

A contribution to fuzzy subspaces

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ABSTRACT. We give a new concept of fuzzy topological subspace, which extends the usual one, and study in it the related concepts of interior, closure and connectedness.

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1. INTRODUCTION

A simplest and at the same time a very important operation of General Topology is transition to a subspace because it let us consider hereditary and local properties, completion and compactification of subspaces, etc.

When A is a subspace of a topological space X , the following assertion is satisfied: F is closed in A iff $F = A \cap T$ for a closed set T of X . This is not true, in general, in considering a similar problem for fuzzy topological spaces (fts). For this reason, in fuzzy research the consideration of subspaces in a fts (X, \mathcal{T}) is restricted by authors only to ordinary subsets of X .

In this paper we extend the concept of a fuzzy topological subspace of X , to fuzzy sets A of X for which the above assertion is satisfied, and then we will be able to extend some concepts and results of the (fuzzy) topological spaces to these subspaces.

The structure of the paper is as follows: After preliminaries, in section 3 we define and study the concept of (fuzzy) subspace, and in sections 4–5 we study in it the concepts of interior, closure and connectedness.

2. PRELIMINARIES

Throughout this paper, I will denote the unit real interval $[0, 1]$. For a non-empty set X , I^X denotes the collection of all mappings from X into I . A member B of I^X is called a fuzzy set of X . The set $\{x \in X : B(x) > 0\}$ is called the support of B and is denoted by $\text{Supp } B$. If B takes only the values 0,

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1, B is called a crisp set in X . From now on, we shall not differentiate between a crisp set B in X and (the ordinary subset of X) $\text{Supp } B$. Nevertheless, the crisp set which always takes the value 1 (respectively, 0) on X is denoted by 1 (respectively, 0).

The union and intersection of a family of fuzzy sets $\{A_i\}_i$ of X is $\bigvee_i A_i$ and $\bigwedge_i A_i$, respectively. The complement of $A \in I^X$, denoted by A' , is defined by the formula $A'(x) = 1 - A(x)$, $x \in X$. For $A, B \in I^X$ we write $A \subset B$ or $B \supset A$ if $A(x) \leq B(x)$, for each $x \in X$. The fuzzy set x_λ of X given by $x_\lambda(y) = 0$ if $y \neq x$, and $x_\lambda(x) = \lambda$ ($\lambda \in]0, 1]$) is called a fuzzy point of X with support x [5]. The fuzzy point x_λ is said to be contained in a fuzzy set A or belonging to A , denoted by $x_\lambda \in A$, if $\lambda \leq A(x)$.

A family \mathcal{T} of fuzzy sets of X containing 0 and 1, is called a fuzzy topology on X [1] if it is closed under arbitrary unions and finite intersections. The pair (X, \mathcal{T}) is called a fuzzy topological space (fts). Each member of \mathcal{T} is called an open (fuzzy) set. The complement of an open set is called a closed (fuzzy) set. \mathcal{T}^c denotes the family of all closed sets of (X, \mathcal{T}) .

3. FUZZY SUBSPACES

Definition 3.1. Let (X, \mathcal{T}) be a fts and $Y \in I^X$. The pair (Y, \mathcal{T}_Y) is called a fuzzy topological subspace of (X, \mathcal{T}) if the family

$$\mathcal{T}_Y = \{G \cap Y : G \in \mathcal{T}\}$$

satisfies the following conditions:

- (c1) For each $H \in \mathcal{T}_Y$ there exist $F_H \in \mathcal{T}^c$ such that $Y - H = F_H \cap Y$.
- (c2) For each $F \in \mathcal{T}^c$ there exist $G_F \in \mathcal{T}$ such that $Y - (F \cap Y) = G_F \cap Y$.

In this case, the members of \mathcal{T}_Y will be called \mathcal{T}_Y -open. If $F \subset Y$ and $Y - F \in \mathcal{T}_Y$, F will be called a \mathcal{T}_Y -closed (fuzzy) set. For shortness we will say Y is a subspace of X .

Notice that \mathcal{T}_Y is closed under finite intersections and arbitrary unions and that 0 and Y (instead of X) are both \mathcal{T}_Y -open and \mathcal{T}_Y -closed. Also, if Y is an ordinary subspace of the topological space X , Y is a subspace of X in the sense of definition 3.1.

Remark 3.2. The above conditions (c1) and (c2) establish that F is \mathcal{T}_Y -closed iff $F = T \cap Y$ for a \mathcal{T} -closed set T of the fts (X, \mathcal{T}) .

Example 3.3. (a) (Ordinary subsets of a fts X are subspaces of X).

Let (X, \mathcal{T}) be a fts and let Y an ordinary subset of X . If $H \in \mathcal{T}_Y$ then $H = G \cap Y$ for some $G \in \mathcal{T}$, thus $Y - H = (1 - G) \cap Y$ and (c1) is satisfied.

If $F \in \mathcal{T}^c$ then $Y - (F \cap Y) = (1 - F) \cap Y$, where $1 - F \in \mathcal{T}$, and (c2) is satisfied. So, Y is a subspace of X .

(b) Let $X \neq \emptyset$. We will denote by f_c the constant function on X given by $f_c(x) = c$, for each $x \in X$, with $0 \leq c \leq 1$. Now, consider the Lowen indiscrete fuzzy topology $\mathcal{T} = \{f_c : c \in [0, 1]\}$, on X . (Notice that $\mathcal{T} = \mathcal{T}^c$.)

Fix a real number $k \in]0, 1[$, and choose a non-empty ordinary subset B of X . Now consider $Y \in I^X$ given by $Y(x) = k$ if $x \in B$, and $Y(x) = 0$ if $x \notin B$. We will see that Y is a subspace of (X, \mathcal{T}) .

In fact, $H \in \mathcal{T}_Y$ iff $H(x) = m$, for each $x \in B$ and $H(x) = 0$ elsewhere, for some $m \in [0, k]$. Then $Y - H$ is given by $(Y - H)(x) = k - m$ if $x \in B$ and $(Y - H)(x) = 0$ if $x \notin B$. So, $Y - H = f_{k-m} \cap Y$ and (c1) is satisfied since $1 - f_{k-m} = f_{1-(k-m)} \in \mathcal{T}$.

Now suppose $F \in \mathcal{T}^c$. Then, also $F \in \mathcal{T}$, and thus $F = f_m$, for some $m \in [0, 1]$. Hence, $Y - (F \cap Y)$ is given by $(Y - (F \cap Y))(x) = k - (m \wedge k)$ for $x \in B$ and $(Y - (F \cap Y))(x) = 0$ if $x \notin B$, and clearly (c2) is satisfied.

(c) (Construction of fuzzy topological subspaces).

Fix a real number $\gamma \in]0, \frac{1}{2}[$, and let \mathcal{L} be a fuzzy topology on X such that $G(x) \leq \gamma$, for each $x \in X$ and $G \in \mathcal{L} \sim \{1\}$. Take $A \in I^X$ such that $A(x) \leq \frac{1}{2}$, for $x \in X$. Clearly, if \mathcal{L} contains at least two proper open sets then A is not a subspace of (X, \mathcal{L}) .

Now, denote

$$\mathcal{L}^* = \{G^* : G^* = 1 - (A - (A \cap G)), G \in \mathcal{L}\}.$$

We will see that $\mathcal{L}^* \cup \{0\}$ is a fuzzy topology on X :

Consider a Family $\{G_i^* : i \in I\}$ of elements of \mathcal{L}^* , and suppose $G_i^* = 1 - (A - (A \cap G_i))$, where $G_i \in \mathcal{L}$, $i \in I$. We will see that $\bigcup_i G_i^* \in \mathcal{L}^*$.

We have

$$\begin{aligned} \bigcup_i G_i^* &= \bigcup_i (1 - (A - (A \cap G_i))) \\ &= (1 - A) + \bigcup_i (A \cap G_i) \\ &= (1 - A) + (A \cap (\bigcup_i G_i)) \\ &= (1 - A) + (A \cap G) \end{aligned}$$

where $G = \bigcup_i G_i \in \mathcal{L}$, and so $\bigcup_i G_i^* \in \mathcal{L}^*$.

Now, we will see \mathcal{L}^* is closed under finite intersection. Take $G_i^* = 1 - (A - (A \cap G_i))$, where $G_i \in \mathcal{L}$, $i = 1, 2$. We have

$$\begin{aligned} G_1^* \cap G_2^* &= (1 - (A - (A \cap G_1))) \cap (1 - (A - (A \cap G_2))) \\ &= (1 - A) + ((A \cap G_1) \cap (A \cap G_2)) \\ &= (1 - A) + (A \cap (G_1 \cap G_2)) \\ &= 1 - A + (A \cap G) \end{aligned}$$

where $G = G_1 \cap G_2 \in \mathcal{T}$, and thus $G_1^* \cap G_2^* \in \mathcal{L}^*$.

Clearly $1 \in \mathcal{L}^*$ and then $\mathcal{L}^* \cup \{0\}$ is a fuzzy topology on X . Now, if $G \in \mathcal{L}$ with $G \neq 1$, then $G(x) < \frac{1}{2}$ for each $x \in X$, and if $G^* \in \mathcal{L}^*$ with $G^* \neq 0$ we have $G^*(x) \geq \frac{1}{2}$, and from these facts it is easy to verify that $\mathcal{T} = \mathcal{L} \cup \mathcal{L}^*$ is a fuzzy topology on X . Now we will see that A is subspace of (X, \mathcal{T}) :

Clearly, $\mathcal{T}_A = \mathcal{L}_A \cup \{A\}$. First we will see condition (c1) is satisfied. Let $H \in \mathcal{T}_A$ and suppose $H = G \cap A$ with $G \in \mathcal{T}$. We distinguish two possibilities:

(i) $G \in \mathcal{L}^* \cup \{1\}$. In this case $H = G \cap A = A$ and $A - H = 0 = 0 \cap A$.

- (ii) $G = \mathcal{L} \sim \{1\}$. In this case $A - H = A - (G \cap A) = (A - (G \cap A)) \cap A$ since $A - (G \cap A) \subset A$, with $1 - (A - (G \cap A)) \in \mathcal{L}^* \subset \mathcal{T}$.

Now, we will prove that condition (c2) is satisfied. Suppose $F \in \mathcal{T}^c$. We distinguish two possibilities:

- (i) $1 - F \in \mathcal{L}^*$. In this case $F = A - (A \cap G)$, with $G \in \mathcal{L}$. So,

$$A - [(A - (A \cap G)) \cap A] = A - (A - (A \cap G)) = A \cap G \in \mathcal{T}_A$$

- (ii) $1 - F \in L \sim \{1\}$. In this case $F(x) \geq \frac{1}{2}$ for each $x \in X$, and thus $F \supset A$. Now, $A - (F \cap A) = 0 \in \mathcal{T}_A$.

Definition 3.4. Let Y be a subspace of the fts (X, \mathcal{T}) , and suppose $B \subset Y$. The pair (B, \mathcal{T}_B) is called a fuzzy topological subspace of (Y, \mathcal{T}_Y) if the family

$$\mathcal{T}_B = \{G \cap B : G \in \mathcal{T}_Y\}$$

satisfy the following conditions

- (c1)' For each $H \in \mathcal{T}_B$ it exists $F_H \in \mathcal{T}_Y^c$ such that $B - H = F_H \cap B$.
(c2)' For each $F \in \mathcal{T}_Y^c$ it exists $G_F \in \mathcal{T}_Y$ such that $B - (F \cap B) = G_F \cap B$.

The terminology \mathcal{T}_B (instead of \mathcal{T}_{Y_B}) is justified in proposition 3.6.

Otherwise, the elements of \mathcal{T}_B are called \mathcal{T}_B -open. If $F \subset B$ and $B - F \in \mathcal{T}_B$, F is called \mathcal{T}_B -closed. For shortness we will say B is a subspace of Y .

Remark 3.5. The above conditions (c1)' and (c2)' establish that F is \mathcal{T}_B -closed iff $F = B \cap T$ for a \mathcal{T}_Y -closed set T of Y .

Proposition 3.6. Suppose B is a subspace of Y and Y is a subspace of the fts (X, \mathcal{T}) . Then, B is a subspace of (X, \mathcal{T}) .

Proof. It is an immediate consequence of remarks 3.2 and 3.5. \square

Proposition 3.7. Let Y be a subspace of the fts (X, \mathcal{T}) . If Y is \mathcal{T} -open (respectively, \mathcal{T} -closed) then $G \in \mathcal{T}_Y$ iff $G \in \mathcal{T}$ (respectively, $F \in \mathcal{T}_Y^c$ iff $F \in \mathcal{T}^c$).

Proposition 3.8. Let B and Y the two subspaces of the fts (X, \mathcal{T}) . If $B \subset Y$, then B is a subspace of Y .

Proof. We have $H \in \mathcal{T}_B$ iff there exists $G^* \in \mathcal{T}$ such that $H = G^* \cap B$. Now, $G \cap B = G \cap Y \cap B$, and since $G = G^* \cap Y \in \mathcal{T}_Y$, then $\mathcal{T}_B = \{G \cap B : G \in \mathcal{T}_Y\}$. We will see that the family \mathcal{T}_B satisfies (c1)' and (c2)'.
Let $H \in \mathcal{T}_B$; then there exists $F_H \in \mathcal{T}^c$ such that

$$B - H = F_H \cap B = (F_H \cap Y) \cap B,$$

where $F_H \cap Y \in \mathcal{T}_Y^c$, and so (c1)' is satisfied.

Now, let $F \in \mathcal{T}_Y^c$; then there exists $G_F \in \mathcal{T}$ such that

$$B - (F \cap B) = G_F \cap B = (G_F \cap Y) \cap B,$$

where $G_F \cap Y \in \mathcal{T}_Y$, and so (c2)' is satisfied. \square

Lemma 3.9. *Let M , N and P be fuzzy sets of X . Then*

$$(M \cap N) - (P \cap (M \cap N)) = (M - (P \cap M)) \cap (N - (P \cap N))$$

Proof. Let $x \in X$ and suppose $M(x) \geq N(x)$. We distinguish three possibilities:

- (1) $N(x) \leq P(x) \leq M(x)$. In this case, the left hand of the above inequality becomes $N(x) - N(x) = 0$, and the right hand becomes $(M(x) - P(x)) \wedge (N(x) - N(x)) = 0$.
- (2) $P(x) \leq N(x) \leq M(x)$. Now the left hand of the inequality becomes $N(x) - P(x)$, and the right hand becomes $(M(x) - P(x)) \wedge (N(x) - P(x)) = N(x) - P(x)$.
- (3) $N(x) \leq M(x) \leq P(x)$. Now, the left hand of the inequality becomes $N(x) - N(x) = 0$, and the right hand becomes $(M(x) - M(x)) \wedge (N(x) - N(x)) = 0$.

Since the announced equality is symmetric respect M and N , the same argument is valid for $N(x) \geq M(x)$, and then the equality is established. \square

Proposition 3.10. *Let A and B be two subspaces of the fts (X, \mathcal{T}) . Then $A \cap B$ is subspace of (X, \mathcal{T}) .*

Proof. Consider the family $\mathcal{T}_{A \cap B} = \{G \cap (A \cap B) : G \in \mathcal{T}\}$. We will see that $\mathcal{T}_{A \cap B}$ satisfies (c1) and (c2).

Let $H \in \mathcal{T}_{A \cap B}$; then $H = G \cap (A \cap B)$ with $G \in \mathcal{T}$. Now, by lemma 3.9,

$$\begin{aligned} (A \cap B) - H &= (A \cap B) - (G \cap (A \cap B)) \\ &= (A - (G \cap A)) \cap (B - (G \cap B)). \end{aligned}$$

But $A - (G \cap A)$ is \mathcal{T}_A -closed and hence $A - (G \cap A) = F_A \cap A$ for some $F_A \in \mathcal{T}^c$. Also $B - (G \cap B) = F_B \cap B$ for some $F_B \in \mathcal{T}^c$ and therefore

$$\begin{aligned} (A \cap B) - H &= (F_A \cap A) \cap (F_B \cap B) \\ &= (F_A \cap F_B) \cap (A \cap B) \end{aligned}$$

and (c1) is satisfied since $F_A \cap F_B$ is \mathcal{T} -closed.

Now, let $F \in \mathcal{T}^c$. By lemma 3.9,

$$\begin{aligned} (A \cap B) - (F \cap (A \cap B)) &= (A - (F \cap A)) \cap (B - (F \cap B)) \\ &= G_A \cap G_B \end{aligned}$$

where $G_A = A - (F \cap A) \in \mathcal{T}_A$ and $G_B = B - (F \cap B) \in \mathcal{T}_B$. So, there are $G_1, G_2 \in \mathcal{T}$ such that $G_A = G_1 \cap A$ and $G_B = G_2 \cap B$ and therefore

$$\begin{aligned} (A \cap B) - (F \cap (A \cap B)) &= (G_1 \cap A) \cap (G_2 \cap B) \\ &= (G_1 \cap G_2) \cap (A \cap B) \\ &= G \cap (A \cap B) \end{aligned}$$

where $G = G_1 \cap G_2 \in \mathcal{T}$, and (c2) is satisfied. \square

Since ordinary subsets in a fts (X, \mathcal{T}) are subspaces, we have the following corollary.

Corollary 3.11. *Let A be a subspace of the fts (X, \mathcal{T}) and Y an ordinary subset of X . Then $A \cap Y$ is a subspace of (X, \mathcal{T}) .*

If A and B are two subspaces of X , in general $A \cup B$ is not a subspace of X , even if $A \cap B = 0$, as shows the following example.

Example 3.12. Let $X = [0, 1]$ and choose three the real numbers a , b and c , such that $0 < c < a < b < 1$. Consider X endowed with the Lowen indiscrete topology \mathcal{T} of example 3.3 (b). Consider the fuzzy sets A and B of X given by

$$A(x) = \begin{cases} a & 0 \leq x \leq \frac{1}{2} \\ 0 & \frac{1}{2} < x \leq 1 \end{cases} \quad B(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ b & \frac{1}{2} < x \leq 1 \end{cases}$$

by (b) of the example 3.3, A and B are subspaces of X . Obviously $A \cap B = 0$ but we will see $A \cup B$ is not a subspace of X .

Consider the constant function f_c on X defined by $f_c(x) = c$, $x \in X$. We have that $f_c \in \mathcal{T}^c$ and since

$$(A \cup B - (f_c \cap (A \cup B)))(x) = (A \cup B - f_c)(x) = \begin{cases} a - c & 0 \leq x \leq \frac{1}{2} \\ b - c & \frac{1}{2} < x \leq 1 \end{cases}$$

it is obvious that condition (c2) cannot be satisfied.

4. INTERIOR AND CLOSURE

In this section, Y will be a subspace of the fts (X, \mathcal{T}) .

According with [1], the \mathcal{T}_Y -interior, denoted by $\text{int}_Y A$, of a fuzzy set A contained in Y is the largest \mathcal{T}_Y -open (fuzzy) set contained in A , and the \mathcal{T}_Y -closure, denoted by $\text{cl}_Y A$, is the smallest \mathcal{T}_Y -closed (fuzzy) set containing A .

Proposition 4.1. *Let $A \subset Y$. Then*

- (i) $\text{int}_X A = \text{int}_Y A \cap \text{int}_X Y$.
- (ii) $\text{cl}_Y A = \text{cl}_X A \cap Y$.

Proof. It is similar to the classic case. □

Through the notion of fuzzy point, it is possible to study the concepts of interior and cluster (adherence) point. According with [5], we give the following definition.

Definition 4.2. A fuzzy set A in (X, \mathcal{T}_Y) is called a \mathcal{T}_Y -neighborhood of the fuzzy point x_λ if there exists $B \in \mathcal{T}_Y$ such that $x_\lambda \in B \subset A$. We also say that x_λ is \mathcal{T}_Y -interior of A . Then, $x_\lambda \in \text{int}_Y A$ iff x_λ is \mathcal{T}_Y -interior of A .

In [3], is given the following definition.

Definition 4.3. The fuzzy set point x_λ is said to belong to B , written $x_\lambda \tilde{\in} B$, iff $B(x) > \lambda$.

According with [3], the following is a distinct definition of an interior point.

Definition 4.4. A fuzzy set A in (Y, \mathcal{T}_Y) is called a \mathcal{T}_Y^\sim -neighborhood of the fuzzy point x_λ if there exists $B \in \mathcal{T}_Y$ such that $x_\lambda \tilde{\in} B \subset A$. Also, x_λ is called a \mathcal{T}_Y^\sim -interior point of A . Then, $x_\lambda \tilde{\in} \text{int}_Y A$ iff x_λ is a \mathcal{T}_Y^\sim -interior point of A .

Notice that if $(\text{int}_Y A)(x) > 0$ then $x_{(\text{int}_Y A)(x)}$ is \mathcal{T}_Y -interior of A , but it is not \mathcal{T}_Y^\sim -interior of A . Nevertheless,

$$\text{int}_Y A = \bigcup \{x_\lambda : x_\lambda \text{ is } \mathcal{T}_Y\text{-interior of } A\} = \bigcup \{x_\lambda : x_\lambda \text{ is } \mathcal{T}_Y^\sim\text{-interior of } A\}.$$

By (i) of proposition 4.1, we have the following corollary.

Corollary 4.5. *Let A be contained in Y . Then, a fuzzy point $x_\lambda \in \text{int}_X Y$ (respectively, $x_\lambda \tilde{\in} \text{int}_X Y$) is a \mathcal{T}_Y -interior (respectively, a \mathcal{T}_Y^\sim -interior) point of A iff it is a \mathcal{T} -interior point of A .*

The following definitions and results are obvious generalizations of the ones given in [5].

Definition 4.6. A fuzzy point x_λ is said to be Y -quasi-coincident with the fuzzy set A of X , denoted by $x_\lambda q_Y A$, if $\lambda + A(x) > Y(x)$.

Definition 4.7. Let $A, B \subset Y$. A is said to be Y -quasi-coincident with B , denoted by $A q_Y B$, if there exists $x \in X$ such that $A(x) + B(x) > Y(x)$.

Definition 4.8. A fuzzy set A in (X, \mathcal{T}_Y) is called a Q_Y -neighborhood of x_λ if there exists $B \in \mathcal{T}_Y$, $B \subset A$, such that $x_\lambda q_Y B$.

Proposition 4.9. *Let $A, B \subset G$. Then $A \subset B$ iff A and $Y - B$ are not Y -quasi-coincident; particularly $x_\lambda \in A$ iff x_λ is not Y -quasi-coincident with $Y - A$.*

Theorem 4.10. *A fuzzy point $x_\lambda \in \text{cl}_Y A$ iff each Q_Y -neighborhood of x_λ is Y -quasi-coincident with A .*

Definition 4.11. A fuzzy point x_λ is called a \mathcal{T}_Y -adherence point of a fuzzy set A if every Q_Y -neighborhood of x_λ is Y -quasi-coincident with A .

According with [3] we give the following definition.

Definition 4.12. The fuzzy point x_λ is called a \mathcal{T}_Y -cluster point of A if for each $G \in \mathcal{T}_Y$ such that $x_{Y(x)-\lambda} \tilde{\in} G$ implies $G \not\subset Y - A$.

Proposition 4.13. *Let $A \subset Y$. The fuzzy point x_λ is \mathcal{T}_Y -cluster point of A iff it is a \mathcal{T}_Y -adherence point of A .*

Proof. Suppose x_λ is \mathcal{T}_Y -cluster point of A . Let $G \in \mathcal{T}_Y$ a Q -neighborhood of x_λ . Then, $\lambda + G(x) > Y(x)$ and thus $x_{Y(x)-\lambda} \tilde{\in} G$, and $G \not\subset Y - A$ since x_λ is a \mathcal{T}_Y -cluster point of A . Therefore, there exists $x \in X$ such that $G(x) > Y(x) - A(x)$, and so G is Y -quasi-coincident with A .

Suppose x_λ is a \mathcal{T}_Y -adherence fuzzy point of A . Let $G \in \mathcal{T}_Y$ such that $x_{Y(x)-\lambda} \tilde{\in} G$. Then $Y(x) - \lambda < G(x)$ and thus G is a neighborhood of x_λ . So, G is Y -quasi-coincident with A . and then there exists $x \in X$ such that $G(x) + A(x) > Y(x)$; therefore $G \not\subset Y - A$ and x_λ is \mathcal{T}_Y -cluster point of A . \square

By (ii) of proposition 4.1 we have the following corollary.

Corollary 4.14. *Let A be contained in Y . Then, a fuzzy point $x_\lambda \in Y$ is a \mathcal{T}_Y -adherence (cluster) point of A iff it is a \mathcal{T} -adherence (cluster) point of A .*

5. CONNECTEDNESS

We will use the concepts of connectedness due to Pu and Liu [5], [6], but with terminology of [7].

Definition 5.1. A fuzzy set D in the fts (X, \mathcal{T}) is called C -disconnected (respectively O -disconnected) if there are $A, B \in \mathcal{T}^c$ (respectively, $A, B \in \mathcal{T}$) such that $A \cap D \neq 0$, $B \cap D \neq 0$, $A \cap B \cap D = 0$ and $A \cup B \supset D$. A fuzzy set is called C -connected (respectively O -connected) if it is not C -disconnected (respectively O -disconnected).

In contrast to General Topology the use of closed and open fuzzy sets in definitions of connectedness of fuzzy sets results in two distinct concepts. Nevertheless, we will see that these concepts agree in subspaces.

According with definition 5.1, we give the following definition.

Definition 5.2. A subspace (Y, \mathcal{T}_Y) of the fts X will be called C -disconnected (respectively O -disconnected) if there are two non-empty \mathcal{T}_Y -closed (respectively \mathcal{T}_Y -open) sets A and B such that $A \cap B = 0$ and $Y = A \cup B$.

The following proposition shows that connected properties are absolute properties in subspaces.

Proposition 5.3. *If Y is a subspace of the fts X , then the fuzzy set Y is C -connected (respectively O -connected) iff the subspace (Y, \mathcal{T}_Y) is C -connected (respectively O -connected).*

Proof. It is straightforward. \square

It is clear that a fts (X, \mathcal{T}) is O -connected iff it is C -connected. Now this fact is extendable to subspaces in the next proposition.

Proposition 5.4. *Let Y be a subspace of the fts (X, \mathcal{T}) . Then Y is C -connected iff it is O -connected.*

Proof. Suppose Y is not O -connected. Then there are two sets $G, H \in \mathcal{T}$ such that

- (1) $G \cap Y \neq 0$, $H \cap Y \neq 0$, $G \cap H \cap Y \neq 0$ and
- (2) $Y = (G \cap Y) \cup (H \cap Y)$.

Now, by (1), for $x \in X$, $(G \cap Y)(x) \neq 0$ iff $(H \cap Y)(x) = 0$ and also $(H \cap Y)(x) \neq 0$ iff $(G \cap Y)(x) = 0$, and thus $Y - (G \cap Y) \neq 0$, $Y - (H \cap Y) \neq 0$, and $(Y - (G \cap Y)) \cup (Y - (H \cap Y)) = Y$.

Also, by (2), for $x \in X$ if $G(x) < Y(x)$ then $H(x) \geq Y(x)$, and if $H(x) < Y(x)$ then $G(x) \geq Y(x)$, and thus $(Y - (G \cap Y)) \cap (Y - (H \cap Y)) = 0$, and then (Y, \mathcal{T}_Y) is not C -connected.

The converse is showed with a similar argument. \square

Definition 5.5 (P1). Two fuzzy sets A_1 and A_2 in a fts (X, \mathcal{T}) are said to be Q -separated if there exist two \mathcal{T} -closed sets H_i ($i = 1, 2$) such that $H_i \supset A_i$ ($i = 1, 2$), and $H_1 \cap A_2 = H_2 \cap A_1 = 0$. It is obvious that A_1 and A_2 are Q -separated iff $\text{cl } A_1 \cap A_2 = \text{cl } A_2 \cap A_1 = 0$.

Note 5.6. A fuzzy set D is C -disconnected [5] iff there exist two non-empty sets A and B , both two contained in $\text{Supp } D$, such that A and B are Q -separated, and $D = A \cup B$. According with this, we give the following definition.

Definition 5.7. A fuzzy subspace (Y, \mathcal{T}_Y) of X is here called disconnected if there exists two non-empty sets, both two contained in Y , such that A and B are Q -separated and $Y = A \cup B$. Y is called connected if it is not disconnected.

Theorem 5.8. *Let Y be a subspace of (X, \mathcal{T}) . They are equivalent:*

- (i) Y is connected.
- (ii) Y is C -connected.
- (iii) Y is O -connected.

Proof. By proposition 5.4 we only have to prove that (i) and (ii) are equivalent. Suppose Y is disconnected. Then there exist two sets $A, B \subset Y$ such that $\text{cl}_X A \cap B = 0$, $\text{cl}_X B \cap A = 0$ and $Y = A \cup B$.

Now, by (ii) of proposition 4.1, we have

$$\text{cl}_Y A = Y \cap \text{cl}_X A = (A \cup B) \cap \text{cl}_X A = (A \cap \text{cl}_X A) \cup (B \cap \text{cl}_X A) = A,$$

and hence A is \mathcal{T}_Y -closed. Similarly, B is \mathcal{T}_Y -closed and then Y is C -disconnected.

Suppose now that Y is C -disconnected. Then, there are two non-empty \mathcal{T}_Y -closed sets A and B , such that $A \cap B = 0$ and $Y = A \cup B$. Now, otherwise by (ii) of proposition 4.1, we have

$$A \cap \text{cl}_X B = (A \cap Y) \cap \text{cl}_X B = A \cap \text{cl}_Y B = A \cap B = 0.$$

Similarly, $B \cap \text{cl}_X A = 0$, and then A and B are Q -separated. \square

Lemma 5.9. *If A and B are Q -separated in the fts X and Y is a connected subspace of X , with $Y \subset A \cup B$, then $Y \subset A$ or $Y \subset B$.*

Proof. If A and B are Q -separated in the fts X , then $A \cap Y$, and $B \cap Y$ are also Q -separated in X , and $Y = (A \cap Y) \cup (B \cap Y)$, then $A \cap Y = 0$ or $B \cap Y = 0$, i.e., $Y \subset B$ or $Y \subset A$. \square

As in the classic case, theorem 5.8 and lemma 5.9 provide with some neat ways of proving a given space X is connected.

Theorem 5.10. *Let X be a fts.*

- (a) *If $X = \bigcup_{\alpha} X_{\alpha}$, where each X_{α} is a connected subspace of X and $\bigcap_{\alpha} X_{\alpha} \neq 0$, then X is connected.*
- (b) *If each pair p, q of fuzzy points of X lies in some connected subspace $E_{p,q}$ of X , then X is connected.*

- (c) If $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is a connected subspace of X and $X_{n-1} \cap X_n \neq 0$, for each $n \geq 2$, then X is connected.

Proof. The proofs are slight modifications of the classic cases. \square

Next theorem is a generalization of theorem 10.1 of [5].

Theorem 5.11. *Let Y be a subspace of the fts (X, \mathcal{T}) and let D be a C -connected fuzzy set in (X, \mathcal{T}) . If $D \subset Y$ then $\text{cl}_Y D$ is also C -connected.*

Proof. Suppose $\text{cl}_Y D$ is C -disconnected. Then, there are two \mathcal{T} -closed sets A and B such that $A \cap \text{cl}_Y D \neq 0$, $B \cap \text{cl}_Y D \neq 0$, $A \cap B \cap Y = 0$ and $A \cup B \supset \text{cl}_Y D$. By the connectedness of D , we may assume that $A \cap D = 0$, that is $D \subset B$. It follows that $\text{cl}_Y D \subset B$ and thus $A \cap \text{cl}_Y D = 0$, which is a contradiction. \square

Definition 5.12. Two fuzzy sets A_1 and A_2 contained in a subspace (Y, \mathcal{T}_Y) of the fts X is said Q -separated in Y if there exist \mathcal{T}_Y -closed sets H_i ($i = 1, 2$) such that $H_i \supset A_i$ ($i = 1, 2$) and $H_1 \cap A_2 = H_2 \cap A_1 = 0$.

Theorem 5.13. *Let \mathcal{A} be a family of C -connected fuzzy sets in fts X . Suppose $\bigcup \mathcal{A}$ is a fuzzy subspace of X . If no two members of \mathcal{A} are Q -separated in $\bigcup \mathcal{A}$, then $\bigcup \mathcal{A}$ is connected.*

Proof. The same proof as theorem 10.2 of [5], but replacing $\text{Supp} \bigcup \mathcal{A}$ by $\bigcup \mathcal{A}$, proves that $\bigcup \mathcal{A}$ is C -connected, and by theorem 5.8 $\bigcup \mathcal{A}$ is connected. \square

5.14 Final considerations. One can extend in a natural way the T_i -fuzzy separation axioms of [5] in fts to subspaces, in such manner that were hereditary properties. Notice that there are many definitions of T_2 -fts in the literature (see [2]), but are particularly interesting the fuzzy separation axioms given in [4], through the concept of R -neighborhood.

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