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Fenestrations induced by perfect tilings

F.G. Arenas and M.L. Puertas*

Abstract. In this paper we study those regular fenestrations (as defined by Kronheimer in [3]) that are obtained from a tiling of a topological space. Under weak conditions we obtain that the canonical grid is also the minimal grid associated to each tiling and we prove that it is a T_0-Alexandroff semirregular trace space. We also present some examples illustrating how the properties of the grid depend on the properties of the tiling and we pose some questions. Finally we study the topological properties of the grid depending on the properties of the space and the tiling.

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1. Introduction

1.1. **Tilings.** We quote from [5] the following considerations about tilings.

A tiling of a topological space X is a covering of X by sets (called tiles) which are the closures of their pairwise-disjoint interiors.

Tilings of $\mathbb{R}^2$ have received considerable attention (see [2] for a wealth of interesting examples and results as well as an extensive bibliography). On the other hand, the study of tilings of general topological spaces is just beginning (see [1], [4], [5] and [6]).

The following definitions will be basic to our discussion. Let $\mathcal{T} = \{T_i : i \in I\}$ be a tiling of a topological space X, we define $I(x) = \{i \in I : x \in T_i\}$. We define the set of frontier points of $\mathcal{T}$ to be the union of the boundaries of the tiles in $\mathcal{T}$ and denote this set by $F(\mathcal{T})$. The protected points of $\mathcal{T}$ constitute the set $P(\mathcal{T}) = \{x \in X : x \in (\bigcup_{i\in I(x)} T_i)^\circ\}$ and the complement of this set in X is $U(\mathcal{T})$, the set of unprotected points of $\mathcal{T}$.

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1.2. **Fenestrations.** According to Kronheimer (see [3]) a fenestration $\mathcal{E}$ of a topological space X is a family of pairwise disjoint open sets whose union is dense in X. The fenestration is said to be regular if the open sets of the family are regular open sets. If $\mathcal{E}$ is a fenestration of X, so is $\mathcal{E}_{reg} = \{\overline{U}^{\circ} : U \in \mathcal{E}\}$, which will be called the regularization of $\mathcal{E}$. Two fenestrations are said to be equivalent if they have the same regularization.

The relation between tilings and fenestrations is given in the following definition.

Definition 1.1. Given a tiling $\mathcal{T} = \{T_i : i \in I\}$ if we set $A_i = T_i^{\circ}$ we clearly obtain a regular fenestration, called $\mathcal{E}(\mathcal{T})$ (the fenestration induced by a tiling).

Note that not every regular fenestration is induced by a tiling, as the following example shows.

Example 1.2. Let $E_n = \{(x, y) \in \mathbb{R}^2 : \frac{1}{n+1} < y < \frac{1}{n}\}$ if $n \in \mathbb{N}$ and $E_0 = \{(x, y) \in \mathbb{R}^2 : y > 1\}$. $\mathcal{E} = \{E_n : n \geq 0\}$ is a regular fenestration of $X = \{(x, y) : y \geq 0\}$; however $\mathcal{E} = \{\overline{E_n} : n \geq 0\}$ is not a tiling of X, since $\bigcup_{n \geq 0} \overline{E_n} = \{(x, y) : y > 0\}$.

Given a fenestration $\mathcal{E}$ of a topological space X, a pseudogrid associated to $\mathcal{E}$ (see again [3]) is a family $\triangle$ of subsets of X such that $\mathcal{E} \subset \triangle$ and $\triangle$ is a partition of X. Each pseudogrid determines a quotient space and the quotient map is open if and only if the pseudogrid is lower semicontinuous, that is, $\mathrm{St}(G, \triangle) = \bigcup\{A \in \triangle : A \cap G \neq \varnothing\}$ is open for every open set G of X. A lower semicontinuous pseudogrid is called a grid.

In [3], Kronheimer studied under what conditions the grid is minimal. This is an interesting point because then the study of the quotient space associated to each grid becomes easier. To this end given a fenestration $\mathcal{E}$ of a space X, there is a canonical way to associate a pseudogrid $\triangle^{\times}$ to $\mathcal{E}$ by identifying two points of X if and only if , for every open neighborhood of either, there exists an open neighborhood of the other one which intersects the same collection of elements of $\mathcal{E}$. It is proved in [3] (Theorem 6.2.) that $\triangle^{\times}$ is the minimal grid associated to $\mathcal{E}$ if and only if $\triangle^{\times}$ is a grid, that is, is a lower semicontinuous decomposition. If it is not a grid, the minimal grid shall be constructed by a different procedure, since minimal grids always exist.

Moreover, if $\triangle^{\times}$ is primitive (that is, for each of its points, the intersection of all its neighborhoods is a regular open set), Kronheimer shows in section 11 of [3] that $\triangle^{\times}$ has properties in the homotopy category which are closely related to those of X. In fact, that is the reason to study the minimal grid associated to any fenestration.

The main result of this paper is constructing in a more explicit way the canonical pseudogrid of the fenestration associated to a tiling and obtaining conditions to ensure that it is the minimal grid and is primitive. We also study how the properties of the tiling reflect as properties of the grid.

2. The minimal grid of a tiling

In the case that the fenestration comes from a tiling we can give a more precise description of the canonical pseudogrid. First, we need to restrict our attention to a particular class of tilings, which will be called perfect and that are general enough to be useful.

Definition 2.1. Let $\mathcal{T} = \{\overline{A_i} : i \in I\}$, with A_i open in X, be a tiling of a topological space X. Given $\sigma \subset I$, let define $A_\sigma = \{x \in X : I(x) = \sigma\}$. We say that the tiling $\mathcal{T}$ is complete if $A_\sigma \subset \overline{A_\tau}$ for every $\tau \subset \sigma$ supposed that A_σ and A_τ are both nonempty, and we say that is perfect if it is complete and $U(\mathcal{T}) = \varnothing$.

Now, our main theorem presents a sufficient condition on a fenestration (induced by a perfect tiling) for its minimal grid to be primitive. The condition is useful in the sense that it is constructive and does not seem excessively restrictive. Moreover, the condition is the conjunction of three simpler conditions (being determined by a tiling, the tiling is complete, $U(\mathcal{T}) = \varnothing$), the omission of any of which causes the theorem fails, as we will see with some examples.

Theorem 2.2. *Let $\mathcal{T}$ be as in Definition 2.1 a perfect tiling of a topological space X and let $\mathcal{E}(\mathcal{T})$ be the induced regular fenestration. Call $\triangle^\times(\mathcal{T})$ the canonical pseudogrid associated to $\mathcal{E}(\mathcal{T})$. Then the T_0-Alexandroff space whose underlying set is $\triangle^\times(\mathcal{T}) = \{\sigma \subset I : A_\sigma \neq \varnothing\}$ and the minimal neighborhood of every $\sigma \in \triangle^\times(\mathcal{T})$ is the set $V(\sigma) = \{\tau \in \triangle^\times(\mathcal{T}) : \tau \subset \sigma\}$ is called the canonical pseudogrid. Moreover, $\triangle^\times(\mathcal{T})$ is semirregular and a lower semicontinuous decomposition; hence $\triangle^\times(\mathcal{T})$ is the minimal grid and is a primitive space.*

Proof. It is clear, using $U(\mathcal{T}) = \varnothing$, that $x \sim y$ if and only if $I(x) = I(y)$, where $\sim$ is the equivalence relation that leads to the canonical pseudogrid. Hence the equivalence classes are just the sets $A_\sigma = \{x \in X : I(x) = \sigma\}$ for each $\sigma \neq \varnothing$ and consequently the quotient is the set of (nonempty) equivalence classes $\triangle^\times(\mathcal{T}) = \{\sigma \subset I : A_\sigma \neq \varnothing\}$.

The quotient map $\pi^\times : X \to \triangle^\times(\mathcal{T})$ is defined as $\pi^\times(A_\sigma) = \sigma$, so if we define $V(\sigma) = \{\tau \in \triangle^\times(\mathcal{T}) : \tau \subset \sigma\}$, we have $(\pi^\times)^{-1}(V(\sigma)) = (\pi^\times)^{-1}(\{\tau \in \triangle^\times(\mathcal{T}) : \tau \subset \sigma\}) = \{x \in X : I(x) \subset \sigma\} = \bigcup_{i\in\sigma} \overline{A_i}$.

As every point is protected, given a point $x \in A_\sigma$, $\{\overline{A_i} : i \in \sigma\}$ is the set of all tiles that contain x, hence $x \in (\bigcup_{i\in\sigma} \overline{A_i})^\circ$. Hence $A_\sigma \subset (\bigcup_{i\in\sigma} \overline{A_i})^\circ$, that is, there is an open set U in X such that $(\pi^\times)^{-1}(\sigma) = A_\sigma \subset U \subset \bigcup_{i\in\sigma} \overline{A_i} = (\pi^\times)^{-1}(V(\sigma))$. Applying $\pi^\times$ in both sides, we obtain $\sigma \subset \pi^\times(U) \subset V(\sigma)$.

Since we have proved that $A_\tau \subset (\bigcup_{i\in\tau} \overline{A_i})^\circ$ for each $\tau \in \triangle^\times(\mathcal{T})$, we obtain $\bigcup_{\tau\subset\sigma} A_\tau \subset (\bigcup_{i\in\sigma} \overline{A_i})^\circ$. In fact the equality holds. To see this, given a point $x \in (\bigcup_{i\in\sigma} \overline{A_i})^\circ$, there exists an open neighborhood U of x with $U \subset \bigcup_{i\in\sigma} \overline{A_i}$. That is, if $i \notin \sigma$, $U \cap \overline{A_i} = \varnothing$, then $x \notin \overline{A_i}$ so if $i \notin \sigma$, we have $i \notin I(x)$, that is, $\tau = I(x) \subset \sigma$, so finally $x \in A_\tau$ for some $\tau \subset \sigma$ and the equality is obtained. Hence $\bigcup_{\tau\subset\sigma} A_\tau$ is open.

Now, to see that the pseudogrid is lower semicontinuous (a grid) we have to check that for every open set G of X, $\mathrm{St}(G) = \bigcup_{\sigma \in \triangle(G)} A_\sigma$ where $\triangle(G) = \{\sigma \subset I : A_\sigma \cap G \neq \varnothing\}$ is open in X.

The condition $A_\sigma \subset \overline{A_\tau}$ for every $\tau \subset \sigma$ given in the hypothesis is clearly equivalent to $A_\tau \cap G \neq \varnothing$ for every $\tau \subset \sigma$ and every open set G of X with $A_\sigma \cap G \neq \varnothing$.

Now, given $x \in \mathrm{St}(G) = \bigcup_{\sigma \in \triangle(G)} A_\sigma$ there is $\sigma \in \triangle(G)$ such that $x \in A_\sigma$; hence we have $x \in A_\sigma \subset \bigcup_{\tau \subset \sigma} A_\tau \subset \mathrm{St}(G)$. Since $\bigcup_{\tau \subset \sigma} A_\tau$ is open, we have that $\mathrm{St}(G)$ is open, as desired.

Since $\pi^\times$ is an open mapping and $\sigma \subset \pi^\times(U) \subset V(\sigma)$, then we have that $V(\sigma)$ is a neighborhood of σ. To see that is the least one (and hence the space is Alexandroff), suppose there is an open set $V \subset V(\sigma)$, with $\sigma \in V \neq V(\sigma)$. Hence there is $\tau \subset \sigma$ with $A_\tau \neq \varnothing$ and $\tau \notin V$. Using that $\tau \subset \sigma$ implies $A_\sigma \subset \overline{A_\tau}$ and that $G = (\pi^\times)^{-1}(V)$ is open in X and $A_\sigma \subset G$ we obtain $G \cap \overline{A_\tau} \neq \varnothing$, so $G \cap A_\tau \neq \varnothing$, that is $(\pi^\times)^{-1}(V) \cap (\pi^\times)^{-1}(\tau) \neq \varnothing$, which is a contradiction with $\tau \notin V$.

Now, to see that the space is T_0 note that, with the given definition, if $\sigma \neq \tau$ clearly $V(\sigma) \neq V(\tau)$.

Finally, to see that the minimal grid (from Theorem 6.2 of [3]) is also semirregular, we have to show that $(\overline{V(\sigma)})^\circ = V(\sigma)$ for every $\sigma \in \triangle^\times(\mathcal{T})$.

Since $\tau \in \overline{V(\sigma)}$ is equivalent to $\sigma \cap \tau \neq \varnothing$ that is the same that $\sigma \in \overline{V(\tau)}$ (note that $\tau \in \overline{V(\sigma)}$ if and only if $V(\tau) \cap V(\sigma) \neq \varnothing$ and that $V(\tau) \cap V(\sigma) = V(\tau \cap \sigma)$), we have that $\overline{V(\sigma)} = \{\tau \in \triangle^\times(\mathcal{T}) : \tau \cap \sigma \neq \varnothing\}$.

Now, given a point $\eta \in \triangle^\times(\mathcal{T})$, $\eta \in (\overline{V(\sigma)})^\circ$ if and only if $V(\eta) \subset \overline{V(\sigma)}$, that is, if $\{i\} \in V(\eta) \subset \overline{V(\sigma)}$ for every $i \in \eta$, hence $\{i\} \cap \sigma \neq \varnothing$, which means $i \in \sigma$, for each $i \in \eta$, so $\eta \subset \sigma$. Hence $\eta \in V(\sigma)$, so we obtain $(\overline{V(\sigma)})^\circ = V(\sigma)$. Then the grid is semirregular. $\square$

Note that the space can always be defined as we have done but only under the conditions of our theorem we can ensure that it is the canonical pseudogrid associated to the tiling (and we obtain in addition that it is also the minimal grid and that is semirregular and primitive).

In the above theorem we use several hypotheses (tiling, complete tiling, tiling without unprotected points) and we obtain several consequences (a canonical construction, T_0-Alexandroffness, lower semicontinuity, semirregularity) altogether. We can ask exactly what hypothesis gives each selected consequence, since for a particular situation we may need to relax the hypotheses to include more cases. In order to clarify this, we present hereafter some illuminating examples and pose some questions.

The first example shows how the description we have given of the canonical pseudogrid depends on the hypothesis.

Example 2.3. Let $T_n = \{(x,y) \in \mathbb{R}^2 : \frac{1}{n+1} < y < \frac{1}{n}\}$ if $n \in \mathbb{N}$, $T_0 = \{(x,y) \in \mathbb{R}^2 : y > 1\}$ and $T_{-1} = \{(x,y) \in \mathbb{R}^2 : y \leq 0\}$. $\mathcal{T} = \{T_n : n \geq -1\}$ is a complete tiling of $X = \mathbb{R}^2$ with $U(\mathcal{T}) = \{(x,0) : x \in \mathbb{R}\}$. However the canonical decomposition associated to $\mathcal{T}$ according to 2.2 should be $Y = \{\{k\}, \{n, n+1\} : k \geq -1, n \geq 0\}$ with the correspondent topology. However, this space is not the canonical pseudogrid, proving that the hypothesis $U(\mathcal{T}) = \varnothing$ cannot be suppressed. In this case, the canonical decomposition is lower semicontinuous in this example, but we can modify it replacing the tile $T_{-1} = \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$ with the tiles $T_{-2} = \{(x, y) \in \mathbb{R}^2 : y \leq 0, x \leq 0\}$ and $T_{-3} = \{(x, y) \in \mathbb{R}^2 : y \leq 0, x \geq 0\}$, and the example obtained has the same properties but the canonical decomposition is not lower semicontinuous (take G an open ball of radius 1 of the point $(2, 0)$).

So we pose the following question:

Problem 2.4. *Can the hypothesis of 2.2 over the tiling be weakened and still obtain that the canonical pseudogrid associated to a tiling is as described in Theorem 2.2?*

The following example shows how semirregularity can be lost if completeness is suppressed.

Example 2.5. We quote here an example cited in 4.9 of [3]. Let D^2 be an open circular disc with center $P = \{p\}$ in $\mathbb{R}^2$. Let A_1, A_2 and A_3 be the three open sectors of 120° and I, J and K be the three open radii respectively contained in $\overline{A_2} \cap \overline{A_3}$, $\overline{A_1} \cap \overline{A_3}$ and $\overline{A_1} \cap \overline{A_2}$. Let i be the midpoint of I and $H = \{i\}$.

Let define $X = A_1 \cup A_2 \cup A_3 \cup H \cup P$ with the induced topology from D^2. $\mathcal{T} = \{A_1 \cup P, A_2 \cup H \cup P, A_3 \cup H \cup P\}$ is a tiling of X. The canonical pseudogrid associated to that tiling is according to Theorem 2.2 and is lower semicontinuous (is the minimal grid, since is the only one). However it is not semirregular (see 4.9 of [3]). In fact the minimal neighborhood of the point $\{1, 2, 3\}$ is $\{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$, that is strictly contained in the expected one $V(\{1, 2, 3\})$, as defined in 2.2 ($V(\{1, 2, 3\})$ also has the element $\{2, 3\}$).

The reason is that this is not a complete tiling, since $\tau = \{2, 3\} \subset \sigma = \{1, 2, 3\}$, $A_\sigma = P$, $A_\tau = H$ and P is not contained in $\overline{H}$

The following two examples show how lower semicontinuity can be lost if completeness is suppressed.

Example 2.6. Let X be $([-1, 1] \times [-1, 1]) \setminus A$ where $A = [-\frac{1}{2}, 0[\times \{0\}$ and let the tiling be $\{A_i : i = 1, 2, 3, 4\}$ where $A_1 = [0, 1] \times [0, 1]$, $A_2 = [0, 1] \times [-1, 0]$, $A_3 = ([-1, 0] \times [-1, 0]) \setminus A$ and $A_4 = ([-1, 0] \times [0, 1]) \setminus A$. Clearly $\tau = \{3, 4\} \subset \sigma = \{1, 2, 3, 4\}$; however A_σ is the point $(0, 0)$, that is not in the closure of $A_\tau = [-1, -\frac{1}{2}[\times \{0\}$. Taking G as an open ball centered in $(0, 0)$ with radius less than $\frac{1}{2}$, we obtain an example of an open set whose star with respect to the partition is not open in X. Note that this is not a complete tiling since the canonical decomposition is not lower semicontinuous.

Note that the key property to ensure the completeness of a tiling is not the connectedness of the set of boundary points of the tiling, as one could feel, looking at the former example. The space and tiling obtained in the above example replacing $A = [-\frac{1}{2}, 0[\times\{0\}$ by $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is complete and the set of boundary points of the tiling is not connected.

Example 2.7. The following example is quoted from 6.6 of [3]. Let consider the tiling $\{A_i : i = 1, 2, 3, 4\}$ of the space $S = [-1, 1] \times [-1, 1]$ where $A_1 = [0, 1] \times [0, 1]$, $A_2 = [0, 1] \times [-1, 0]$, $A_3 = [-1, 0] \times [-1, 0]$ and $A_4 = [-1, 0] \times [0, 1]$. Let X be the Möbius band obtained from S by identifying the points $(x, -1)$ with $(-x, 1)$ for each $-1 \leq x \leq 1$ and write ρ for the natural map of S onto X. $\mathcal{T} = \{B_i = \rho(A_i) : i = 1, 2, 3, 4\}$ is a tiling of X without unprotected points.

However it is not perfect, since it is not complete. In fact $B_{\{1,2,3,4\}}$ has two points: $P = \rho((0, 0))$ and the point $Q = \rho((0, -1)) = \rho((0, 1))$. Now, $B_{1,3} = \rho([-1, 0] \times \{-1\}) = \rho([0, 1] \times \{1\})$ is not empty; we have that $Q \in \overline{B_{1,3}}$, but $P \notin \overline{B_{1,3}}$, so $\{1, 3\} \subset \{1, 2, 3, 4\}$ but $B_{\{1,2,3,4\}} \not\subset \overline{B_{1,3}}$. And we have that $\triangle^{\times}(\mathcal{T})$ is a semirregular minimal trace space but it is not a lower semicontinuous decomposition (6.6 of [3]).

Hence, to ensure that the tiling gives a lower semicontinuous decomposition, it is not enough to have a tiling without unprotected points. However, the example 2.5 shows that we can obtain a lower semicontinuous decomposition from a tiling without unprotected points that is not complete. On the other hand the example 2.7 shows that we can obtain a semirregular decomposition from a tiling without unprotected points that is not complete.

So we can pose the following question.

Problem 2.8. *Can the hypotheses of 2.2 over the tiling be weakened and still obtain that the canonical pseudogrid associated to a tiling is semirregular and lower semicontinuous?*

3. Topological properties of the minimal grid of a tiling

In this final section we ask how topological properties (other than semirregularity and lower semicontinuity) of the minimal grid of a tiling $\triangle^{\times}(\mathcal{T})$ can be deduced from the topological properties of the space X and properties of the tiling. The following notation is quoted from [3]. For a topological space X, if the set of isolated points $X^{\Lambda} = \{x \in X : \{x\} \text{ is open in } X\}$ is dense in X we shall call it the trace of X and X itself a trace space. Every pseudogrid is a trace space, whose trace is the image of the fenestration.

We summarize the results in the following theorem.

Theorem 3.1. *Let $\mathcal{T}$ be as in Definition 2.1 a perfect tiling of a topological space X and let $Y = \triangle^{\times}(\mathcal{T})$ be the minimal grid associated to $\mathcal{E}(\mathcal{T})$ according to 2.2.*

(1) *If X is compact, connected, locally compact or locally connected, so is Y. Moreover, $W(Y) \leq W(X)$, where $W(X)$ is the weight of X.*
(2) *If $\mathcal{T}$ is a tiling, then $|\mathcal{T}| \leq W(X)$, and we also have that $W(Y) = |Y| \leq 2^{|\mathcal{T}|}$.*
(3) *$\mathcal{S} = \{\overline{\{y\}} : y \in Y^{\Lambda}\}$ is a tiling of Y.*

(4) *The image under the quotient map of the set of singular points $S(\mathcal{T})$ (frontier points such that every neighborhood of which intersects infinitely many tiles in $\mathcal{T}$) of the tiling is the set of cluster points (see* [7]*) of the trace (hence a tiling without singular points gives a grid whose trace has no cluster points). If $S_0(\mathcal{T}) = \{x \in S(\mathcal{T}) : I(x) \text{ is infinite}\} = \varnothing$, Y is a locally finite space.*

Proof. (1) The quotient map is open.

(2) First, if $\mathcal{T}$ is a tiling, given $x \in A_i$, there is $B \in \mathcal{B}$ such that $x \in B \subset A_i$, since A_i is open. Since the sets A_i are pairwise disjoint, the cardinal of the base $\mathcal{B}$ is at least the cardinal of the set of A_i's and this is valid for any base.

Finally, since Y is T_0-Alexandroff, the cardinal of the space is the same as the weight, and since the points of Y are subsets of I, is not greater than $2^{|\mathcal{T}|}$.

(3) Clear.

(4) Since a cluster point of Y^Λ is one whose neighborhoods have infinitely many points of Y^Λ, the first assertion is clear.

If $S_0(\mathcal{T}) = \varnothing$, $I(x)$ is finite for every $x \in X$, hence $\{\sigma \subset I : A_\sigma \neq \varnothing\}$ is a set of finite subsets of I, hence $V(\sigma)$ is finite for every $\sigma \in \triangle^\times(\mathcal{T})$, hence a locally finite space as defined in section 3 of [3].

□

Note that the tiling defined in Example 2.3 has $U(\mathcal{T}) \neq \varnothing$ and $S_0(\mathcal{T}) = \varnothing$ and the minimal grid (that is not of the form we have obtained in 2.2) is not locally finite.

If Y is locally connected, we can construct a base formed by open sets that are simultaneously regular open and connected, so we can pose the following question.

Problem 3.2. *Find a relation between the connectivity of the tiles and the local connectedness of Y.*

It is clear that $\triangle^\times(\mathcal{T}_{|A}) \neq \triangle^\times(\mathcal{T})_{|\pi(A)}$ that is, the grid associated to the restriction of the tiling to A is not the restriction of the grid to the image of A, since $\mathcal{T}_{|A}$ may not be a tiling under the conditions of 2.2 even if $\mathcal{T}$ is (see example 2.5, with $X = D^2$ and $A = X$). We can also ask if $\triangle^\times(\mathcal{T}_1 \times \mathcal{T}_2) = \triangle^\times(\mathcal{T}_1) \times \triangle^\times(\mathcal{T}_2)$.

Problem 3.3. *Find two perfect tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ of topological spaces X_1 and X_2 under the conditions of 2.2 such that $\triangle^\times(\mathcal{T}_1 \times \mathcal{T}_2) \neq \triangle^\times(\mathcal{T}_1) \times \triangle^\times(\mathcal{T}_2)$.*

It is clear that two topologically equivalents tilings (in the sense of section 1 of [6]) give the same induced canonical pseudogrid. Now, we say that two perfect tilings are $\triangle^\times$-equivalent if and only if there is an homeomorphism between their induced canonical pseudogrids.

Problem 3.4. *Are there two non-topologically equivalent perfect tilings of $\mathbb{R}^n$ such that they are $\triangle^\times$-equivalent?*

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F.G. Arenas
Area of Geometry and Topology
Faculty of Experimental Sciences
Universidad de Almería
04120 Almería
Spain

E-mail address: farenas@ual.es

M.L. Puertas
Area of Applied Mathematics
Faculty of Experimental Sciences
Universidad de Almería
04120 Almería
Spain

E-mail address: mpuertas@ual.es