma \in \mathbf{\Gamma}$ such that $U_\Gamma^{-1}(x) \subseteq X \setminus L$. Since $U_\Gamma^{-1}(x) = \bigcap\{A \in \Gamma : x \in A\} \in \mathcal{L}$, then \mathcal{L} is an α -lattice.

Therefore $W(X, \mathcal{L})$, the Wallman compactification associated with \mathcal{L} , is a T_1 compactification of X . Let $\Delta(\Gamma) = \{B_A : A \in \Gamma\}$, where $B_A = \{\mathcal{F} \in W(X, \mathcal{L}) : A \in \mathcal{F}\}$ for every $A \in \Gamma$ and every $\Gamma \in \mathbf{\Gamma}$. Let $\mathbf{\Delta}(\mathbf{\Gamma}) = \{\Delta(\Gamma) : \Gamma \in \mathbf{\Gamma}\}$. We denote $W(X, \mathcal{L})$ by $W(X, \mathbf{\Gamma})$ hereafter.

Theorem 5.2. $\mathbf{\Delta}(\mathbf{\Gamma})$ is a finite directed fractal structure over $W(X, \mathcal{L})$ compatible with its (usual) topology.

Proof. It is clear that $\Delta(\Gamma)$ is a finite closed covering for all $\Gamma \in \mathbf{\Gamma}$ and hence $U_{\Delta(\Gamma)}(\mathcal{F})$ is open whenever $\mathcal{F} \in W(X, \mathcal{L})$. On the other hand, since $B_{L_1 \cup L_2} = B_{L_1} \cup B_{L_2}$ we have that for each $A \in \Gamma_1$ with $\Gamma_1 \in \mathbf{\Gamma}$ and $\Gamma_2 \in \mathbf{\Gamma}$ with $\Gamma_2 \prec \prec \Gamma_1$ it follows that $A = \bigcup \{C \in \Gamma_2 : C \subseteq A\}$, so $B_A = \bigcup \{B_C : C \subseteq A\}$. Therefore $\Delta(\Gamma_2) \prec \prec \Delta(\Gamma_1)$.

In order to prove that $\mathbf{\Delta}(\mathbf{\Gamma})$ is compatible with the usual topology of $W(X, \mathcal{L})$, let \mathcal{F} be a \mathcal{L} -ultrafilter such that $\mathcal{F} \notin B_L$. Let $L = \bigcup_{k \in K} \bigcap_{l \in L} A_{kl}$ with K, L finite, $A_{kl} \in \Gamma_{kl}$ and $\Gamma_{kl} \in \mathbf{\Gamma}$, and let $\Gamma \in \mathbf{\Gamma}$ such that $\Gamma \prec \prec \Gamma_{kl}$ for each $k \in K$ and $l \in L$. Let us prove that $U_{\Delta(\Gamma)}(\mathcal{F}) \subseteq W(X, \mathcal{L}) \setminus B_L$. Let \mathcal{G} be an ultrafilter such that $\mathcal{G} \in U_{\Delta(\Gamma)}(\mathcal{F})$ and suppose that $\mathcal{G} \in B_L$. Since $B_L = \bigcap_{k \in K} \bigcup_{l \in L} B_{A_{kl}}$, then for all $l \in L$, there exists $k = k(l) \in K$ such that $\mathcal{G} \in B_{A_{kl}}$. Then, since $\mathcal{F} \in U_{\Delta(\Gamma)}^{-1}(\mathcal{G}) = \bigcap \{B_A \in \Delta(\Gamma) : \mathcal{G} \in B_A; A \in \Gamma\}$, we have that $\mathcal{F} \in B_{A_{kl}}$ for each $l \in L$, and then $\mathcal{F} \in \bigcap_{l \in L} \bigcup_{k \in K} B_{A_{kl}} = B_L$. The contradiction proves the desired result.

Therefore $\mathbf{\Delta}(\mathbf{\Gamma})$ is a finite fractal structure over $W(X, \mathcal{L})$ compatible with its topology. \square

Remark 5.3. Note that $B_A = Cl_{W(X, \mathbf{\Gamma})} A$ and $B_A \cap X = A$ for each $A \in \Gamma \in \mathbf{\Gamma}$, and hence $\mathbf{\Delta}(\mathbf{\Gamma})_X = \mathbf{\Gamma}$.

Lemma 5.4. Let $(X, \mathbf{\Gamma})$ be a half complete T_1 GF-space. Then $X = G(X)$ (recall Remark 3.3).

Proof. Let $x \in X$ and suppose that there exists $(\rho_\Gamma(y_\Gamma)) \in \varprojlim \{G_\Gamma : \Gamma \in \mathbf{\Gamma}\}$ such that $\rho_\Gamma(y_\Gamma) \leq_\Gamma \rho_\Gamma(x)$ for all $\Gamma \in \mathbf{\Gamma}$. Since X is half complete, by Lemma 5.1 there exists $y \in X$ such that $\rho_\Gamma(y) \leq_\Gamma \rho_\Gamma(y_\Gamma) \leq_\Gamma \rho_\Gamma(x)$, and hence we have that $x \in U_\Gamma(y)$ for each $\Gamma \in \mathbf{\Gamma}$, and since X is T_1 , it follows that $x = y$, and hence $\rho_\Gamma(y_\Gamma) = \rho_\Gamma(x)$ for all $\Gamma \in \mathbf{\Gamma}$. Therefore $\rho(x) \in G(X)$.

Conversely, let $(\rho_\Gamma(x_\Gamma)) \in G(X)$. Since X is half complete then there exists $x \in X$ such that $\rho_\Gamma(x) \leq_\Gamma \rho_\Gamma(x_\Gamma)$ for each $\Gamma \in \mathbf{\Gamma}$, and since $(\rho_\Gamma(x_\Gamma)) \in G(X)$ it follows that $\rho_\Gamma(x_\Gamma) = \rho_\Gamma(x)$ for each $\Gamma \in \mathbf{\Gamma}$, whence $(\rho_\Gamma(x_\Gamma)) \in \rho(X)$. \square

Theorem 5.5. Let $\mathbf{\Gamma}$ be a finite directed fractal structure over X such that the induced quasi-uniformity is point symmetric. Then $(G(X, \mathbf{\Gamma}), \mathcal{U}_{G(\mathbf{\Gamma})})$ is quasi-isomorphic to $(W(X, \mathbf{\Gamma}), \mathcal{U}_{\mathbf{\Delta}(\mathbf{\Gamma})})$.

Proof. First, let us prove that $U_{\Delta(\Gamma)}^*(\mathcal{F}) \cap X \neq \emptyset$.

$$U_{\Delta(\Gamma)}^*(\mathcal{F}) = \bigcap \{B_A \in \Delta(\Gamma) : A \in \mathcal{F}; A \in \Gamma\} \setminus \bigcup \{B_A \in \Delta(\Gamma) : A \notin \mathcal{F}; A \in \Gamma\},$$

and hence $\bigcap\{A \in \Gamma : A \in \mathcal{F}\} \in \mathcal{F}$ and $\bigcup\{A \in \Gamma : A \notin \mathcal{F}\} \notin \mathcal{F}$, and then

$$\bigcap\{A \in \Gamma : A \in \mathcal{F}\} \setminus \bigcup\{A \in \Gamma : A \notin \mathcal{F}\} \neq \emptyset$$

(if $\bigcap\{A \in \Gamma : A \in \mathcal{F}\} \setminus \bigcup\{A \in \Gamma : A \notin \mathcal{F}\} = \emptyset$, then $\bigcap\{A \in \Gamma : A \in \mathcal{F}\} \subseteq \bigcup\{A \in \Gamma : A \notin \mathcal{F}\}$ and hence $\bigcup\{A \in \Gamma : A \notin \mathcal{F}\} \in \mathcal{F}$). Let

$$x \in \bigcap\{A \in \Gamma : A \in \mathcal{F}\} \setminus \bigcup\{A \in \Gamma : A \notin \mathcal{F}\}.$$

Then $\mathcal{F}_x \in U_{\Delta(\Gamma)}^*(\mathcal{F})$ (where \mathcal{F}_x is the \mathcal{L} -ultrafilter generated by x).

Let $g_\Gamma : G_\Gamma \rightarrow G_{\Delta(\Gamma)}$ be defined by $g_\Gamma(U_\Gamma^*(x)) = U_{\Delta(\Gamma)}^*(\mathcal{F}_x)$. Let see that g_Γ is a poset isomorphism.

Note first that given $A \in \Gamma$ it holds that $\mathcal{F}_y \in B_A$ if and only if $A \in \mathcal{F}_y$, or equivalently $y \in A$. Then

$$y \in U_\Gamma(x) \text{ if and only if } x \in U_\Gamma^{-1}(y) = \bigcap\{A \in \Gamma : y \in A\},$$

or what is the same, $\mathcal{F}_x \in \bigcap\{B_A \in \Delta(\Gamma) : \mathcal{F}_y \in B_A : A \in \Gamma\}$, that is, $\mathcal{F}_y \in U_{\Delta(\Gamma)}(\mathcal{F}_x)$. From this equivalence we can deduce that g_Γ is well defined, injective and order-preserving. Since we have proved that $U_{\Delta(\Gamma)}^*(\mathcal{F}) \cap X \neq \emptyset$, then it is clear that g_Γ is surjective. Therefore g_Γ is a poset isomorphism. Moreover, given $\Gamma, \Lambda \in \mathbf{\Gamma}$ with $\Lambda \prec \prec \Gamma$ it holds that

$$\begin{aligned} \phi_{\Lambda\Gamma} \circ g_\Lambda(U_\Lambda^*(x)) &= \phi_{\Lambda\Gamma}(U_{\Delta(\Lambda)}(\mathcal{F}_x)) \\ &= U_{\Delta(\Gamma)}(\mathcal{F}_x) \\ &= g_\Gamma(U_\Gamma^*(x)) \\ &= g_\Gamma \circ \phi_{\Lambda\Gamma}(U_\Lambda^*(x)), \end{aligned}$$

and hence $\phi_{\Lambda\Gamma} \circ g_\Lambda = g_\Gamma \circ \phi_{\Lambda\Gamma}$ for each $\Lambda \prec \prec \Gamma$.

Then we have that $\varprojlim\{G_\Gamma : \Gamma \in \mathbf{\Gamma}\} = \varprojlim\{G_{\Delta(\Gamma)} : \Gamma \in \mathbf{\Gamma}\}$. Now, since $G(X, \mathbf{\Gamma})$ and $W(X, \mathbf{\Gamma})$ are compact it follows by Lemma 5.4 that $G(X, \mathbf{\Gamma})$ (resp. $W(X, \mathbf{\Gamma})$) is the set of closed point of $\varprojlim\{G_\Gamma : \Gamma \in \mathbf{\Gamma}\}$ (resp. $\varprojlim\{G_{\Delta(\Gamma)} : \Gamma \in \mathbf{\Gamma}\}$).

Since $\mathcal{U}_{G(\Gamma)}$ (resp. $\mathcal{U}_{\Delta(\Gamma)}$) is the transitive quasi-uniformity induced by $\varprojlim\{G_\Gamma : \Gamma \in \mathbf{\Gamma}\}$ (resp. $\varprojlim\{G_{\Delta(\Gamma)} : \Gamma \in \mathbf{\Gamma}\}$), then $(G(X, \mathbf{\Gamma}), \mathcal{U}_{G(\Gamma)})$ is quasi-isomorphic to $(W(X, \mathbf{\Gamma}), \mathcal{U}_{\Delta(\Gamma)})$, what proves the result. \square

6. STONE-ĆECH COMPACTIFICATION AS A GF-COMPACTIFICATION

Next, we are going to consider certain directed fractal structures which induced quasi-uniformities are the Pervin quasi-uniformity and the finest transitive quasi-uniformity.

Definition 6.1. *Let Γ be a covering of a topological space X . We say that Γ is compatible with the topology of X (or simply compatible) if $U_\Gamma(x) = X \setminus \bigcup\{A \in \Gamma : x \notin A\}$ is open for all $x \in X$.*

If Γ is a compatible covering, we will denote

$$U_\Gamma = \{(x, y) \in X \times X : y \in U_\Gamma(x)\}.$$

Note that U_Γ is an entourage of X .

Note that it is true that $(U_\Gamma)^{-1}(x) = \bigcap \{A \in \Gamma : x \in A\}$ (see Proposition 2.3). Also note that if Γ is a covering and $U_\Gamma(x)$ is a neighborhood of x for every $x \in X$ then $U_\Gamma(x)$ is open for every $x \in X$, since if $y \in U_\Gamma(x)$ then $U_\Gamma(y) \subseteq U_\Gamma(x)$.

Proposition 6.2. *Let Γ be a covering of a topological space X . Then Γ is compatible if and only if it is a closed closure-preserving covering.*

Proof. If Γ is a closed closure-preserving covering, then it is clear that

$$U_\Gamma(x) = X \setminus \bigcup \{A \in \Gamma : x \notin A\}$$

is open for each $x \in X$ and hence Γ is compatible.

On the other hand, let Γ be a compatible covering, let $\{A_\lambda : \lambda \in \Lambda\}$ be any subfamily of Γ and let $x \in \overline{\bigcup_{\lambda \in \Lambda} A_\lambda}$. Then $U_\Gamma(x) \cap \bigcup_{\lambda \in \Lambda} A_\lambda$ is nonempty, so there exists $\lambda_0 \in \Lambda$ and $y \in A_{\lambda_0}$ such that $y \in U_\Gamma(x) \cap A_{\lambda_0}$, but then $x \in (U_\Gamma)^{-1}(y) = \bigcap \{A \in \Gamma : y \in A\}$, and hence $x \in A_{\lambda_0}$. Therefore $\bigcup_{\lambda \in \Lambda} A_\lambda$ is closed, and hence Γ is a closed closure-preserving covering. \square

Let Γ_1, Γ_2 be two coverings of a space X . We denote

$$\Gamma_1 \wedge \Gamma_2 = \{A \cap B : A \in \Gamma_1; B \in \Gamma_2\}.$$

The proof of the following lemma is straightforward.

Lemma 6.3. *Let Γ_1 and Γ_2 be two compatible coverings of a topological space. Then $U_{\Gamma_1 \wedge \Gamma_2} = U_{\Gamma_1} \cap U_{\Gamma_2}$ and hence $\Gamma_1 \wedge \Gamma_2$ is a compatible covering.*

Lemma 6.4. *Let (X, \mathcal{U}) be a quasi-uniform space, and let $U \in \mathcal{U}$ such that $U \circ U = U$. Then $\Gamma_U = \{U^{-1}(x) : x \in X\}$ is a compatible covering of X and $U_{\Gamma_U} = U$.*

Proof. First, let us show that Γ_U is a closed covering. Let $x \in X$, and let $y \in \overline{U^{-1}(x)}$, then it is clear that there exists $z \in U(y) \cap U^{-1}(x)$, but then it follows that $x \in U \circ U(y) = U(y)$, whence $y \in U^{-1}(x)$, and hence $U^{-1}(x)$ is closed for all $x \in X$. Then Γ_U is a closed covering. By Proposition 3.5 it follows that $U_{\Gamma_U} = U$, and hence Γ_U is a compatible covering. \square

Proposition 6.5. *Let X be a topological space, and let $\mathbf{\Gamma}$ be the directed fractal structure consisting of all compatible coverings of X . Then $\mathbf{\Gamma}$ is the directed fractal structure induced by the finest transitive quasi-uniformity (and hence the quasi-uniformity induced by $\mathbf{\Gamma}$ is the finest transitive quasi-uniformity of X).*

Proof. Let \mathcal{FT} be the finest transitive quasi-uniformity of X , and let $U \in \mathcal{FT}$ with $U \circ U = U$. By Lemma 6.4 it follows that Γ_U is a compatible covering of X and that $U_{\Gamma_U} = U$. Then $\mathbf{\Gamma}_{\mathcal{FT}} \subseteq \mathbf{\Gamma}$.

By Lemma 6.3 it follows that $\Gamma_1 \wedge \Gamma_2 \in \mathbf{\Gamma}$ for each $\Gamma_1, \Gamma_2 \in \mathbf{\Gamma}$ and it is clear that $\Gamma_1 \wedge \Gamma_2 \prec\prec \Gamma_1, \Gamma_2$, and hence $\mathbf{\Gamma}$ is a base of a directed fractal structure over X . Let Γ_1, Γ_2 be coverings such that Γ_1 is compatible and $\Gamma_1 \prec\prec \Gamma_2$.

Then it is easy to check that $U_{\Gamma_1} \subseteq U_{\Gamma_2}$, and hence it follows that Γ_2 is a compatible covering. Therefore $\mathbf{\Gamma}$ is a directed fractal structure over X .

Let $\mathcal{U}_{\mathbf{\Gamma}}$ be the quasi-uniformity induced by $\mathbf{\Gamma}$, then we have proved that $U \in \mathcal{U}_{\mathbf{\Gamma}}$ whenever $U \in \mathcal{FT}$, and therefore $\mathcal{FT} \subseteq \mathcal{U}_{\mathbf{\Gamma}}$. On the other hand, it is clear that $\mathcal{U}_{\mathbf{\Gamma}}$ is transitive, whence \mathcal{FT} is finest than $\mathcal{U}_{\mathbf{\Gamma}}$, and hence $\mathcal{FT} = \mathcal{U}_{\mathbf{\Gamma}}$.

Since $\mathcal{FT} = \mathcal{U}_{\mathbf{\Gamma}}$ it follows that $\mathbf{\Gamma}_{\mathcal{U}_{\mathbf{\Gamma}}} = \mathbf{\Gamma}_{\mathcal{FT}}$. By Corollary 3.6 we have that $\mathbf{\Gamma} \subseteq \mathbf{\Gamma}_{\mathcal{U}_{\mathbf{\Gamma}}} = \mathbf{\Gamma}_{\mathcal{FT}}$ and hence $\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathcal{FT}}$. \square

Proposition 6.6. *Let $\mathbf{\Gamma}$ be the finite directed fractal structure consisting of all compatible finite coverings of X . Then $\mathbf{\Gamma}$ is the quasi-uniformity induced by the Pervin quasi-uniformity of X (and hence the quasi-uniformity induced by $\mathbf{\Gamma}$ is the Pervin quasi-uniformity).*

Proof. Let \mathcal{P} be the finest transitive quasi-uniformity of X , and let $U \in \mathcal{P}$ with $U \circ U = U$. By Lemma 6.4 it follows that Γ_U is a compatible covering of X and that $U_{\Gamma_U} = U$. By Proposition 3.9 $\mathbf{\Gamma}_U$ is finite since U is totally bounded. Then $\mathbf{\Gamma}_{\mathcal{P}} \subseteq \mathbf{\Gamma}$.

By Lemma 6.3 it follows that $\Gamma_1 \wedge \Gamma_2 \in \mathbf{\Gamma}$ for each $\Gamma_1, \Gamma_2 \in \mathbf{\Gamma}$ and it is clear that $\Gamma_1 \wedge \Gamma_2 \prec\prec \Gamma_1, \Gamma_2$, and hence $\mathbf{\Gamma}$ is a base of a directed fractal structure over X . Let Γ_1, Γ_2 be coverings such that Γ_1 is compatible and finite and $\Gamma_1 \prec\prec \Gamma_2$. Then it is easy to check that $U_{\Gamma_1} \subseteq U_{\Gamma_2}$, and hence it follows that Γ_2 is a compatible covering. Note that it is also finite. Therefore $\mathbf{\Gamma}$ is a finite directed fractal structure over X .

By Proposition 3.9, it follows that $\mathcal{U}_{\mathbf{\Gamma}}$ is totally bounded. Since the Pervin quasi-uniformity is the finest totally bounded transitive quasi-uniformity of X and $\mathcal{P} \subseteq \mathcal{U}_{\mathbf{\Gamma}}$, then it follows that $\mathcal{P} = \mathcal{U}_{\mathbf{\Gamma}}$.

Since $\mathcal{P} = \mathcal{U}_{\mathbf{\Gamma}}$ it follows that $\mathbf{\Gamma}_{\mathcal{U}_{\mathbf{\Gamma}}} = \mathbf{\Gamma}_{\mathcal{P}}$. By Corollary 3.6 we have that $\mathbf{\Gamma} \subseteq \mathbf{\Gamma}_{\mathcal{U}_{\mathbf{\Gamma}}} = \mathbf{\Gamma}_{\mathcal{P}}$ and hence $\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathcal{P}}$. \square

We denote by $\mathbf{\Gamma}_{\mathcal{P}}$ the directed fractal structure induced by the Pervin quasi-uniformity, which by Proposition 6.6 is the finite directed fractal structure consisting of all finite closed coverings of X (note that a finite covering is compatible if and only if it is closed).

Lemma 6.7. *Let X be a topological space. Then $\mathcal{L}_{\mathbf{\Gamma}_{\mathcal{P}}}$ is the family of closed sets of X .*

Proof. Let F be a closed subspace of X , and let $\Gamma = \{F, X \setminus F^{\circ}\}$, then it is clear that $\Gamma \in \mathbf{\Gamma}_{\mathcal{P}}$ and hence $F \in \mathcal{L}_{\mathbf{\Gamma}_{\mathcal{P}}}$. \square

From the previous lemma, the proof of the following theorem is straightforward. This theorem allow us to introduce the Stone-Ćech compactification as a GF-compactification.

Theorem 6.8. *Let X be a topological space. Then $G(X, \mathbf{\Gamma}_{\mathcal{P}})$ is the Stone-Ćech compactification of X if and only if X is normal.*

Finally, we obtain the Stone-Ćech compactification of any Tychonoff space X as a GF-compactification.

Theorem 6.9. *Let X be a Tychonoff space, and let Γ_Z be the family of all finite covering by zero-sets. Then Γ_Z is a finite starbase directed fractal structure and $G(X, \Gamma_Z)$ is the Stone-Čech compactification of X .*

Proof. Let $x \in X$ and U be an open set containing x , then there exists a continuous map $f : X \rightarrow [0, 1]$ with $f(x) \in \{0\}$ and $f(X \setminus U) \subseteq \{1\}$. Let $F = f^{-1}([0, \frac{1}{2}])$ and $V = f^{-1}([\frac{1}{2}, 1])$. Then it is clear that $x \in V \subseteq F \subseteq U$, and hence $\Gamma_1 = \{F, X \setminus V\}$ is a finite covering by zero-sets, so $\Gamma_1 \in \Gamma_Z$. It is also clear that $\text{St}(x, \Gamma_1) = F \subseteq U$.

On the other hand, if $\Gamma_1, \Gamma_2 \in \Gamma_Z$ then it is clear that $\Gamma_1 \wedge \Gamma_2 \in \Gamma_Z$ (note that intersection and union of zero-sets is a zero-set), and since $\Gamma_1 \wedge \Gamma_2 \prec\prec \Gamma_1, \Gamma_2$ it follows that Γ_Z is a base of a directed fractal structure over X . If $\Gamma_1 \prec\prec \Gamma_2$ with $\Gamma_1 \in \Gamma_Z$ then it is easy to check that $\Gamma_2 \in \Gamma_Z$, and hence Γ_Z is a finite starbase directed fractal structure over X .

Finally, since \mathcal{L}_{Γ_Z} is the family of zero-sets of X , then it follows that $W(X, \mathcal{L}_{\Gamma_Z})$ is the Stone-Čech compactification of X , and by Theorem 5.5 it follows that $G(X, \Gamma_Z)$ is the Stone-Čech compactification of X (note that the quasi-uniformity induced by Γ_Z is locally symmetric by Proposition 3.9). \square

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