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Extendible spaces

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Abstract. The domain theoretic notion of lifting allows one to extend a partial order in a trivial way by a minimum. In the context of Quantitative Domain Theory (e.g. [BvBR98] and [FK93]) partial orders are represented as quasi-metric spaces. For such spaces, the notion of the extension by an extremal element turns out to be non trivial.
To some extent motivated by these considerations, we characterize the directed quasi-metric spaces extendible by an extremum. The class is shown to include the S-completable directed quasi-metric spaces.
As an application of this result, we show that for the case of the invariant quasi-metric (semi)lattices, weightedness can be characterized by order convexity combined with the extension property.

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1. Background

A function $d\colon X \times X \to \mathcal{R}_0^+$ is a *quasi-pseudo-metric* iff

1) $\forall x \in X.\, d(x,x) = 0$
2) $\forall x, y, z \in X.\, d(x,y) + d(y,z) \geq d(x,z)$.

A *quasi-pseudo-metric space* is a pair (X, d) consisting of a set X together with a quasi-pseudo-metric d on X.

In case a quasi-pseudo-metric space is required to satisfy the T_0-separation axiom, we refer to such a space as a *quasi-metric* space.

In that case, condition 1) and the T_0-separation axiom can be replaced by the following condition:

1′) $\forall x, y.\, d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$.

The *conjugate* d^{-1} of a quasi-pseudo-metric d is defined to be the function $d^{-1}(x,y) = d(y,x)$, which is again a quasi-pseudo-metric (e.g. [FL82]). The conjugate of a quasi-pseudo-metric space (X, d) is the quasi-pseudo-metric space (X, d^{-1}). The *(pseudo-)metric* d^* induced by a quasi-(pseudo-)metric d is defined by $d^*(x,y) = max\{d(x,y), d(y,x)\}$.

We discuss a few examples of quasi-pseudo-metric spaces.

The function $d_1 \colon \mathcal{R}^2 \to \mathcal{R}_0^+$, defined by $d_1(x,y) = y - x$ when $x < y$ and $d_1(x,y) = 0$ otherwise, and its conjugate are quasi-pseudo-metrics. We refer to d_1 as the "left distance" and to its conjugate as the "right distance". These quasi-pseudo-metrics correspond to the nonsymmetric versions of the standard metric m on the reals, where $\forall x, y \in \mathcal{R}.\, m(x,y) = |x - y|$.

Note that the right distance has the usual order on the reals as associated order, that is $\forall x, y \in \mathcal{R}.\, x \leq_{d_1^{-1}} y \Leftrightarrow x \leq y$, while for the left distance we have $\forall x, y \in \mathcal{R}.\, x \leq_{d_1} y \Leftrightarrow x \geq y$.

The function $d_2 \colon (\overline{\mathcal{R}} - \{0\})^2 \to \mathcal{R}_0^+$, defined by $d_2(x,y) = \frac{1}{y} - \frac{1}{x}$ when $y < x$ and 0 otherwise, and its conjugate are quasi-pseudo-metrics.

The *complexity space* (C, d_C) has been introduced in [Sch95] (cf. also [Sch96] and [RS99]. Here

$$C = \{f \colon \omega \to \mathcal{R}^+ | \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty\}$$

and d_C is the quasi-pseudo-metric on C defined by

$$d_C(f,g) = \sum_{n=0}^{\infty} 2^{-n} [(\frac{1}{g(n)} - \frac{1}{f(n)}) \vee 0]$$

whenever $f, g \in C$. The complexity space (C, d_C) is a quasi-metric space with a maximum $\top$, which is the function with constant value ∞.

The *dual complexity space* is introduced in [RS99] as a pair (C^*, d_{C^*}), where $C^* = \{f : \omega \to \mathcal{R}_0^+ \mid \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty\}$, and d_{C^*} is the quasi-metric defined on C^* by $d_{C^*}(f,g) = \sum_{n=0}^{\infty} 2^{-n}[(g(n) - f(n)) \vee 0]$, whenever $f, g \in C^*$. We recall that (C, d_C) is isometric to (C^*, d_{C^*}) by the isometry $\Psi : C^* \to C$, defined by $\Psi(f) = 1/f$ (see [RS99]). Via the analysis of its dual, several quasi-metric properties of (C, d_C), in particular Smyth completeness and total boundedness, are studied in [RS99].

A quasi-pseudo-metric space (X, d) is *totally bounded* iff $\forall \epsilon > 0\, \exists x_1 \ldots x_n \in X\, \forall x \in X\, \exists i \in \{1, \ldots, n\}.\, d^*(x_i, x) < \epsilon$.

The *associated preorder* $\leq_d$ of a quasi-pseudo-metric d is defined by $x \leq_d y$ iff $d(x,y) = 0$.

A *dual property* of a given property P of quasi-metric spaces is a property Q such that P holds for a quasi-metric space (X, d) iff Q holds for the conjugate quasi-metric space (X, d^{-1}). A property is *self-dual* iff it is its own dual property. As an example, we remark that the property of total boundedness is (trivially) self-dual.

We write that a quasi-pseudo-metric space *encodes* a preorder when $\forall x, y \in X.\, d(x,y) \in \{0,1\}$. In that case we also write that the *encoded preorder* is the preorder $(X, \leq_d)$. Conversely, for a given preorder $(X, \leq)$, one can define a quasi-pseudo-metric space $(X, d_{\leq})$ which encodes the preorder, in the obvious way.

A preorder $(X, \leq)$ is *upper directed* iff $\forall x, y \in X\, \exists z \in X.\, z \geq x$ and $z \geq y$.

A preorder $(X, \leq)$ is *lower directed* iff $\forall x, y \in X\, \exists z \in X.\, z \leq x$ and $z \leq y$.

A preorder $(X, \leq)$ is *bi-directed* iff it is both upper directed and lower directed.

As most results will be stated for upper directed spaces, we refer to these spaces in the following simply as *directed* spaces.

A quasi-pseudo-metric space is *(upper/lower) directed* iff its associated preorder is (upper/lower) directed. A quasi-pseudo-metric space *has a maximum (minimum)* iff the associated preorder has a maximum (minimum).

We recall ([Sch96], Lemma 5) that quasi-pseudo-metrics satisfy the following property, which we refer to as the "*Monotonicity Lemma*": if (X, d) is a quasi-pseudo-metric space then $\forall x, y, z \in X.\, (x' \leq_d x$ and $y' \geq_d y) \Rightarrow d(x', y') \leq d(x, y)$.

A quasi-metric space (X, d) is *weightable* iff there exists a function $w\colon X \to \mathcal{R}_0^+$ such that $\forall x, y \in X.\, d(x, y) + w(x) = d(y, x) + w(y)$. The function w is called a *weighting function*, $w(x)$ is the *weight* of x and the quasi-metric d is *weightable by the function* w. A *weighted* space is a triple (X, d, w) where (X, d) is a quasi-metric space weightable by the function w. A *weightless point* of a weighted quasi-metric space is a point of zero weight. A space (X, d) is weightable with respect to a point $y \in X$ iff the function w defined by $\forall x \in X.\, w(x) = d(y, x)$ is a weighting function of (X, d).

Example 1.1. The quasi-metric space $(\mathcal{R}_0^+, d_1)$ is weightable by the identity function, $w_1(x) = x$. The quasi-metric space $(\overline{\mathcal{R}^+}, d_2)$ is weightable by the function $w_2(x) = \frac{1}{x}$. The complexity space (C, d_C) is weightable by the function w_C where $\forall f \in C.\, w_C(f) = \sum_n \frac{2^{-n}}{f(n)}$. Each of the examples is weightable with respect to a point $(0, \infty$, and $\top$ respectively).

We recall that the conjugate quasi-metric space $(\mathcal{R}_0^+, d_1^{-1})$ is *not* weightable ([Sch96, p. 352]). For more information on conjugates of weightable spaces we refer the reader to [KV94].

An *extension* of a weighted space (X, d, w) is a weighted space (X', d', w') such that the quasi-metric space (X', d') is an extension of the quasi-metric space (X, d) and such that $w'|X$ coincides with w.

A function $f\colon X \to \mathcal{R}_0^+$ is *fading* iff $inf_{x \in X} f(x) = 0$.

A weighted quasi-metric space is of *fading weight* iff its weighting function is fading.

Example 1.2. The spaces $(\mathcal{R}^+, d_1, w_1)$, $(\mathcal{R}^+, d_2, w_2)$, the complexity space (C, d_C, w_C) and the Baire space $(\mathcal{N}^{<\omega}, b, w_b)$ are weighted spaces of fading weight.

We recall the following proposition from [Sch00].

Proposition 1.3. *The weighting functions of a weightable quasi-metric space are exactly the strictly decreasing functions $f + c$, where $c \geq 0$ and where f is the unique fading weighting of the space.*

The following section focuses on directed quasi-metric spaces and the extension by extremal elements ("lifting", cf. [Sch97]) is analyzed in this context. We recall that not every directed quasi-metric space is extendible by a maximum ([Sch97]). This leads to a characterization of the class of directed spaces which are extendible by a maximum, via the class of "M-spaces". A dual result for the case of lower directed spaces is obtained via the class of "m-spaces" and the results are combined for the case of bi-directed spaces, resulting in a characterization via the class of "bi-extendible spaces".

2. Extendible spaces

We focus initially on the case of (upper) directed spaces.

Definition 2.1. An M-space is a directed quasi-metric space (X, d) which satisfies the condition $\forall y \in X\, \exists \alpha \in \mathcal{R}_0^+\, \forall x \geq_d y.\, d(x, y) \leq \alpha$. We refer to this condition as "the M-space condition".

For future reference we introduce the following definition.

Definition 2.2. If (X, d) is an M-space then its M-extension (X_1, d_1) is defined as follows:
If (X, d) has a maximum, we define (X_1, d_1) to be the quasi-metric space (X, d). Otherwise we choose $x_1 \notin X$ and define $X_1 = X \cup \{x_1\}$ and d_1 to be the extension of d defined by $\forall x \in X_1.\, d_1(x, x_1) = 0$ and $\forall x \in X.\, d_1(x_1, x) = \sup\{\lim_n d(x_n, x) \mid (x_n)_n \in x\uparrow\}$, where $x\uparrow = \{(x_n)_{n \geq 1} \mid x_1 \geq_d x \text{ and } \forall n \geq 1.\, x_n \leq_d x_{n+1}\}$.

The following theorem, which is a version of Theorem 13 of [Sch96], adapted to the context of M-spaces, provides a necessary and sufficient condition for directed quasi-metric spaces to be extendible by a maximum.

Since the proof is similar to the proof of Theorem 13 of [Sch96], we only present a sketch. The only difference is that the "functional boundedness" condition used in [Sch96], i.e. $\exists f \colon X \to \mathcal{R}^+\, \forall x, y \in X.\, d(x, y) \leq f(y)$, has been replaced by the equivalent M-space condition, for which the original proof still holds.

Theorem 2.3. *A directed quasi-metric space is extendible by a maximum iff the space is an M-space.*

Proof. If (X, d) is a quasi-metric space extendible by a maximum, say x_1, to a space (X_1, d_1) where $X_1 = X \cup \{x_1\}$, then we have that $\forall x, y \in X. d(x, y) = d_1(x, y) \leq d_1(x_1, y)$ which implies the M-space condition.

To show the converse, note that if an M-space (X, d) has a maximum x_1 then the result holds trivially.

In case the quasi-metric space (X, d) does not posses a maximum, let (X_1, d_1) be its M-extension. Using the M-space condition one can verify that the space (X_1, d_1) is a quasi-metric space with maximum x_1, which extends the quasi-metric space (X, d).

□

An easy verification shows that M-extensions of an M-space are unique up to isometry. Hence in the following we will refer to *the* M-extension of an M-space.

Remark 2.4. If (X, d) is an M-space then for any $x \in X$ there exists a sequence $(y_n)_n \in x\uparrow$ such that $d_1(x_1, x) = \lim_n d(y_n, x)$. Indeed, since $d_1(x_1, x) = \sup\{\lim_n d(x_n, x) | (x_n)_n \in x\uparrow\}$, there exists a sequence of sequences in $x\uparrow$, say $[(x_n^k)_n]_k$, such that $d(x_1, x) = \sup_k\{\lim_n d_1(x_n^k, x)\}$. Since the quasi-metric space (X, d) is directed, we can define a sequence $(y_n)_n$ where $y_1 = x_1^1$ and $\forall n \geq 1. y_{n+1}$ is an element such that $y_{n+1} \geq_d y_n, x_n^1, \ldots, x_n^n$. Note that $(y_n)_n \in x\uparrow$ and that $\forall k. \lim_n d(x_n^k, x) \leq \lim_n d(y_n, x)$. So $d_1(x_1, x) = \lim_n d(y_n, x)$.

We refer to such a sequence $(y_n)_n$ as a "representative sequence for x".

Example 2.5. Any directed subspace (Y, d) of a space (X, d') which possesses a maximum x_1 is an M-space. Indeed, note that by the Monotonicity Lemma we have that $\forall x, y \in Y. d(x, y) = d'(x, y) \leq d'(x_1, y)$ and hence the M-space condition holds. This implies in particular that the complexity spaces are M-spaces since they are directed subspaces of the complexity space $(\mathcal{C}, d)$ ([Sch97]).

The quasi-metric spaces $(\mathcal{R}^+, d_1)$ and $(\mathcal{R}^+, d_2)$ are M-spaces.

Finally, we remark that any bounded directed quasi-metric space is an M-space.

The fact that not every directed quasi-metric space is extendible by a maximum is shown by the counterexample $(\mathcal{R}_0^+, d_1^{-1})$, which violates the M-space condition (cf. [Sch97]).

Definition 2.6. A quasi-metric space (X, d) is order-convex iff $\forall x, y, z \in X. x \geq_d y \geq_d z \Rightarrow d(x, y) + d(y, z) = d(x, z)$.

Example 2.7. It is easy to verify that every weightable quasi-metric space is order convex (cf. [Sch96]). The quasi-metric space $(\mathcal{R}_0^+, d_1^{-1})$ is an example of an order convex space which is not weightable ([Sch96]).

Remark 2.8. It is easy to verify that the notion of order-convexity is self-dual.

Lemma 2.9. *The M-extension of an order convex M-space is order convex.*

Proof. Let (X,d) be an order convex M-space. In order to verify that the extension (X_1,d_1) is order convex, we verify that $\forall x,y,z \in X_1. x \geq_{d_1} y \geq_{d_1} z \Rightarrow d_1(x,y)+d_1(y,z) = d_1(x,z)$ by a distinction of cases.

If x,y and z all belong to X then the result follows immediately by order convexity of the quasi-metric space (X,d). The verifications of the cases where $y = x_1$ or $z = x_1$ are straightforward. So we only need to consider the case where $x = x_1$ and $y,z \in X$.

If $y,z \in X$ such that $y \geq_{d_1} z$, then, since d_1 and d coincide on X, we also have $y \geq_d z$. For some representative sequence $(y_n)_n \in y\!\uparrow$ (cf. Remark 2.4) we have that $d_1(x_1,y)+d_1(y,z) = \lim_n d(y_n,y)+d_1(y,z) = \lim_n(d(y_n,y)+d(y,z)) = \lim_n d(y_n,z)$, where the last equality follows by order convexity of (X,d). So to obtain the result it suffices to show that $\lim_n d(y_n,z) = d_1(x_1,z)$. Since $(y_n)_n \in y\!\uparrow$ and $y \geq_d z$, we have that $(y_n)_n \in z\!\uparrow$ and thus the inequality $\lim_n d(y_n,z) \leq d_1(x_1,z)$ holds.

We assume by way of contradiction that $\lim_n d(y_n,z) < d_1(x_1,z)$. Let $(z_n)_n$ be a representative sequence for z. Then we have that $\lim_n d(z_n,z) > \lim_n d(y_n,z)$.

Then there exists a sequence $(z_n)_n \in z\!\uparrow$ such that $\lim_n d(z_n,z) > \lim_n d(y_n,z)$.

Using directedness, we can define a sequence $(u_n)_n$ inductively such that $\forall n. u_n \geq z_n, y$ and $\forall n. u_{n+1} \geq_d u_n$. Thus $(u_n)_n \in y\!\uparrow$.

So for this sequence we still have that $\lim_n d(u_n,z) > \lim_n d(y_n,z)$. By order convexity we have that $d(u_n,z) = d(u_n,y)+d(y,z)$ and $d(y_n,z) = d(y_n,y)+d(y,z)$, so we obtain that $\lim_n d(u_n,y) > \lim_n d(y_n,y) = d_1(x_1,y)$, which yields a contradiction. □

In the following we focus on the case of lower and bi-directed spaces.

Definition 2.10. An m-space is a lower directed quasi-metric space (X,d) which satisfies the condition $\forall x \in X\, \exists \alpha \in \mathcal{R}_0^+\, \forall y \leq_d x. d(x,y) \leq \alpha$. We refer to this condition as "the m-space condition". A quasi-metric space is bi-extendible iff it is an M-space and an m-space. In that case we also write that the quasi-metric is bi-extendible.

Remark 2.11. The notions of an M-space and of an m-space are dual.

Definition 2.12. If (X,d) is an m-space then its m-extension (X_1,d_1) is defined as follows:
If (X,d) has a minimum, we define (X_0,d_0) to be the quasi-metric space (X,d). Otherwise we choose $x_0 \notin X$ and define $X_0 = X \cup \{x_0\}$ and d_0 to be the extension of d defined by $\forall x \in X_0. d_0(x_0,x) = 0$ and $\forall x \in X. d_0(x,x_0) = \sup\{\lim_n d(x,x_n) \mid (x_n)_n \in x\!\downarrow\}$, where $x\!\downarrow = \{(x_n)_{n\geq 1} \mid x_1 \leq_d x \text{ and } \forall n \geq 1. x_{n+1} \leq_d x_n\}$.

One can easily verify that m-extensions are unique up to isometry, so we will refer in the following to *the* m-extension of an m-space.

The following theorem provides a dual version of Theorem 2.3.

Theorem 2.13. *A lower directed space quasi-metric is extendible by a minimum iff the space is an m-space.*

The reader may wish to omit the proof of the following technical proposition on first reading.

Proposition 2.14. *For bi-extendible quasi-metric spaces the operations of m-extension and M-extension commute.*

Proof. Let (X,d) be a bi-extendible quasi-metric space. We let (X_{10}, d_{10}) denote the m-extension of the M-extension of (X,d), while (X_{01}, d_{10}) denotes the M-extension of the m-extension of (X,d).

Since both kinds of extensions are unique up to isometry, we can select the names of the extremal points added during the extensions and guarantee that $X_{01} = X_{10}$. We will denote this set by $\overline{X}$ in the following and denote the minimum by $\overline{x}_0$ and the maximum by $\overline{x}_1$ respectively.

In order to verify that the quasi-metrics d_{10} and d_{01} coincide, we need to verify that $\forall \overline{x}, \overline{y} \in \overline{X}.\, d_{10}(\overline{x}, \overline{y}) = d_{01}(\overline{x}, \overline{y})$. We distinguish seven cases. The cases where $\overline{x}, \overline{y} \in X$ and where either $\overline{x} \in \{\overline{x}_0, \overline{x}_1\}$ or $\overline{y} \in \{\overline{x}_0, \overline{x}_1\}$, or where $\overline{x} = \overline{x}_0$ and $\overline{y} = \overline{x}_1$, follow by straightforward verifications, so we omit the details.

For the remaining case, we need to verify that $d_{10}(\overline{x}_1, \overline{x}_0) = d_{01}(\overline{x}_1, \overline{x}_0)$; that is we need to verify that:
$sup\{lim_n[sup\{lim_m d(x^n_m, x_n) \mid (x^n_m)_m \in x_n\!\uparrow\}] \mid (x_n)_n$ is a decreasing sequence$\} =$
$sup\{lim_k[sup\{lim_l d(x_k, x^k_l) \mid (x^k_l)_l \in x_k\!\downarrow\}] \mid (x_k)_k$ is an increasing sequence$\}$.

We will verify that:
$sup\{lim_n[sup\{lim_m d(x^n_m, x_n) \mid (x^n_m)_m \in x_n\!\uparrow\}] \mid (x_n)_n$ is a decreasing sequence$\} \leq$
$sup\{lim_k[sup\{lim_l d(y_k, y^k_l) \mid (y^k_l)_l \in y_k\!\downarrow\}] \mid (y_k)_k$ is an increasing sequence$\}$.

The converse inequality is shown in a similar way.

The following notation is used: for each natural number n and a given decreasing sequence $(x_n)_n$, we let $A_n = sup\{lim_m d(x^n_m, x_n) \mid (x^n_m)_m \in x_n\!\uparrow\}$ and for each natural number k and a given increasing sequence $(y_k)_k$, we let $B_k = sup\{lim_l d(y_k, y^k_l) \mid (y^k_l)_l \in y_k\!\downarrow\}$.

Clearly it suffices to show that for every decreasing sequence $(x_n)_n$ there exists an increasing sequence $(y_k)_k$ such that, for A_n corresponding to the sequence $(x_n)_n$ and for B_k corresponding to the sequence $(y_k)_k$, $lim_n A_n \leq lim_k B_k$.

Let $(x_n)_n$ be a decreasing sequence then for each natural number n, we select a representative sequence $(u^n_m)_m$ for x_n (cf. Remark 2.4), which allows us to rewrite A_n as: $A_n = lim_m d(u^n_m, x_n)$.

It is easy to see that the representative sequence can be chosen inductively as follows: select a representative sequence $(u^1_m)_m$ for x_1. For each $n \geq 1$, we select a representative sequence $(v^{n+1}_m)_m$ for x_{n+1} and replace it, using directedness, by a sequence $(u^{n+1}_m)_m$, such that $u^{n+1}_1 \geq_d v^{n+1}_1, u^n_1$ and for each $m \geq 1$, $u^{n+1}_{m+1} \geq_d v^{n+1}_{m+1}, u^n_{m+1}, u^{n+1}_m$. Clearly the sequence $(u^n_m)_m$ is still a representative sequence for x_n since it belongs to $x_n\!\uparrow$ and by the Monotonicity Lemma: $\forall n.\, d(u^n_m, x_n) \geq d(v^n_m, x_n)$.

So we have that $A_n = lim_m d(u^n_m, x_n)$ where, by construction, for each natural number m, the sequence u^n_m is increasing in n.

Let $(\epsilon_n)_n$ be a sequence which converges to 0, then, since $A_n = lim_m d(u^n_m, x_n)$, we can select for each n an index m_n such that $A_n - d(u^n_{m_n}, x_n) < \epsilon_n$. Without loss of generality we can assume that the sequence $(m_n)_n$ is increasing.

In particular we obtain that the sequence $(u^n_{m_n})_n$ is an increasing sequence for which $lim_n A_n = lim_n d(u^n_{m_n}, x_n)$.

We remark that for each n, $d(u^n_{m_n}, x_n) \leq sup\{lim_l d(u^n_{m_n}, x^n_l) | \, (x^n_l)_l \in u^n_{m_n}\!\downarrow\}$.

This follows since $u^n_{m_n} \geq_d x_n$ and thus there exists a sequence $(x^n_l)_l \in u^n_{m_n}\!\downarrow$ such that for some l, $x^n_l = x_n$. By the Monotonicity Lemma, $d(u^n_{m_n}, x^n_l)$ is increasing in l. So we obtain that for each n, $d(u^n_{m_n}, x_n) \leq lim_l d(u^n_{m_n}, x^n_l)$ and thus for each n, $d(u^n_{m_n}, x_n) \leq sup\{lim_l d(u^n_{m_n}, x^n_l) | \, (x^n_l)_l \in u^n_{m_n}\!\downarrow\}$.

The result now follows since $lim_n A_n = lim_n d(u^n_{m_n}, x_n) \leq lim_n sup\{lim_l \, d(u^n_{m_n}, x^n_l) | \, (x^n_l)_l \in u^n_{m_n}\!\downarrow\} = lim_n B_n$. This concludes the proof, where we define the desired sequence $(y_k)_k$ to be the sequence $(u^n_{m_n})_n$.

□

Definition 2.15. If (X, d) is a bi-extendible space then its extension is defined to be the m-extension of its M-extension, or alternatively the M-extension of its m-extension.

The fact that the notion of an extension is well defined follows by Proposition 2.14. Again, one can verify that extensions of bi-extendible spaces are unique up to isometry, so we will refer in the following to *the* bi-extension of a bi-extendible space. The bi-extension of a bi-extendible space (X, d) is denoted by (X_2, d_2) and its elements are always overlined in order to clearly distinguish them from other elements.

Combining Theorem 2.3 and Theorem 2.13, we obtain the following result.

Theorem 2.16. *A bi-directed space is extendible by a minimum and a maximum iff the space is bi-extendible.* □

In the next subsection we show that the S-completable directed spaces form a class of M-spaces. Similar results are obtained for the case of lower directed and bi-directed spaces.

2.1. **S-completable directed spaces.** The S-completable (topological) quasi-uniform spaces have been defined in [Sün93] as the (topological) quasi-uniform spaces of which the Smyth completion is again a quasi-uniform space; a condition which in general is violated as indicated in [Sün93].

An alternative characterization of S-completable (topological) quasi-uniform spaces in terms of Cauchy nets has been given in Theorem 5 of [Sün95].

We adopt this characterization in what follows as an alternative definition of the S-completable spaces, as this approach does not require any reference to the more abstract context of the theory of topological quasi-uniform spaces.

The definition given below is based on an adaptation of this characterization to the specific case of the quasi-metric spaces, which suffices for our purposes.

Definition 2.17. A quasi-metric space (X, d) is S-completable iff every Cauchy net on (X, d) is biCauchy.

We remark that the Smyth-completability condition can be simplified for the case of quasi-metric spaces to a requirement on sequences rather than on nets.

Proposition 2.18. *A quasi-metric space is S-completable iff every Cauchy sequence on the space is biCauchy.*

For a proof of this result, we refer the reader to [KS97] or also [Sch96].

Two main examples of classes of S-completable quasi-metric spaces have been discussed in the literature: the weightable spaces (e.g. [Kün93]) and the totally bounded spaces (e.g. [Sün95]). In [Kün93] it is shown that every totally bounded quasi-metric space (X, d) can be replaced by an equivalent weightable quasi-metric space (X, d'). Hence the weightable quasi-metric spaces include all cases of S-completable quasi-metric spaces thus far encountered in the literature.

The Smyth-completability of weightable quasi-metric spaces has been demonstrated by Künzi ([Kün93], Proposition 15). We remark that the above characterization for Smyth-completability consists of a weak symmetry requirement (cf. also [KS97]).

To end the section, we show that the S-completable directed quasi-metric spaces form a class of M-spaces.

Proposition 2.19. *Every S-completable directed quasi-metric space is an M-space.*

Proof. Let (X, d) be a S-completable directed quasi-metric space.

To show that (X, d) is an M-space we need to verify the condition: $\forall y \in X\, \exists \alpha \in \mathcal{R}^+\, \forall x \geq_d y.\, d(x, y) \leq \alpha$.

Assume by way of contradiction that $\exists y \in X\, \forall \alpha \in \mathcal{R}^+\, \exists x \geq_d y.\, d(x, y) > \alpha$.

Define the sequence $(x_n)_n$ by induction as follows: $x_1 = y$ and $\forall n \geq 1$, let x'_{n+1} be an element such that $x'_{n+1} \geq_d y$ and $d(x'_{n+1}, y) > d(x_n, y) + 1$ and let x_{n+1} be an element such that $x_{n+1} \geq_d x_n, x'_{n+1}$. The sequence $(x_n)_n$ is increasing with respect to the associated order $\leq_d$ and thus a Cauchy sequence. The sequence is not biCauchy however, since $\forall n \geq 1.\, d(x_{n+1}, x_n) \geq d(x_{n+1}, y) - d(x_n, y) \geq d(x'_{n+1}, y) - d(x_n, y) > (d(x_n, y) + 1) - d(x_n, y) = 1$. We obtain a contradiction with the Smyth-completability of the space (X, d) and thus the space (X, d) is an M-space.

□

The converse of Proposition 2.19 is not true in general as illustrated by the quasi-metric space $(\mathcal{N}, d_{\leq})$ which encodes the partial order $(\mathcal{N}, \leq)$, where $\leq$ is the standard order on the natural numbers. This space is an M-space, but is clearly not S-completable. However, cf. Corollary 2.33 for a converse under suitable hypotheses.

Corollary 2.20. *Every weightable directed space is an M-space. Every totally bounded directed space is an M-space.*

We have focused thus far on the case of (up) directed quasi-metric spaces and join semilattices. It is straightforward to verify that the results translate dually to the case of lower directed quasi-metric spaces and meet semilattices.

We state the dual notions and some of the corresponding results, where we omit the proofs.

A quasi-metric space (X, d) is *co-weightable* iff its conjugate (X, d^{-1}) is weightable. A *co-weighting function* of a quasi-metric space is a weighting function of its conjugate. A *co-weighted* space (X, d, w) is a triple consisting of a set X, a quasi-metric d on X and a co-weighting function w.

A quasi-metric space (X, d) is *bi-weightable* iff it is weightable and co-weightable. We remark that any weighted space (X, d, w) of bounded weight, where say $\forall x \in X. w(x) \leq K$, is co-weighted by the weighting function $K - w$ ([Kün93]). Hence any weighted space of bounded weight is bi-weightable. Similarly one obtains that any co-weighted space of bounded co-weight is bi-weightable.

We say that a quasi-metric space (X, d) is *co-S-completable* iff its conjugate is S-completable. Hence a space is co-S-completable iff every right Cauchy sequence if biCauchy.

We immediately obtain dual versions for Proposition 2.19 and for Corollary 2.20.

Theorem 2.21. *Every co-S-completable lower directed quasi-metric space is an m-space.*

Corollary 2.22. *Every co-weightable lower directed quasi-metric space is an m-space. Every totally bounded lower directed quasi-metric space is an m-space.*

We call a quasi-metric space bi-S-completable in case it is S-completable and co-S-completable.

We omit the straightforward results for bi-directed spaces corresponding to the preceding theorem and its corollary.

2.2. **Weightable directed spaces.** We recall some basic facts regarding the theory of upper weightable spaces (cf. [Sch97]) and we discuss the connections with weightable directed spaces.

Definition 2.23. If (X, d) is a quasi-metric space then (X, d) is upper weightable iff there exists a weighting function w for (X, d) such that $\forall x, y \in X. d(x, y) \leq w(y)$. We refer to such a function w as an upper weighting function. A weighted space (X, d, w) is upper weighted iff w is an upper weighting function. An upper weightable space is strongly upper weighted iff all of its weighting functions are upper weighting functions. A quasi-metric space is upper weightable with respect to a point iff it is weightable with respect to this point via an upper weighting.

Example 2.24. The quasi-metric space $(\mathcal{R}_0^+, d_1)$ is upper weightable by the function w_1, the quasi-metric space $(\overline{\mathcal{R}^+}, d_2)$ is upper weightable by the function w_2 and the complexity space (C, d_C) is upper weightable by the function w_C. Each of the spaces is upper weightable with respect to a point ($0, \infty$ and $\top$ respectively).

It is easy to verify, using the Correspondence Theorem, that the notion of an upper weighted space (X, d, w) is equivalent to a partial metric space (X, p) such that

$$\forall x, y \in X.\, p(x, y) \leq p(x, x) + p(y, y).$$

We recall from [Sch97] that a weighted space is upper weighted iff it has a directed weighted extension. Still, upper weightable spaces and weightable directed spaces need not behave similarly. For instance, an upper weightable space need not have a fading *upper* weighting as opposed to a weightable directed space. The following result ([Sch97], Proposition 16) sheds some light on the case of weightable directed spaces.

Proposition 2.25. *A weightable directed space has a fading upper weighting function and thus is strongly upper weighted.*

Since weightable directed spaces are strongly upper weighed, by Proposition 1.3 we know that the upper weighting functions of a weightable directed space are determined by a unique fading upper weighting.

The following proposition provides a characterization of this fading upper weighting and hence of all upper weightings of a weightable directed space.

Proposition 2.26. *If (X, d, f) is a fading weighted directed space with M-extension (X_1, d_1), say with a maximum x_1, then the function f_1 defined by $\forall x \in X.\, f_1(x) = d_1(x_1, x)$ and $f_1(x_1) = 0$ is the fading weighting of the M-extension and extends the weighting f.*

Proof. Let (X, d, f) be a directed quasi-metric space of fading weight.

Let $x \in X$ and let $(y_n)_n$ be a representative sequence for x in $x\uparrow$, that is $d_1(x_1, x) = \lim_n d(y_n, x)$.

Since the space (X, d, f) is of fading weight and directed, we can construct a representative sequence $(x_n)_n \in x\uparrow$ such that $\lim_n f(x_n) = 0$.

Indeed, let $(z_n)_n$ be any sequence of elements of X such that $\lim_n f(z_n) = 0$. It is easy to verify that a sequence $(x_n)_n$ of $x\uparrow$ can be defined inductively, using directedness, such that $\forall n.\, x_n \geq_d y_n, z_n$.

The sequence $(x_n)_n$ is still a representative sequence for x by the Monotonicity Lemma. By the fact that f is decreasing, we also obtain that the sequence of weights $(f(x_n))_n$ converges to 0. So $d_1(x_1, x) = \lim_n d(x_n, x)$ where $\lim_n f(x_n) = 0$.

By the weighting equality and since $\forall n.\, x_n \geq_d x$ we have that $\forall n.\, f(x) = f(x_n) + d(x_n, x)$. Hence by taking the limit, we obtain that $f(x) = d_1(x_1, x)$.

So the function f_1, as defined in the proposition, extends f. Clearly the function f_1 is still fading.

In order to show that f_1 is a weighting for (X_1, d_1) we only need to verify that $\forall x \in X_1.\, d_1(x_1, x) + f_1(x_1) = d_1(x, x_1) + f_1(x)$; that is $d_1(x_1, x) = f_1(x)$. If $x \in X$ the equality follows from the above. If $x = x_1$ the equality reduces to the trivial identity $0 = 0$. □

Remark 2.27. The reader familiar with [Sch97] can easily verify that the *lifting* (X_0, d_0, f_0) of a weighted directed space (X, d, f) of fading weight, as defined in [Sch97], is such that the quasi-metric space (X_0, d_0) and (X_1, d_1) are isometric and such that f_0 and f_1 coincide. Hence we will refer to the fading weighting f_1 obtained in the above proposition as the lifting of the fading weighting f. The space (X_1, d_1, f_1) will be referred to as the lifting of the weighted space (X, d, f).

Combining Propositions 1.3 and 2.26, we obtain the following corollary.

Corollary 2.28. *If (X, d) is a weightable directed space then its weighting functions are exactly the strictly decreasing upper weighting functions $f_0 + c$ where $c \geq 0$ and where f_0 is the unique fading weighting of (X, d), defined by: $\forall x \in X.\, f_0(x) = \sup\{\lim_n d(x_n, x) \mid (x_n)_n \in x\!\uparrow\}$.*

It is straightforward to supply dual versions of the results on weightable directed spaces for the case of co-weightable lower directed spaces. One can show that the co-weightings of a lower directed space are strictly increasing and consist exactly of the functions $f_0 + c$, where $c \geq 0$ and f_0 is the unique fading co-weighting of the space.

Proposition 2.29. *(Co-)weightable bounded (lower) directed quasi-metric spaces are bi-weightable.*

Proof. We verify the case for a weightable bounded directed space. The case for co-weightable bounded lower directed spaces follows by dualization.

Let (X, d) be a weightable bounded directed quasi-metric space and let K be a bound for the quasi-metric d. We recall that any bounded directed quasi-metric space is an M-space.

Let (X_1, d_1) be the M-extension of the space (X, d), with a maximum x_1. It is easy to verify that the M-extension is still bounded by K.

Let f be the fading weighting of (X, d). Then we have that $\forall x \in X.\, d_1(x, x_1) + f_1(x) = d_1(x_1, x) + f_1(x_1)$, where f_1 is the lifting of the fading weighting f. Since $f_1(x_1) = 0$, we obtain that $\forall x \in X.\, f(x) \leq K$. Hence the fading weighting f_1 is bounded by a constant K and thus the space (X_1, d_1) is co-weightable by $K - f_1$. We conclude that the space (X, d) is co-weightable by the co-weighting $K - f$.

□

Proposition 2.30. *(Co-)weightable bi-extendible quasi-metric spaces (X, d) are bi-weightable. The weighting functions of the bi-extension (X_2, d_2), with minimum $\overline{x}_0$ and maximum $\overline{x}_1$, are the functions $f = f_2 + c$, where $c \geq 0$ and where f_2 is the fading weighting for (X_2, d_2), defined by: $\forall \overline{x} \in X_2.\, f_2(\overline{x}) = d_2(\overline{x}_1, \overline{x})$. We refer to this weighting as the lifting of the fading weighting f*

to the bi-extension. The co-weighting functions for (X_2, d_2) are the functions $\hat{f} = \hat{f}_2 + c$, where $c \geq 0$ and $\forall \overline{x} \in X_2.\, \hat{f}_2(\overline{x}) = d_2(\overline{x}, \overline{x}_0)$. In particular we have that $f_2(\overline{x}_0) = \hat{f}_2(\overline{x}_1) = d_2(\overline{x}_1, \overline{x}_0)$ and $\hat{f}_2(\overline{x}_0) = f_2(\overline{x}_1) = 0$.

The fading weighting f_2 and the fading co-weighting $\hat{f}_2$ are related as follows:

$$\forall \overline{x} \in X_2.\, f_2(\overline{x}) = d_2(\overline{x}_1, \overline{x}_0) - \hat{f}_2(\overline{x}).$$

Proof. Let (X, d) be a bi-extendible quasi-metric space. By the Monotonicity Lemma it is easy to see that the bi-extension (X_2, d_2) is a bounded quasi-metric space, where $\forall \overline{x}, \overline{y} \in X_2.\, d_2(\overline{x}, \overline{y}) \leq d_2(\overline{x}_1, \overline{x}_0)$. Hence, by Proposition 2.29, we obtain that any (co)-weightable bi-extendible space is bi-weightable.

So, without loss of generality, it suffices to verify the case for weightable bi-extendible quasi-metric spaces.

Let (X, d) be a weightable bi-extendible quasi-metric space, say with a fading weighting f.

We first verify that the bi-extension is weightable. Let (X_2, d_2) be the bi-extension of (X, d) and let $\hat{f}_2$ be the fading co-weighting on (X_2, d_2) obtained from f via an application of Proposition 2.26 and its dual version. We sketch the construction. First a fading weighting f_1 is obtained via Proposition 2.26, where f_1 extends f to the lifting (X_1, d_1). We remark that f_1 is bounded on X_1, since $\forall x \in X_1.\, f_1(x) = d_1(x_1, x) = d_2(\overline{x}_1, x) \leq d_2(\overline{x}_1, \overline{x}_0)$. So we can obtain the co-weighting $K - f_1$ on (X_1, d_1), where $K = d_2(\overline{x}_1, \overline{x}_0)$. Clearly, this function is fading, by the definition of d_2 and by the fact that $f_1(x) = d_1(x_1, x)$. So, by the dual version of Proposition 2.26, we can obtain the fading co-weighting f_2 on (X_2, d_2), which extends $K - f_1$. It is easy to verify that this fading co-weighting is still bounded by K and thus the fading weighting for the bi-extension is the function $f_2 = K - \hat{f}_2$. Hence we obtain that the bi-extension is weightable.

Since f_2 is fading and decreasing, we have that $f_2(\overline{x}_1) = 0$. By the weighting equality we also obtain that $d_2(\overline{x}_1, \overline{x}_0) = f_2(\overline{x}_0) - f_2(\overline{x}_1) = f_2(\overline{x}_0)$.

Finally, we remark that since $\overline{x}_1$ is the maximum of the bi-extension, we obtain that $\forall \overline{x} \in X_2.\, f_2(\overline{x}) + d_2(\overline{x}, \overline{x}_1) = f_2(\overline{x}_1) + d_2(\overline{x}_1, \overline{x})$ and thus $f_2(\overline{x}) = d_2(\overline{x}_1, \overline{x})$. Similarly one can show that $\forall \overline{x} \in X_2.\, \hat{f}_2(\overline{x}) = d_2(\overline{x}, \overline{x}_0)$. □

2.3. **Invariance.** A quasi-metric space (X, d) is called a semilattice iff the associated partial order $(X, \leq_d)$ is a semilattice.

A join semilattice (X, d) is *invariant* iff $\forall x, y, z \in X.\, d(x \sqcup z, y \sqcup z) \leq d(x, y)$. In that case we also write that the quasi-pseudo-metric d is invariant. The notions of an *invariant meet semilattice* and of an *invariant lattice* are defined in the obvious way. One can easily verify that invariant join semilattices are quasi-pseudo-metric join semilattices and that similar results hold for the case of invariant meet semilattices and for invariant lattices.

It is convenient to recall the following alternative characterization of invariance ([Sch00]).

A join semilattice (X, d) is invariant iff $\forall x, y \in X.\, d(x \sqcup y, y) = d(x, y)$. A meet semilattice (X, d) is invariant iff $\forall x, y \in X.\, d(x, x \sqcap y) = d(x, y)$.

The invariant quasi-metric semilattices include many well known examples of Quantitative Domain Theory, including the Baire partial metric spaces, the complexity spaces, the interval domain and the totally bounded Scott domains ([Sch00]).

We will show that in the context of the invariant join semilattices the notion of an order convex M-space and of a weightable space coincide (Theorem 2.31). This result is a version of Theorem 13 of [Sch96], adapted to the context of the M-spaces.

We include the proof since the theorem is central and we remark that again, the "functional boundedness" condition used in [Sch96] has been replaced by the weaker M-space condition. The reader familiar with [Sch96] will notice that the order convexity condition corresponds to the "descending path condition" discussed in [Sch96]. We prefer to use the terminology "order convex" in order to be consistent with the terminology used in [Gie80]. The following theorem is a version of Theorem 15 of [Sch96] in the context of M-spaces.

Theorem 2.31. *An invariant join semilattice is weightable iff the underlying quasi-metric space is an order convex M-space.*

Proof. Let (X, d) be an invariant join semilattice.

We assume that (X, d) is weightable. Since the space is directed, (X, d) is upper weightable and thus satisfies the M-space condition. Since (X, d) is also directed it is an M-space. The fact that (X, d) is order convex follows immediately since (X, d) is weightable ([Sch96]).

To show the converse, we assume that (X, d) is an order convex M-space. By (the proof of) Theorem 2.3, there exists an M-extension (X_1, d_1) of (X, d), where $X_1 = X \cup \{x_1\}$ and x_1 is the maximum of the quasi-metric space (X_1, d_1).

Since upper weightability is a hereditary property, in order to show that the quasi-metric space (X, d) is upper weightable it suffices to show that the space (X_1, d_1) is upper weightable. We will verify that the space (X_1, d_1) is upper weightable by the weighting function w_0, defined by $\forall x \in X_1.\, w_0(x) = d_1(x_1, x)$.

Note that (X_1, d_1) is order convex by Lemma 2.9 and thus $\forall x, y \in X_1.\, x \geq_{d_1} y \Rightarrow d_1(x_1, x) + d_1(x, y) = d_1(x_1, y)$. Equivalently we have that $\forall x, y \in X_1.\, x \geq_{d_1} y \Rightarrow d_1(x, y) = w_0(y) - w_0(x)$.

It is straightforward to verify that (X_1, d_1) is an invariant join semilattice. So we have that $\forall x, y \in X_1.\, d_1(x, y) = d_1(x \sqcup y, y)$ and $d_1(y, x) = d_1(y \sqcup x, x)$. Since we have shown that $\forall x, y \in X_1.\, x \geq_{d_1} y \Rightarrow d_1(x, y) = w_0(y) - w_0(x)$, we obtain that $\forall x, y \in X_1.\, d_1(x, y) - d_1(y, x) = d_1(x \sqcup y, y) - d_1(x \sqcup y, x) = (w_0(y) - w_0(x \sqcup y)) - (w_0(x) - w_0(x \sqcup y)) = w_0(y) - w_0(x)$. So (X_1, d_1) is weighted with respect to w_0 and hence (X, d) is weightable with respect to the function $w_0|X$.

□

From an inspection of the above proof, we obtain the following corollary.

Corollary 2.32. *The M-extension of an invariant join semilattice which is an order convex M-space is an invariant join semilattice which is upper weightable with respect to its maximum.*

Corollary 2.33. *Invariant join semilattices which are order convex M-spaces are S-completable.*

Proof. Follows by Theorem 2.31. □

We remark that the M-space condition is necessary in the corollary. This is illustrated by the quasi-metric space $(\mathcal{R}_0^+, d_1^{-1})$ which is an order convex invariant join semilattice, but not an M-space and not S-completable. The space is not S-completable since the increasing sequence of the natural numbers is an example of a Cauchy sequence which is not biCauchy. The space also provides an example of an order convex directed space which is not an M-space.

We state the dual version for Theorem 2.31. As pointed out before, the notion of order convexity is self-dual.

Theorem 2.34. *An invariant meet semilattice is co-weightable iff the underlying quasi-metric space is an order convex m-space.*

We remark that the co-weighting in this case is the function which expresses the distance from a point to the minimum of the m-extension.

We obtain the following combination of Theorems 2.31 and 2.34 for the case of lattices.

Theorem 2.35. *An invariant lattice is bi-weightable iff the underlying quasi-metric space is an order convex bi-extendible space.*

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