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Finite approximation of stably compact spaces

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Abstract. Finite approximation of spaces by inverse sequences of graphs (in the category of so-called *topological graphs*) was introduced by Smyth in [21, 20], and developed further in [22, 25, 27, 29]. The idea was subsequently taken up by Kopperman and Wilson, who developed their own purely topological approach using inverse spectra of finite T_0-spaces in the category of *stably compact* spaces [12]. Both approaches are, however, restricted to the approximation of (compact) Hausdorff spaces and therefore cannot accommodate, for example, the upper space and (multi-) function space constructions. We present a new method of finite approximation of stably compact spaces using finite *stably compact graphs*, which when the topology is discrete are simply finite directed graphs. As an extended example, illustrating the problems involved, we study (ordered spaces and) *arcs*.

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1. Introduction

It is a standard technique in some areas of topology to consider a space as the limit of an inverse sequence of simpler, or at least more well-understood spaces. An alternative approach, developed by Smyth in [20, 21], is that of considering a topological space as the limit of an inverse sequence of undirected graphs, which are the structures considered in digital topology and tolerance geometry. (Approximation of spaces by graphs had been used in an informal fashion by Bandt and co-workers in the study of fractals; see [3].) In digital topology, undirected graphs are considered as computationally tractable *approximations* of Euclidean space, and the inverse limit construction is a formal bridge between a space and its approximations.

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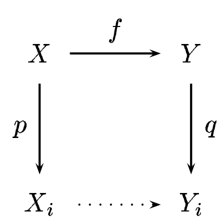
Smyth showed that a space is compact metric if and only if it can be approximated by an inverse sequence of finite undirected

graphs. This "approximation" requires some explanation. The inverse limit construction must take place within a category in which both spaces and graphs are objects. *Topological graphs* were introduced for this purpose, and in [20, 27] these are topological spaces together with closed tolerance (i.e. reflexive and symmetric) relations. The morphisms are continuous, relation-preserving maps, and the resultant category has limits of inverse sequences. Approximation of a space by an inverse sequence of finite graphs then means that the space is the quotient of the inverse limit topology by the inverse limit relation. A fair amount can be done within this framework, see [22, 25, 27, 29, 24].

The idea was subsequently taken up by Kopperman and Wilson, who developed a purely topological approach [12]. They consider inverse sequences of finite T_0-spaces, which, like graphs, are structures that are fundamental in digital topology. In this framework, a space is said to be approximated by an inverse sequence if it is the *T_2-reflection* of the limit, and every compact metric space can be approximated thus. (In fact, they showed that every compact Hausdorff space can be approximated by an inverse *spectrum* of finite T_0-spaces.) All this takes place within the category of *stably compact spaces*, stable compactness being the right notion of compactness for non-Hausdorff spaces.

Both approaches are, however, restricted to the approximation of compact Hausdorff spaces, and so cannot accommodate, for example, the upper space and (subsequent) multi-function space constructions, because these in general yield non-Hausdorff spaces even from Hausdorff spaces. Indeed, consideration of multi-functions is particularly compelling in the context of approximation by inverse sequences. Let X and Y be limits of inverse sequences, and let $p : X \to X_i$ and $q : Y \to Y_i$ be projection mappings. How can we represent a mapping $f : X \to Y$ by a mapping from X_i to Y_i when there is quite possibly no such map that makes the diagram commute?



We argued in [25] (see also [24]) that it is appropriate to represent f by the relation $(p \times q)(graph(f))$ in that f is then the limit of its representations. We will return to this point in Section 4.

Unlike compact Hausdorff spaces, stably compact spaces *are* closed under the constructions mentioned above, but it is difficult to see how stably compact spaces in general can admit a purely topological approximation along the lines of [12]. One would presumably want to consider the T_0-reflection rather than the T_2-reflection of the limit. The limit of an inverse sequence of T_0-spaces is, however, already T_0, and it is well-known that only the spectral spaces can be constructed in this manner. It appears that the only remaining option is to consider inverse sequences of finite spaces that are not necessarily T_0 (i.e. finite preorders) and then to consider the T_0-reflection of the limit, but it is not difficult to show that this gives the limit of the T_0-reflections of the spaces in the sequence (i.e. T_0-reflections preserve inverse limits), and so again only the spectral spaces can be constructed in this way.

Graph-theoretic approximation can, however, be generalized to accommodate approximation of stably compact spaces, and this is the subject of this work. We consider approximation by *directed* graphs (instead of undirected graphs as for the Hausdorff case), and we will show that a space is stably compact if and only if it can be approximated by an inverse sequence of directed graphs.

The approximation of continua by connected finite graphs was studied in some detail in [27, 29, 22]. We believe that this study can fruitfully be generalized to the stably compact case, and in Section 3 we provide a partial illustration of this theme by considering the case of *arcs*.

Apart from a discussion of general linearly ordered spaces in Section 3 (to provide the background for the study of arcs), and a general quotienting result in the concluding section, we confine attention to compact spaces in this paper. For Hausdorff spaces, we know that approximation by (no longer finite) graphs is not confined to the compact case: see [29] for locally compact spaces, and [23] for Polish spaces. For non-Hausdorff spaces, however, we so far understand only the stably compact case. The main approximation result (namely, that finite directed graphs suffice) is set out in Section 4, and Section 5 investigates how the approximations carry over for various basic constructions on stably compact spaces.

2. Preliminaries

2.1. **Notation and terminology.** A relation R on a topological space X is *closed* if it is closed with respect to the product topology. That $(x, y) \in R$ is often written as xRy. R^{op} denotes the relation $\{(x, y) \mid (y, x) \in R\}$. For any $C \subseteq X$, $R(C)$ denotes the set $\{x \mid \exists c \in C.\ cRx\}$. The set C is called *R-saturated* if $C = R(C)$. The *R-saturated topology* is the collection of all R-saturated open sets. In the case that R is a pre-order, R-saturated sets (R^{op}-saturated sets) are called *upper* (*lower*) sets, and the R-saturated topology (R^{op}-saturated topology) is called the *upper* (*lower*) topology.

The specialization pre-order on a space X is denoted $\sqsubseteq_X$, i.e. $x \sqsubseteq_X y$ if every neighbourhood of x is also a neighbourhood of y. When we speak of upper (lower) sets in a topological space we mean w.r.t. this order. Also, in this case, the upper set $\sqsubseteq_X (C)$ is denoted $up(C)$ and the lower set $(\sqsubseteq_X)^{op}(C)$ is denoted $down(C)$.

2.2. **Stably compact spaces.** A *compact ordered space* is a compact Hausdorff space together with a closed partial order [13]. The corresponding *stably compact topology* is the upper topology. A *stably compact space* is any topological space that can be constructed thus.

Let X be a stably compact space. The *cocompact topology* is the collection of the complements of the compact upper subsets of X, and the resultant space is denoted X^{op}, which is sometimes called the *dual* of X. Sets that are compact (open) with respect to the cocompact topology are called *op*-compact (*op*-open). The *patch topology* is the coarsest topology that refines both the original and the cocompact topologies, and the resultant space, which is called the *patch space*, is denoted $patch(X)$. Sets that are compact (open) with respect to the patch topology are called *patch-compact* (*patch-open*).

The equivalence of stably compact and compact ordered spaces is given by that $(patch(X), \sqsubseteq_X)$ is a compact ordered space for which X is the corresponding stably compact space. Moreover, $(patch(X), (\sqsubseteq_X)^{op})$ is a compact ordered space for which X^{op} is the corresponding stably compact space. Therefore $X = (X^{op})^{op}$, $patch(X) = patch(X^{op})$ and $\sqsubseteq_{X^{op}} = (\sqsubseteq_X)^{op}$.

The compact upper subsets of X are precisely the compact lower subsets of X^{op}, and are also precisely the patch-compact upper sets. If C is patch-compact then $up(C)$ is compact and $down(C)$ is *op*-compact.

Nachbin's (very useful) separation result for compact ordered spaces in the context of stably compact spaces is:

Lemma 2.1 (Nachbin [13])**.** *Let C, D be patch-compact sets such that $up(C)$ and D are disjoint. Then there exist disjoint open U and op-open V that contain C, D respectively.*

Some care has to be taken over the above "equivalence" of compact ordered and stably compact spaces when it comes to the morphisms. A continuous map $f : X \to Y$ between stably compact spaces is *perfect* if it is also continuous with respect to the cocompact topologies. Equivalently, f is perfect if it is continuous with respect to the patch topologies and is order-preserving with respect to the specialization orders. Although we consider the category of stably compact spaces and continuous maps, it is the category of stably compact spaces and perfect maps that is equivalent to the category of compact ordered spaces and continuous, order-preserving maps.

We have not been able to find the following result in the literature:

Lemma 2.2. *For any stably compact space X, all three of X, X^{op} and $patch(X)$ are 2nd-countable if and only if any one of them is.*

Proof. Let X have a countable basis $\mathcal{B}$, which may be assumed to be closed under finite unions. Say that the *compact-upper hull* of a set is the smallest compact upper set that contains it. Then the collection of complements of the compact-upper hulls of the elements of $\mathcal{B}$ is a basis of X^{op}: if C is compact upper and $x \notin C$ then, by Nachbin's result, there is a compact-upper D and a $U \in \mathcal{B}$ with $x \notin D \supseteq U \supseteq C$. Then X is 2nd-countable iff X^{op} is, and if both are then so is $patch(X)$. Now let $\mathcal{B}$ be a countable basis of $patch(X)$, again closed under finite unions. For any $U \in \mathcal{B}$, the set $\{x \mid up(x) \subseteq U\}$ is open in X, and this gives a countable basis of X: if V is open in X and $x \in V$ then there is some $U \in \mathcal{B}$ such that $up(x) \subseteq U \subseteq V$. □

2.3. **Inverse sequences.** The limit of an inverse sequence is, strictly speaking, a categorical notion (see any basic textbook on category theory). The *limit* is an object in a limiting cone, and we speak of *the* limit because any two limits are isomorphic. Conversely, if X is the limit of some inverse sequence in some category, with projections p_i, and if $h : Y \to X$ is an isomorphism, then Y is equally well the limit, with projections $p_i \circ h$. This is all deliberately over-concise, as we will avoid categorical language here, although it is vital to at least check that "limits" are indeed categorical limits when the structures under consideration (in this work, stably compact graphs) are non-standard.

The main reason why we don't need to consider category theory explicitly is that the limit of an inverse sequence

$$X_1 \leftarrow^{f_1} X_2 \leftarrow^{f_2} \cdots$$

(an inverse sequence will often be denoted as (X_i, f_i)) in the category of topological spaces has a very concrete and standard description (see e.g. [4]): it is the set $X_\omega = \{(x_1, x_2, \ldots) \mid \forall i.\ x_i \in X_i,\ f_{i+1}(x_{i+1}) = x_i\}$ of all *threads* of the sequence, together with the topology that has as a basis the collection $\{p_i^{-1}(U) \mid U \text{ open in } X_i,\ i = 1, 2, \ldots\}$, where

$$p_i : X_\omega \to X_i,\ (x_1, x_2, \ldots) \mapsto x_i, \qquad i = 1, 2, \ldots$$

are the *projections.*

It is a classical result (see [4]) that the limit of an inverse sequence of compact Hausdorff spaces is compact Hausdorff. The limit of an inverse sequence of compact but not necessarily Hausdorff spaces is not necessarily compact, however (see [4, Ex.1.9.7(b)] for a counterexample). This is hardly surprising as classical compactness is a very weak property in the non-Hausdorff setting: for example, any arbitrarily pathological space is trivially compact so long as its specialization order has a bottom element. One possibly interesting exception is that the limit of an inverse sequence of any finite spaces is compact, because the limit topology is refined by the limit topology for the discrete topologies on the spaces. At any rate, the following result is further evidence, if it were needed, that stable compactness is the right notion of compactness for non-Hausdorff spaces. The result generalizes the above-mentioned classical result because the Hausdorff stably compact spaces are precisely the compact Hausdorff spaces.

Proposition 2.3. *The limit of an inverse sequence of stably compact spaces and perfect bonding maps is stably compact, and each projection on the limit is perfect.*

The result was proved in [28] (see also [12]) via consideration of the inverse sequence $patch(X_1) \leftarrow^{f_1} patch(X_2) \leftarrow^{f_2} \dots$; each bonding map is patch-continuous because perfect. The limit of this sequence is a compact Hausdorff space, on which the limit relation (see later) $\bigcap(p_i \times p_i)^{-1}(\sqsubseteq_{X_i})$ is a closed partial order. Thus we have a compact ordered space, and the corresponding stably compact space turns out to be the limit of original sequence. Moreover, the projections are perfect because they are patch-continuous and order-preserving.

The *dual* of an inverse sequence $\Phi = (X_i, f_i)$ of stably compact spaces and perfect bonding maps is

$$\Phi^{op} = ((X_i)^{op}, f_i)$$

Proposition 2.4. *The limit of Φ^{op} is the dual of the limit of Φ.*

Proof. We need to show that $B^{op} = \{p_i^{-1}(U) \mid U \text{ open in } (X_i)^{op},\ i = 1, 2, \dots\}$ is a basis of $(X_\omega)^{op}$. Each element of B^{op} is *op*-open because the projections p_i are perfect. Let C be compact upper in X_ω and let $y \notin C$. For each $x \in C$ there is some basic open set in X_ω that contains x but not y. It is simple to prove that the basis of X_ω is closed under finite unions, so there is some basic open set $p_i^{-1}(U)$ that contains C but not y. Then $up(p_i(C))$ and $p_i(y)$ are disjoint, so by Nachbin's separation result there exists some *op*-open V that contains $p_i(y)$ and is disjoint from $p_i(C)$. Then $y \in p_i^{-1}(V) \in B^{op}$, and $p_i^{-1}(V)$ is disjoint from C. □

The final result here concerns products. Consider any category that has finite products, and let $\Phi = (X_i, f_i)$ and $\Psi = (Y_i, g_i)$ be inverse sequences in this category. Their product is the inverse sequence

$$\Phi \times \Psi = (X_i \times Y_i, f_i \times g_i)$$

The following is a very straightforward categorical result, and the proof is omitted.

Proposition 2.5. *Let X_ω, Y_ω be the respective limits of Φ, Ψ, with p_i, q_i the projections. Then $X_\omega \times Y_\omega$ is the limit of $\Phi \times \Psi$, with $p_i \times q_i$ the projections.*

3. Ordered spaces and arcs

3.1. **Introduction. Topological graphs.** In this section our aim is to study, as a sort of extended example, the approximation of stably compact arcs by finite "arcs". The material illustrates one of the main themes of the paper, namely that for the (finite) approximation of stably compact spaces, finite topological spaces (or pre-orders) do not suffice: one has to consider relations that are not necessarily pre-orders.

By a *topological graph* we understand a structure (X, R), where X is a topological space and R is a binary relation on X. Of course, such a structure can be expected to be useful only when R interacts with the topology of X in a significant way. In particular, we have (generalizing the compact ordered spaces):

Definition 3.1. A *Hausdorff graph* is a structure (X, R), where X is a Hausdorff space and R is a closed reflexive relation on X.

An ordinary graph may be considered as an example of a Hausdorff graph by taking the discrete topology on the vertex set; on the other side any Hausdorff space provides an example by taking R as the identity.

An arbitrary topological graph (X, R) is said to be *connected* if there is no partition of X into R-disjoint open sets. (Two subsets A, B are R-disjoint if there is no selection (x, y) from the two sets with xRy.) The sub-topological graph on any $X' \subseteq X$ is defined in the obvious way, as is then a *cut-point* of a connected topological graph. These obviously agree with the usual definitions, in the examples given.

The term *Hausdorff ordered space* shall refer to a Hausdorff graph (X, R) in which R is a partial order; similarly for Hausdorff *pre-ordered* spaces. The following observation generalizes a statement made about compact ordered spaces in the preceding section:

Proposition 3.2. *Let (X, R) be a Hausdorff pre-ordered space. Then the specialization order of the upper topology coincides with R.*

Proof. Suppose that xRy. Then, trivially, every open upper set containing x contains y. Again, suppose that $\neg xRy$. Then (since R is closed) $\{z \mid \neg zRy\}$ is an open upper set containing x but not y. □

In the remainder of this section, we shall typically be concerned with structures having two distinct orders as well as a topology. The overall linear order (of a "line") will always be denoted $<$. The topology $\mathcal{T}$ will usually be the order topology derived from $<$. The second order, denoted $\sqsubseteq$, is closed w.r.t. $\mathcal{T}$. ($\sqsubseteq$ is typically a partial order, but may more generally be a pre-order, or may indeed be generalized to a reflexive relation.) The upper and lower topologies are considered w.r.t. $\sqsubseteq$, $\mathcal{T}$.

3.2. **Linearly ordered spaces. Arcs.** Our goal in this section is an analysis of (stably compact) arcs and their approximation by finite structures. We approach this via a discussion of lines in general.

We begin with a totally ordered set $(X, <)$ endowed with its order topology: that is, the topology having as subbase the collection of sets (rays) of the form $(x, \rightarrow)$, $(x, \leftarrow)$, where $(x, \rightarrow)$ is $\{y \mid y < x\}$. (A *ray* is a subset of X that is either $<$-saturated or $<^{op}$-saturated. A ray S is *delimited* if it has an end-point, that is, a point x such that S is one of the four sets $\{y \mid x < y\}$, $\{y \mid x \leq y\}$ etc.) Distinct points x, y are *adjacent* if there is no point z such that $x < z < y$ or $y < z < x$. Suppose that $\{U, V\}$ is a partition of X into open rays. Then it is easy to see that either the rays U, V are both delimited, or neither is. The space X is said to be *complete* if (in any such partition) only the case that U, V are both delimited can occur. An equivalent condition (for completeness) is that every $<$-directed set that is bounded above has a least upper bound. (Of course there is a further equivalent condition in terms of $<^{op}$-directed sets.)

The order topology of $(X, <)$ is evidently Hausdorff. Even if the space is complete, however, it fails to be connected if there is any pair of adjacent vertices in X. Our procedure will be to create a line out of X by introducing a (partial) second ordering $\sqsubseteq$ having the property that, if (and only if) two points are adjacent in the ordering $<$, they are comparable in the ordering $\sqsubseteq$. In this way the space X is endowed with a "fence" structure (see 3.8, 3.12 for precise definitions).

Example 3.3. *Khalimsky space.* Let X be a (not necessarily proper) interval of the integers, $\mathbb{Z}$, in their usual ordering. Obviously, the order topology $\mathcal{T}$ is discrete. Now we let $\sqsubseteq$ be a "zig-zag" ordering of X: x, y are comparable w.r.t. $\sqsubseteq$ iff $|x - y| \leq 1$ (from which it follows, for example, that if $n, n+1, n+2 \in X$ and $n \sqsubseteq n+1$, then $n+1 \sqsupseteq n+2$). Then $(X, \mathcal{T}, \sqsubseteq)$ is a Hausdorff ordered space; its upper topology is what is usually called the (one-dimensional) Khalimsky topology.

Example 3.4. *Smyth interval.* [11] Let I be the usual unit interval. For each dyadic rational $r \in I$, introduce two elements r^-, r^+, so that r^- immediately precedes r, and r^+ immediately succeeds r in the (total) ordering. (In the case of $0, 1$, we introduce 0^+ and 1^-, but omit 0^- and 1^+.) Let $\mathcal{T}$ be the order topology of the resulting totally ordered set X. It is easily checked that $(X, \mathcal{T})$ is compact and totally disconnected (a Stone space). Now we let $x \sqsubseteq y$ hold (for distinct x, y) just in case x is a dyadic, r, and y is r^- or r^+. The stably compact space, say sI, of the compact ordered space $(X, \mathcal{T}, \sqsubseteq)$ may be considered as a spectral compactification (or "spectralization") of I; see [19]. More significant in the present context, however, is that sI is the inverse limit of finite Khalimsky spaces [22].

Abstracting from the preceding examples, let $(X, <)$ be a complete ordered set with order topology $\mathcal{T}$. Let R be a reflexive relation on X such that (for distinct x, y) xRy holds only if x is adjacent to y (w.r.t. $<$).

Proposition 3.5. *With X, $<$, $\mathcal{T}$, R as above, we have: $(X, \mathcal{T}, R)$ is connected if and only if every pair x, y of adjacent elements is related by R (that is, $(x, y) \in R \cup R^{op}$).*

In seeking to characterize "lines", and arcs in particular, we have to consider connectivity, and especially cut-points. We define these for our topological ordered spaces, and other topological relational structures; but it is of interest to determine when these concepts can be characterized purely topologically. For connectedness this is extremely simple:

Proposition 3.6. *Let (X, R) be a topological graph. The following are equivalent:*

(1) *(X, R) is connected;*
(2) *X is connected in the R-saturated topology;*
(3) *X is connected in the R^{op}-saturated topology.*

Proof. Let $\{U, V\}$ be a partition of X into open sets. Then U, V are R-disjoint if and only if both $R(U) \subseteq U$ and $R(V) \subseteq V$. Thus (1) $\equiv$ (2), and (1) $\equiv$ (3) is proved similarly. □

Separations induced by cut-points require a more careful discussion. Let (X, R) be a connected topological graph. We say that (U, x, V) is a *division* of (X, R) if x is a cut-point, and (U, V) is a separation of the sub-topological graph on $X \backslash \{x\}$. (An ordinary topological space is taken account of here, by taking $R = id$.)

The spaces with which we are concerned in this section generally have the feature that $\sqsubseteq$-chains are of length at most 2 (as in Examples 3.3, 3.4). The reason for this is that if we have a chain of length 3, say $x \sqsubset y \sqsubset z$, then y cannot be a cut-point.

Proposition 3.7. *Let $(X, \sqsubseteq)$ be a Hausdorff pre-ordered space in which every $\sqsubseteq$-chain has length ≤ 2, $x \in X$, $U, V \subseteq X$. The following are equivalent:*

(1) *(U, x, V) is a division of the Hausdorff pre-ordered space $(X, \sqsubseteq)$;*
(2) *(U, x, V) is a division w.r.t either the upper topology or the lower topology of $(X, \sqsubseteq)$.*

Proof. (2) $\Rightarrow$ (1): Evident.

(1) $\Rightarrow$ (2): Let (U, x, V) be a division of the ordered space. Suppose first that x is not comparable (via $\sqsubseteq$) with any element of $U \cup V$. Then U, V are both upper, as well as lower, open sets, and so (U, x, V) is a division w.r.t. both the upper and the lower topology. Next suppose that x is comparable with some element of $U \cup V$: say we have that $u \in U$ with $u \sqsubseteq x$. Then there is no element v of $U \cup V$ such that $x \sqsubseteq v$. This means that U, V are both lower open sets, and so we have a division w.r.t. the lower topology.

Likewise, the case that we have $v \in U \cup V$ with $x \sqsubseteq v$ yields a division w.r.t. the upper topology. □

Our next topic is the comparison of fenced spaces with the selective spaces of [11]. In order to facilitate the comparison, it is convenient to consider what we shall call "partially fenced" spaces.

Definition 3.8. A *partially fenced space* is a tuple $(X, <, \mathcal{T}, \sqsubseteq)$, where $<$ is a total order on X, $\mathcal{T}$ is the order topology, and $\sqsubseteq$ is a partial order on X such that $x \sqsubseteq y$ holds only if x, y are adjacent (or $x = y$).

Remark 3.9. Under the stated conditions, $(X, \mathcal{T}, \sqsubseteq)$ is a Hausdorff ordered space, so that, in particular, $\sqsubseteq$ is the specialization order of the upper topology.

Definition 3.10. A *selective space* is a triple $(X, <, \mathcal{T}')$, where $<$ is a total order on X and $\mathcal{T}'$ is a topology coarser than the order topology such that:

(1) $(X, \mathcal{T}')$ is T_0;
(2) $x \lhd y$ only if x, y are adjacent (or $x = y$), where $\lhd$ is the specialization order of $\mathcal{T}'$.

The resemblance between the two notions is rather evident. Indeed we have:

Proposition 3.11. *Let $\mathcal{T}'$ be a topology on the totally ordered set $(X, <)$. Then $(X, <, \mathcal{T}')$ is a selective space if and only if $\mathcal{T}'$ is the upper topology of a partial fencing $(\mathcal{T}, \sqsubseteq)$ on X.*

Proof. IF: immediate, in view of the Remark following Def. 3.8.

ONLY IF: Consider the partially fenced space $(X, <, \mathcal{T}, \lhd)$ where $\mathcal{T}$ (as usual) is the order topology. Every subbasic (ray) open set of $(X, \mathcal{T}')$ is trivially an upper open set of $(X, \mathcal{T}, \lhd)$. On the other hand, suppose we have an order-open ray, say $(x, \rightarrow)$, which is not open in $\mathcal{T}'$. Then x has an immediate $<$-successor, say y (since, by an easy argument, any ray $(z, \rightarrow)$ is open in case z lacks an immediate successor). Any subbasic open set which contains y is then either of the form $(z, \leftarrow)$ where $y < z$, or of the form $(z, \rightarrow)$ where $z < x$. Every such set contains x, from which we conclude that $y \lhd x$. Hence $(x, \rightarrow)$ is not an (open) upper set of $(X, \mathcal{T}, \lhd)$. □

Definition 3.12. A *fenced space* is a partially fenced space $(X, <, \mathcal{T}, \sqsubseteq)$ in which every pair of adjacent elements is comparable w.r.t $\sqsubseteq$.

Theorem 3.13. *The following are equivalent, for a tuple $(X, <, \mathcal{T}, \sqsubseteq)$:*

(1) *$(X, \mathcal{T}, \sqsubseteq)$ is connected (as a Hausdorff ordered space);*
(2) *$(X, <)$ together with the upper topology is a connected selective space;*
(3) *X is complete and fenced.*

Proof. (1) $\equiv$ (2) by Props 3.6 and 3.11.

(1) $\Rightarrow$ (3): It is clear that, if X is either incomplete or else has a pair of adjacent elements not related by $\sqsubseteq$, we get a separation of X into a left ray and a right ray, each of which is open in the upper topology.

(3) $\Rightarrow$ (1): Suppose that we have a separation (U, V) of X, and that X is complete. W.l.g. we may suppose that we have points $u \in U$, $v \in V$ such that $u < v$. Let $p = \bigvee\{x \in U \mid x < v\}$. Since V is a union of order-open intervals, we must have $p \in U$. By the same remark (applied to U), there is an element $q \in V$, adjacent to p on the right. Clearly, p and q are not related by $\sqsubseteq$; thus, X is not fenced. □

By a result of [10] (Theorem 9.18), the linear ordering of a connected selective space X is intrinsic, in the sense that it can be recovered (up to inversion) from the topology of X. This leaves open the question whether an autonomous characterization of the topologies involved, not mentioning the linear order, can be achieved. We shall return to this point in a moment.

A fourth notion, equivalent to the three presented in Theorem 3.13, was actually the first to be developed, though not fully published at the time. This was the CLOTS (connected linearly ordered topological space) of [17].

Definition 3.14. A CLOTS is a connected, totally ordered T_0-space $(X, <)$ such that a subbase of open sets is given by the sets $(x, \rightarrow), (x, \leftarrow)$ where $\{x\}$ is closed.

Proposition 3.15. *Let $(X, <)$ be connected, totally ordered, T_0. The X is a CLOTS if and only if it is selective.*

Proof. IF: This is, in effect, Theorem 9.13 of [10].

ONLY IF: Let X be a CLOTS. Suppose, for a contradiction, that we have points x, y, z with $x < y < z$ and $x \sqsubseteq z$. Every subbasic open set which contains y is either of the form $(p, \rightarrow)$ (and hence contains z), or else of the form $(p, \leftarrow)$ and thus contains x (and therefore also z, since $x \sqsubseteq z$). So $y \sqsubseteq z$. Since X is T_0, there must be a subbasic open set, say $(q, \rightarrow)$ (q closed), containing z but not y. Thus $q \in [y, z)$, and $(q, \leftarrow)$ is an open set containing x but not z: contradiction. □

The development which has led to the ideas presented in this section may be said to have begun with the studies of digital topology by Khalimsky, Kopperman and associates. The requirement was for "digital" lines and arcs which were T_0 but not necessarily T_2, and might well be finite. Concretely, there were the Khalimsky spaces and, as a general concept, the COTS [10]: a COTS is a connected topological space X in which, for any three distinct points, one of them (say x) lies between the other two, in the sense that they lie is distinct components of $X \backslash \{x\}$.

Clearly, the one-dimensional Khalimsky spaces and the real line are examples of COTS.

A desideratum proposed by Smyth (as part of a general theory of the approximation of "continuous" spaces by digital constructs) was that the class of spaces involved should be closed under inverse limits. This is not satisfied by the COTS, as witnessed by Examples 3.3 and 3.4 above (sI is not a COTS, since the points r^+, r^- are not cut-points). The question arose, whether there is a suitable class of spaces, closed under inverse limits, and including at least the locally connected COTS. (That COTS were not required to be locally connected was perhaps an anomaly; cf. [22], Sec 2.) This led to the CLOTS of [17], and to the (connected) selective spaces of [11].

As compared with standard continuum theory and its (intrinsic) *separation order*, the CLOTS and selective space theories have the order as an explicit part of the structure. This feature may appear to be unavoidable: as we have seen, the points r^+, r^- do not separate the space sI, and the prospects for constructing the ordering of the space as a separation order seem poor. Nevertheless, Smyth [22] and Webster [28] (independently) proposed means for achieving this. In [28] the idea is to work with the upper and the lower topology: in particular, a point x is a di-cut-point (of a stably compact space X) if it is a cut-point of X or of X^{op}. This leads to a definition of the *di-separation order* as a generalization of the usual separation order. In the case of sI, it is easy to check that every point is either an end point or a di-cut-point. The linear ordering of sI is indeed obtained (up to inversion) as the di-separation order. The approach in [22], on the other hand, is to work with Hausdorff ordered spaces (or, still more generally, with Hausdorff graphs). Again we note that in sI, viewed in this way, every non-end point is a cut point. Indeed, Proposition 3.7 assures us that, in spaces of the kind being considered here, the two approaches agree.

Is it, then, possible to characterize those topological ordered (or, relational) structures for which a suitable total separation order can be defined intrinsically? The answer is given in [22], as a direct generalization of the locally connected COTS:

Definition 3.16. A connected, locally connected topological graph $(X, \mathcal{T}, R)$ is *linear* if, for any three distinct points in X, there is a division (U, p, V) of X such that exactly one of the three points is in each of $U, V, \{p\}$.

It is proved in [22] that linear graphs are equivalent, via the separation order, to complete fenced spaces (called, in [22], "linear orders with adjacency"). It would take us too far afield to consider this equivalence in detail, and for convenience we shall continue here to work with the "fenced" spaces, in which the total order is explicitly given.

It is by now widely accepted that, in asymmetric topology, the stably compact spaces play the role taken by the compact (Hausdorff) spaces in Hausdorff topology [18, 2]. The study of stably compact continua, called skew continua in [12], has been initiated in [12, 28]. It turns out that the classical theory generalizes in a fairly straightforward way to the stably compact situation. An attractive feature of the stably compact theory is that we have non-trivial *finite* continua, which can under certain conditions serve to approximate continua in general. We are here concerned with *arcs* as examples of stably compact continua. The spaces involved may be thought in the first instance in terms of their compact ordered aspect. With little or no extra work, the compact ordered formulation can be generalized to compact graphs; this generalization will turn out to have some advantages.

In view of the results in the preceding subsection, we can work with a rather simple characterization of the arcs we are concerned with:

Definition 3.17. An *arc* is a 2nd-countable compact fenced space.

This definition leaves something to be desired, as it assumes that the overall (separation) order of the space is given explicitly. But we know from the previous observations that this feature is not essential. The advantage of Def. 3.17 is that it enables us to describe, in a simple manner, all the arcs which can arise (our main purpose in this subsection).

Example 3.18. Let C be the set of infinite binary sequences, with the lexicographic order $<$. Then the order topology $\mathcal{T}$ coincides with the Cantor topology of C. For every pair x, y of adjacent elements of C, put $x \sqsubseteq y$ iff $x < y$; that is, we have $\sigma 01^\omega \sqsubseteq \sigma 10^\omega$ for every finite string σ. Clearly, $\sqsubseteq$ is $\mathcal{T}$-closed, and $(C, \mathcal{T}, \sqsubseteq)$ is an arc. Notice, however, that this space is not a Priestley space (equivalently, the upper topology is not spectral): C has no non-trivial clopen upper sets.

Definition 3.19. A *Stone arc* is a fenced space $(X, <, \mathcal{T}, R)$ where $(X, \mathcal{T})$ is a 2nd-countable Stone space.

A finite (Stone) arc is simply a finite fence (in the terminology of poset theory) with discrete topology - if we require the fencing relation to be a partial order. If the relation is allowed to be a pre-order, we get the same class of finite arcs, with the single addition of the "arc" $a \equiv b$ consisting of two points, each related to the other. Allowing the fencing relation to be any reflexive relation means that we get the class of finite linear (reflexive) graphs. (To be sure, these remarks are a little imprecise, as we have not explicitly specified the total order involved.)

Returning to Example 3.18 we see that, since it is not a Priestley space, the Stone arc $(C, \mathcal{T}, \sqsubseteq)$ cannot be expressed as the inverse limit of finite arcs in the strict sense (i.e. with partial orders as fences). However, let us take as C_n the graph with vertex-set $\{0,1\}^n$ (the binary strings, or numerals, of length n), and relation R the reflexive closure of the successor relation on C_n. Then it is easy to check informally that C is the inverse limit of these simple linear graphs C_n.

Here we have assumed that the morphisms for the structures $(X, <, \mathcal{T}, R)$ are the maps which preserve $\leq$ (i.e. $< \cup =$) and R, and are continuous w.r.t. $\mathcal{T}$. Consideration of inverse limits shows that every 2nd-countable Stone space can be (totally) ordered in such a way that its topology coincides with the order topology: we have only to express it as an inverse limit of a sequence of finite discrete spaces, all of them endowed with total orders in such a way that the bonding maps are morphisms as just described.

Further considering these inverse limits, we note the following. Let Σ be an inverse sequence of finite arcs where the bonding maps are graph morphisms which also preserve the linear ($<$-)orders. Then the inverse limit of Σ is a Stone arc. We claim that, conversely, any Stone arc $(X, <, R)$ can be represented in this way. The main point is to show that $(X, <)$ has a base of (order-)open intervals which are also closed. Consider an arbitrary clopen subset K of X. By compactness, K has a least element q. Moreover, K is a join of (order-)open intervals, which may be assumed finite in number (again by compactness). It follows that either $q = \bot_X$, or q has an immediate predecessor p. Thus $\leq (K)$ $(= (p, \rightarrow))$ is clopen. Hence, any subbasic open ray, say $(y, \rightarrow)$, as a join of clopen subsets (by the Stone property), is actually a join of *clopen* subbasic rays. From this we have the desired base of clopen intervals, which we use to construct a sequence of finite (ordered) discrete spaces having $(X, <)$ as inverse limit. The relation R induces a relation on each of these finite spaces as its image (via the projection). The remainder of the argument is straightforward, and we obtain:

Theorem 3.20. *The Stone arcs are exactly the inverse limits of sequences of finite linear graphs.*

Example 3.21. Let C be as in Example 3.18, except that this time we put $x \equiv y$ (that is, both $x \sqsubseteq y$ and $y \sqsubseteq x$) for each adjacent pair. We now have a compact (Stone) pre-ordered space, where the pre-order is actually an equivalence relation. As with Example 3.18, we do not have a Priestley space, but the structure, as a Stone arc, is easily expressed as an inverse limit of finite linear graphs.

The significant new aspect of this example is the equivalence relation. It is natural to think of quotienting the structure by this equivalence, and the result is the unit interval - see Example 4.9. In the next Section, we shall provide a general setting for this observation.

4. Stably Compact Graphs

It is well-known that a convenient category of stably compact spaces is obtained if one takes the continuous maps $f : X \to UY$ (UY being the upper space over Y considered later) as morphisms. More precisely, one shows that the upper space construct U is (functorial and) monadic over the category $\mathbf{K}$ (of stably compact spaces and continuous maps), and then takes the Kleisli category with respect to this monad. With $[X \to UY]$ as the exponential, and a suitable tensor product (which in its object part is just the product $X \times Y$ in $\mathbf{K}$), the category turns out to be symmetric monoidal closed. See in particular Schalk [15] and Sünderhauf [26] (notice that Sünderhauf's treatment is couched in terms of quasi-uniform spaces). A recent discussion is provided by Jung & Kegelmann & Moshier [2], where the emphasis is on Stone duality.

Although this categorical work provides part of the context for what we are doing here, we shall not use it explicitly. We take from it, however, the key observation that continuous maps from X to UY are the same as closed subsets of $X \times Y^{op}$. This observation, which will be sharpened below (Prop 5.10), partly motivates the following definition:

Definition 4.1. A *stably compact graph* is a stably compact space X together with a reflexive relation R that is closed in the product space $X \times X^{op}$. A *graph morphism* between stably compact graphs is a continuous, relation-preserving map.

Example 4.2.

(1) *Stably compact spaces* Any stably compact space together with its specialization order is a stably compact graph (this is a simple consequence of Nachbin's separation result).
(2) *Topological graphs* The stably compact Hausdorff spaces are precisely the compact Hausdorff spaces. Any compact Hausdorff space together with a closed reflexive relation is a stably compact graph. This includes the compact ordered spaces.
(3) *Directed graphs* Any finite discrete space together with a reflexive relation is a stably compact graph.
(4) *Finite T_0-graphs* The finite stably compact spaces are precisely the finite T_0-spaces. Any finite T_0-space X together with a relation R such that $\sqsubseteq_X \subseteq R$ is a stably compact graph.

Some basic properties that will be of use later on are:

Lemma 4.3. *Let (X, R) be any stably compact graph. Then:*

(1) *If $x \sqsubseteq_X y$ then xRy;*
(2) *If $x' \sqsubseteq_X x$ and xRy and $y \sqsubseteq_X y'$ then $x'Ry'$;*
(3) *The* dual (X^{op}, R^{op}) *is a stably compact graph;*
(4) *For any patch-compact C, the set $R(C)$ is compact upper;*
(5) *For any patch-open U, the set $int_R(U) = \{x \mid R(x) \subseteq U\}$ is open.*

Proof. If $(x, y) \notin R$ then there is some open U and some *op*-open V that contain x, y respectively such that $(U \times V)$ is disjoint from R. Then (2) is an immediate consequence of this, and (1) is an immediate consequence of this together with that R is reflexive. (3) follows from that $(X^{op})^{op} = X$. R is obviously a closed relation on $patch(X)$, and it is a straightforward exercise that $R(C)$ is therefore patch-compact. (1) gives that $R(C)$ is upper, which then gives (4). For (5), let C denote the complement of U. Then $int_R(U)$ is the complement of $R^{op}(C)$. Now C is compact upper in X^{op}, so by (4) applied to (X^{op}, R^{op}), $R^{op}(C)$ is compact upper in X^{op}. □

What is the role of stably compact graphs in approximating stably compact spaces? The case study in the preceding section provided some evidence for the significance of *Stone* graphs. This is confirmed by Proposition 4.7 below: any stably compact space can be represented by a Stone pre-ordered space $(X, \leq)$, which in turn is an inverse limit of finite discrete graphs. Why then do we need to consider so rich a structure as the stably compact graph? The answer lies in the constructions which we want to be able to carry out, specifically those mentioned at the beginning of this section. Thus, the upper "space" of a compact Hausdorff graph (X, R) will be a structure (UX, R') where UX is the (stably compact) upper space of X, and R' is a relation yet to be determined. By working in the category of stably compact graphs, we will be able to accommodate these constructions, and treat the corresponding finite approximations in a uniform manner.

4.1. **Inverse sequences.** Consider an inverse sequence of stably compact graphs and perfect graph morphisms (a graph morphism is perfect if it is perfect as a map between the underlying topological spaces)

$$\Delta = (X_1, R_1) \leftarrow^{f_1} (X_2, R_2) \leftarrow^{f_2} \cdots$$

Let X_ω denote the limit of the sequence of spaces $\Phi = (X_i, f_i)$, and let p_i denote the projections. The *limit relation* on X_ω is

$$R_\omega = \bigcap (p_i \times p_i)^{-1}(R_i)$$

Proposition 4.4. *(X_ω, R_ω) is a stably compact graph, and is the limit of the inverse sequence Δ in the category of stably compact graphs.*

Proof. From the last two results in Section 2.2, $X \times X^{op}$ is the limit of the inverse sequence of spaces $\Phi \times \Phi^{op}$, and R_ω is closed in this space because it is the intersection of the closed sets $(p_i \times p_i)^{-1}(R_i)$.

For any cone $((Y, S), q_1, q_2, \dots)$ over Δ, $(Y, q_1, q_2, \dots)$ is a cone over (X_i, f_i) in the category of topological spaces. The corresponding mediating map is $Y \to X_\omega,\ y \mapsto (q_1(y), q_2(y), \dots)$, which is easily seen to be relation-preserving with respect to S and R_ω, and is therefore a mediating map in the category of stably compact graphs. □

4.2. **Quotients.** In this section we give a "quotienting" construction for a particular class of stably compact graphs, namely the stably compact *pre-orders*, which are the stably compact graphs in which the relation is a pre-order. The quotient of a stably compact pre-order is a stably compact space, and may legitimately be called the stably compact *reflection*, for reasons discussed below.

Let (X, R) be a stably compact pre-order, and consider the corresponding equivalence relation

$$x \equiv_R y \text{ if } xRy \text{ and } yRx$$

Let $Y = X/\equiv_R$ denote the set of equivalence classes, and let $\phi : X \to Y$ be the set-theoretic quotient map. The quotient topology on Y is the collection

$$\{U \mid \phi^{-1}(U) \text{ open and } R\text{-saturated}\}$$

and we say that Y together with this topology is the *quotient* of (X, R).

Proposition 4.5. *The quotient of a stably compact pre-order is a stably compact space. Moreover, the map* $\phi : (X, R) \to (Y, \sqsubseteq_Y)$ *is a perfect graph morphism.*

Proof. It is simple to prove that $\equiv_R$ is a closed relation on $patch(X)$, and the quotient of a compact Hausdorff space by a closed equivalence relation is again compact Hausdorff. Let $\psi : patch(X) \to patch(X)/\equiv_R$ be the quotient map, and consider the relation

$$\psi(x) \sqsubseteq \psi(y) \text{ if } xRy$$

It is again simple to prove that this relation is well-defined, and is a closed partial order, so we have a compact ordered space. That Y is the corresponding stably compact space is given by that: U is open and $\sqsubseteq$-saturated iff $\psi^{-1}(U)$ is open and R-saturated (and therefore $\sqsubseteq_X$-saturated) iff $\phi^{-1}(U)$ is open in X and R-saturated iff U is open in Y.

Therefore $patch(X)/\equiv_R \,=\, patch(Y)$ and $\sqsubseteq \,=\, \sqsubseteq_Y$. Then the map $\phi : X \to Y$ is perfect because the corresponding map between the corresponding compact ordered spaces is continuous and order-preserving. □

Observe also that ϕ is a topological quotient map (onto and U open iff $\phi^{-1}(U)$ open) with respect to the quotient topology on Y and the R-saturated topology on X.

Recall (Example 4.2) that a stably compact space Y together with its specialization order $\sqsubseteq_Y$ is a stably compact graph, and (as any continuous map is order-preserving w.r.t. the specialization orders) the category of stably compact spaces is a full subcategory of the category of stably compact graphs.

Entirely analogous to the T_i-reflection construction considered by Kopperman and Wilson, the above quotienting construction has the property that, where $(Y, \sqsubseteq_Y)$ is the quotient of the stably compact preorder (X, R), with ϕ the quotient map, then for any stably compact space Z and any morphism $h : (X, R) \to (Z, \sqsubseteq_Z)$, there is a unique morphism (continuous map) $k : (Y, \sqsubseteq_Y) \to (Z, \sqsubseteq_Z)$ such that $h = k \circ \phi$. (That h is relation-preserving together with that $\sqsubseteq_Z$ is a partial order (stably compact spaces are T_0) gives that the map $k(\phi(x)) = h(x)$ is well-defined, and the proof that k is continuous is straightforward.)

4.3. **Finite approximation of stably compact spaces.**

Definition 4.6. An inverse sequence of stably compact graphs and perfect bonding maps *approximates* a stably compact space Y if its limit is a stably compact preorder whose quotient is Y.

We are interested in approximation by *finite* stably compact graphs. Say that a *finite (directed) graph* is a stably compact graph whose underlying topological space is finite discrete. Then any stably compact space can be approximated by finite graphs (see below). However, it is entirely natural in certain contexts (to be discussed later on) to consider approximation by finite, but not necessarily discrete, stably compact graphs. So to be clear about the distinction: a *finite graph* is a finite stably compact graph whose underlying topology is discrete.

Proposition 4.7. *Every 2nd-countable stably compact space can be approximated by an inverse sequence of finite graphs.*

Proof. Let Y be a 2nd-countable stably compact space. Then $patch(Y)$ is 2nd-countable, and so is compact metric (= compact, Hausdorff, 2nd-countable). Where X_ω denotes the Cantor space, let $\phi : X_\omega \to patch(Y)$ be a topological quotient map (it is a classical result that the compact metric spaces are precisely the quotients of Cantor space by closed equivalence relations). The relation

$$R_\omega = \{(x, y) \mid \phi(x) \sqsubseteq_Y \phi(y)\}$$

gives a stably compact preorder (X_ω, R_ω) whose quotient is Y. Now X_ω is the limit of an inverse sequence $X_1 \leftarrow^{f_1} X_2 \leftarrow^{f_2} \cdots$ of finite discrete spaces; let p_i be the projection maps and, for each i, let

$$R_i = (p_i \times p_i)(R_\omega)$$

This gives an inverse sequence $(X_1, R_1) \leftarrow^{f_1} (X_2, R_2) \leftarrow^{f_2} \cdots$ of finite graphs whose limit is (X_ω, R_ω). □

The requirement that the limit relation be a preorder is equivalent to the condition that the collection of relations $(p_i \times p_i)^{-1}(R_i)$ is a base of a quasi-uniformity, and can be guaranteed by requiring that each bonding map has the following property. Say that a graph morphism $f : (X, R) \to (Y, S)$ is *strong* if it is also relation-preserving with respect to $R \circ R$ and S. It is simple to prove that the limit relation in the limit of an inverse sequence of stably compact graphs and strong maps is a preorder; it can also be shown that any stably compact space can be approximated by an inverse sequence of finite graphs and strong bonding maps, but we will not go into the proof here.

Example 4.8. *Cantor space* Let X_i denote the set $\{0,1\}^i$ with the discrete topology, and let f_i denote the truncation map $(x_1,\dots,x_i,x_{i+1}) \mapsto (x_1,\dots,x_i)$. Then the Cantor space is the limit X_ω of the resultant inverse sequence. Every stably compact space can be approximated by an inverse sequence of graphs whose underlying inverse sequence of spaces is the this. Moreover, every compact Hausdorff space can be approximated by such an inverse sequence in which each graph is undirected (i.e. the relation in each graph is symmetric).

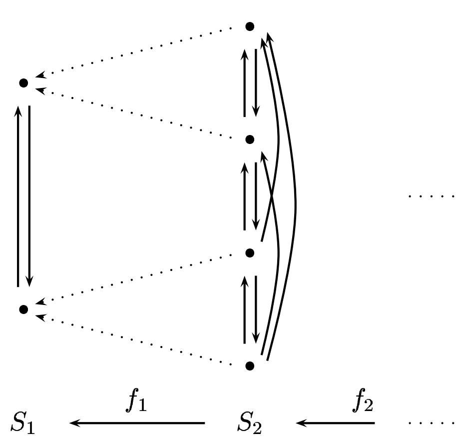
Example 4.9. *The unit interval* Let $<_i$ denote the lexicographic order on X_i in the previous example, let A_i denote the union of the corresponding adjacency relation with the identity, and let I_i denote the graph (X_i, A_i). Then the inverse sequence of graphs $I_1 \leftarrow^{f_1} I_2 \leftarrow^{f_2} \cdots$ approximates the unit interval. This was proved in [20], but to see this, consider the map

$$\phi : X_\omega \to I,\ (x_1, x_2, \dots) \mapsto \frac{x_1}{2} + \frac{x_2}{4} + \cdots$$

This is a topological quotient map, and is one-to-one apart from the fact that, for each dyadic point $0 < d < 1$, $\phi^{-1}(d) = \{(x,1,1\dots),\ (y,0,0\dots)\}$, for some $x, y \in X_i$ such that xA_iy and $x <_i y$. It follows that the unit interval is the quotient by the limit of the adjacency relations.

This example also gives the relationship between the digital and Euclidean planes mentioned in the introduction. For each i, the product graph (see later) $I_i \times I_i$ is a digital plane with the "8-connectivity" relation (see e.g. [14]). The inverse sequence of product graphs approximates (see Prop 5.2) the unit square.

Example 4.10. *The Scott unit interval* This is the stably compact space corresponding to the unit interval considered with its usual linear order. Let S_i denote the graph $(X_i, A_i \cup <_i)$. From the previous example it is clear that the Scott unit interval is approximated by the inverse sequence of graphs:



5. Constructions on stably compact graphs

In this section we give various constructions on stably compact graphs, and show that they preserve approximation.

The first result is that concerning duality. The *dual* of a stably compact graph (X, R) is (X^{op}, R^{op}), which (Lemma 4.3) is a stably compact graph. Any perfect graph morphism is clearly also a graph morphism when considered as a map between the respective dual graphs: the dual Δ^{op} of an inverse sequence Δ is then defined in the obvious way. It is simple to prove (given Prop. 2.4) that the limit of Δ^{op} is the dual of the limit of Δ. It is also very straightforward to show that if a stably compact space Y is the quotient of a stably compact pre-order, then Y^{op} is the quotient of the dual pre-order. Therefore:

Proposition 5.1. *If Δ is an inverse sequence of stably compact graphs that approximates the stably compact space Y, then Δ^{op} approximates Y^{op}.*

5.1. **Products.** The product $X \times Y$ of two stably compact spaces is stably compact (see [9]), and the corresponding compact ordered space is $(patch(X) \times patch(Y), \sqsubseteq_X \times \sqsubseteq_Y)$, the product order being the product relation defined below. From this it follows that $(X \times Y)^{op} = X^{op} \times Y^{op}$.

The *product* of the stably compact graphs (X, R) and (Y, S) is the product space $X \times Y$ together with the product relation

$$R \times S = \{((x, y), (x', y')) \mid xRx' \text{ and } ySy'\}$$

It is straightforward that this is indeed a stably compact graph, and is the product in the category of stably compact graphs.

Proposition 5.2. *If Δ, Λ are inverse sequences of stably compact graphs that approximate the stably compact spaces X, Y respectively, then $\Delta \times \Lambda$ approximates $X \times Y$.*

Proof. Given the result that products preserve limits of inverse sequences in any category, it suffices to show that if X, Y are the respective quotients of the stably compact pre-orders (X', R), (Y', S), then $X \times Y$ is the quotient of their product. Let $\phi : X' \to X$ and $\psi : Y' \to Y$ be the respective quotient maps. It is straightforward that the product map $\phi \times \psi$ is perfect, and we claim it is a topological quotient map with respect to the $(R \times S)$-saturated topology. The product map is clearly onto. The preimage of a basic open set $U \times V$ is $\phi^{-1}(U) \times \psi^{-1}(V)$, which is $(R \times S)$-saturated because $\phi^{-1}(U)$ is R-saturated and $\psi^{-1}(V)$ is S-saturated. Now let $(\phi \times \psi)^{-1}(W)$ be an $(R \times S)$-saturated open set. If this set contains (x, y) then it contains $R(x) \times S(y)$, and then by compactness of $R(x)$ and $S(y)$ (Lemma 4.3) there are open sets U, V such that $R(x) \times S(y) \subseteq U \times V \subseteq (\phi \times \psi)^{-1}(W)$. We may assume that U, V are, respectively, R-saturated and S-saturated (consider the saturated open sets $int_R(U)$, $int_S(V)$ - see Lemma 4.3), therefore $(\phi \times \psi)(U \times V) = \phi(U) \times \psi(V)$ is an open set contained in W, so W is open.

It remains only to check that $X \times Y$ is the quotient by the equivalence relation $\equiv_{R\times S}$, i.e. that $(\phi \times \psi)(x,y) = (\phi \times \psi)(x',y')$ if and only if $(x,y) \equiv_{R\times S} (x',y')$, the proof of which is simple. □

5.2. **The upper graph.** The *upper space* over a stably compact space X is the set UX of compact upper subsets of X together with the topology that has the basis

$$\{\Box U \mid U \text{ open in } X\}$$

where $\Box U$ denotes the set $\{C \in UX \mid C \subseteq U\}$. For any continuous map $f : X \to Y$, the map

$$Uf : UX \to UY,\ C \mapsto up(f(C))$$

is continuous, and is perfect if f is perfect. The upper space construction is functorial. The upper inverse sequence over an inverse sequence $\Phi = (X_i, f_i)$ of stably compact spaces and perfect bonding maps is the inverse sequence

$$U\Phi = UX_1 \leftarrow^{Uf_1} UX_2, \leftarrow^{Uf_2} \cdots$$

The following result appears to be new; we are not aware of any corresponding work in the literature apart from that on the Vietoris space in [16].

Proposition 5.3. *The upper space construction preserves limits of inverse sequences of stably compact spaces and perfect bonding maps.*

Proof. A definition that will be used in this and a later proof is that a *decreasing sequence* in Φ is a sequence $(C_i, C_{i+1}, \dots)$, beginning at any i, such that each $C_j \subseteq X_j$, and such that each $C_{j+1} \subseteq f_j^{-1}(C_j)$. The *limit* of this sequence is the intersection in X_ω of

$$p_i^{-1}(C_i) \supseteq p_{i+1}^{-1}(C_{i+1}) \supseteq \dots$$

(where, as usual, X_ω is the limit of Φ with p_i the projections). Examples of decreasing sequences are the threads of $U\Phi$. Associating a thread with its limit and considering the properties of this association is the main technique in this proof.

Lemma 5.4. *The limit of a decreasing sequence of patch-closed sets is patch-closed, and is non-empty if each set in the sequence is non-empty.*

Proof. Each projection p_i is perfect so the limit, as the intersection of patch-closed sets, is patch-closed. It is a classical result (given in e.g. [4]) that the limit of an inverse sequence of non-empty compact Hausdorff spaces is non-empty. Now the decreasing sequence can be regarded as such an inverse sequence in an obvious way, and it is then a straightforward exercise to show that this implies that the above limit is non-empty.

To $C \in UX_\omega$ we assign the decreasing sequence $(Up_1(C), Up_2(C), \dots)$, which is a thread of $U\Phi$ because the upper space construction is functorial. Then C is the limit of its associated thread: clearly it is contained in the limit, and, as was shown in Prop 2.4, for any $y \notin C$ there is a basic open set that contains C but not y. The assignment is therefore one-to-one.

Let $(C_1, C_2, \dots)$ be any thread in $U\Phi$ and let C be its limit. Then $C \in UX_\omega$, and we claim that the thread is that associated with C. Now each $Up_i(C) \subseteq C_i$, so suppose for contradiction that for some i this is a proper subset inclusion. By Nachbin's separation result, let U be an open set that contains $Up_i(C)$ but not C_i. Then $C \subseteq p_i^{-1}(U)$, but consider the decreasing sequence of patch-closed sets $(C_i \backslash U, C_{i+1} \backslash f_i^{-1}(U), \dots)$. A simple induction argument shows that each set in this sequence is non-empty, so its limit is non-empty. But the limit is a subset of $C \backslash p_i^{-1}(U)$, which is a contradiction.

We therefore have a bijection between UX_ω and the set of threads of $U\Phi$. Now UX_i has the basis $\{\Box U \mid U \text{ open in } X_i\}$, and it follows easily that the inverse limit topology on the set of threads has the basis $\{(C_1, C_2, \dots) \mid C_i \in \Box U,\ U \text{ open in } X_i,\ i = 1, 2, \dots\}$. The sets $p_i^{-1}(U)$ form a basis of X_ω, and it is a simple exercise in compactness to show that UX_ω then has the basis $\{\Box(p_i^{-1}(U)) \mid U \text{ open in } X_i,\ i = 1, 2, \dots\}$. It is then easy to see that the bijection maps basic open sets in one topology to basic open sets in the other, and so is a homeomorphism. Therefore UX_ω is the limit of $U\Phi$, and it is almost immediate (see the remarks at the beginning of Section 2.3) that the maps Up_i are the corresponding projections. □

Regarding the other hyperspace constructions, we will not consider the Vietoris space in this work, although the results and proof techniques for the upper space go through for the Vietoris space. Neither will we consider much the lower space (apart from the description it gives of the dual of the upper space), although the duals of the results for the upper space apply immediately to the lower space.

Definition 5.5. The *upper graph* over a stably compact graph (X, R) is the upper space UX together with the *upper relation* $UR = \{(C, D) \mid D \subseteq R(C)\}$.

Spelling out the upper relation, we have

$$C(UR)D \equiv \forall y \in D. \exists x \in C.\ xRy$$

Evidently, this is an analogue, or generalization, of the upper ("Smyth") order of domain theory. "Power relation" constructs of this kind are also considered in the algebra literature: see for example Brink [5]. Usually it is only the "convex", or strong, relation

$$D \subseteq R(C) \ \&\ C \subseteq R^{op}(D)$$

and its generalization to n-ary relations which is considered. At the other extreme we have the *weak* relation:

$$CR_wD \equiv \exists x \in C. \exists y \in D.\ xRy$$

Although this has no analogue in domain theory, it proves to be very useful in approximation studies [29, 24]. The upper relation of Definition 5.5 has been chosen because it facilitates the upper space approximation theorem (Prop 5.9).

To show that the upper graph is indeed a stably compact graph, we need a description of $(UX)^{op}$. This is provided by the *lower space* over X, which is the set LX of compact lower subsets of X together with the topology that has as a subbase the collection

$$\{\Diamond U \mid U \text{ open in } X\}$$

where $\Diamond U$ denotes the set $\{C \in LX \mid C \cap U \neq \varnothing\}$. Now the compact upper subsets of X are precisely the compact lower subsets of X^{op}, and it is a standard result (see e.g. [15]) that $(UX)^{op} = L(X^{op})$.

Proposition 5.6. *The upper graph is a stably compact graph.*

Proof. If $(C, D) \notin UR$ then there is some $y \in D$ such that $y \notin R(C)$. Then by compactness of C there is some open U in X and some open V in X^{op} such that $C \subseteq U$, $y \in V$ and $(U \times V)$ is disjoint from R. Then $\Box U$ is an open neighbourhood of C in UX, $\Diamond V$ is an open neighbourhood of D in $L(X^{op}) = (UX)^{op}$, and $(\Box U \times \Diamond V)$ is disjoint from UR. □

Proposition 5.7. *The upper graph construction is functorial.*

Proof. Given that the upper space construction is functorial, we need only show that for any graph morphism $f : (X, R) \to (Y, S)$, the map Uf is relation-preserving with respect to UR and US. If $(C, D) \in UR$ then $f(D) \subseteq S(f(C))$ because f is relation-preserving. For any $x \in Uf(D)$ there is some $d \in D$ such that $f(d) \sqsubseteq_Y x$, and some $c \in C$ such that $f(c)Sf(d)$. Therefore (Lemma 4.3) $f(c)Sx$, therefore $Uf(D) \subseteq S(f(C)) = S(Uf(C))$. □

The upper inverse sequence $U\Delta$ over an inverse sequence of stably compact graphs Δ is defined in the obvious way.

Proposition 5.8. *The upper graph construction preserves limits of inverse sequences.*

Proof. Given the result for inverse sequences of upper spaces, and the fact that the projections Up_i are relation-preserving, we need only show that $(C, D) \in UR_\omega$ if, for all i, $(Up_i(C), Up_i(D)) \in UR_i$. Suppose, then, that $(C, D) \notin UR_\omega$. Then there is some $x \in D$ such that $(R_\omega)^{op}(x)$ and C are disjoint. Now $(R_\omega)^{op}(x)$ is by definition the limit of the decreasing sequence of patch-compact sets $((R_1)^{op}(p_1(x)), (R_2)^{op}(p_2(x)), \dots)$, and C is the limit of the decreasing sequence of patch-compact sets $(Up_1(C), Up_2(C), \dots)$, therefore we have a decreasing sequence of patch-compact sets

$$((R_1)^{op}(p_1(x)) \cap Up_1(C), (R_2)^{op}(p_2(x)) \cap Up_2(C), \dots)$$

whose limit is $(R_\omega)^{op}(x) \cap C$, which is empty.

Therefore (by the Lemma in the proof of Prop 5.3) there is some i such that $(R_i)^{op}(p_i(x))$ and $p_i(C)$ are disjoint. □

Proposition 5.9. *If Δ is an inverse sequence of stably compact graphs that approximates the stably compact space Y then the inverse sequence $U\Delta$ approximates UY.*

Proof. Given the previous result it suffices to show that if (X, R) is a stably compact pre-order whose quotient space is Y, then UY is the quotient of the upper graph (UR is obviously a pre-order). Recall that the quotient map $\phi : X \to Y$ is perfect, so we have a perfect map $U\phi : UX \to UY$.

We claim that $U\phi$ is a topological quotient map with respect to the UR-saturated topology on UX. $U\phi$ is onto because ϕ is perfect and onto: for any $C \in UY$, $\phi^{-1}(C) \in UX$, and $C = U\phi(\phi^{-1}(C))$. For any U open in Y, $\phi^{-1}(U)$ is open and R-saturated, therefore $\Box\phi^{-1}(U) = (U\phi)^{-1}(\Box U)$ is open and UR-saturated. Let $\mathcal{U}$ be any set in UY such that $(U\phi)^{-1}(\mathcal{U})$ is open and UR-saturated. For any $C \in \mathcal{U}$ there is some V open in X such that $\phi^{-1}(C) \in \Box V \subseteq (U\phi)^{-1}(\mathcal{U})$. Now $\phi^{-1}(C)$ is R-saturated, so we may assume without loss of generality (consider the R-saturated open set $int_R(V)$) that V is R-saturated, in which case $\Box V$ is UR-saturated. Now $\phi(V)$ is open in Y because ϕ is a topological quotient map with respect to the R-saturated topology on X, and then $C \in (U\phi)(\Box V) = \Box\phi(V) \subseteq \mathcal{U}$, therefore $\mathcal{U}$ is open.

It remains only to show that UY is the quotient of the UR-saturated topology by the equivalence relation $\equiv_{UR}$, i.e. that $(U\phi)(C) = (U\phi)(D)$ iff $(C, D) \in \equiv_{UR}$. This is given by the easily checked fact that, for any C, $(U\phi)(C) = \phi(R(C))$, and so $(U\phi)(C) = (U\phi)(D)$ iff $R(C) = R(D)$. □

5.3. **Function spaces.** The natural topology on the set $[X \to Y]$ of continuous functions between any two stably compact spaces is the *compact-open* topology, which is the topology that has a subbase of the sets

$$(C, U) = \{f \mid C \subseteq f^{-1}(U)\}$$

for any compact $C \subseteq X$ and any open $U \subseteq Y$. But it is well-known that this topology is not necessarily stably compact. The function space $[X \to UY]$ (considered with the corresponding compact-open topology) *is* stably compact however, and so can be approximated by finite graphs. We will not give the construction here; instead we will give a simpler construction, based on results already obtained, of a space of relations that is isomorphic to this function space. This isomorphism takes $f \in [X \to UY]$ to its graph

$$graph(f) = \{(x, y) \mid y \in f(x)\}$$

which is an element of $U(X^{op} \times Y)$. Let $\mathbf{2}$ denote the Sierpinski space: there is an isomorphism (see Escardó [7])

$$UX \cong [X^{op} \to \mathbf{2}]$$

which maps C to its characteristic map $x \mapsto 0$ iff $x \in C$. Moreover, for any three stably compact spaces, we have the standard (Currying) homeomorphism (see [7])

$$[X \to [Y \to Z]] \cong [(X \times Y) \to Z]$$

The following result and its proof were communicated to the authors by Martin Escardó.

Proposition 5.10 (Escardó). *The map* $[X \to UY] \to U(X^{op} \times Y)$, $f \mapsto graph(f)$ *is an isomorphism.*

Proof. We have the following sequence of isomorphisms

$$\begin{aligned} [X \to UY] &\cong [X \to [Y^{op} \to \mathbf{2}]] \\ &\cong [(X \times Y^{op}) \to \mathbf{2}] \\ &\cong [(X^{op} \times Y)^{op} \to \mathbf{2}] \\ &\cong U(X^{op} \times Y) \end{aligned}$$

The resultant isomorphism between the first and last spaces in this sequence is the map $f \mapsto graph(f)$. □

Now let Δ and Λ be inverse sequences that approximate the stably compact spaces X and Y respectively. From previous results we then have that the inverse sequence $U(\Delta^{op} \times \Lambda)$ approximates $U(X^{op} \times Y)$.

6. Conclusion

An idea implicit in the preceding is that a topological space Y may be "represented" by a space X together with a relation R, in a way that generalizes ordinary quotienting (in which a space is represented by another space together with an equivalence relation). More specifically, the idea is as follows. We are trying to construct the space Y and (surjective) representation map $\phi : X \to Y$ in such a way that the *inequation* $\phi(x) \sqsubseteq_Y \phi(y)$ holds whenever xRy (rather than $\phi(x) = \phi(y)$ as in the usual quotienting).

We may remark that "quotienting by a pre-order" is an operation that is very natural in algebraic and domain-theoretic work in computer science, and has frequently been studied in those settings: see for example Courcelle & Raoult [6], Hennessy [8], Abramsky & Jung [1].

In the context of T_0-spaces we have the following (recall that the T_0-ification, or "T_0-reflection", of a space X is its quotient by the equivalence relation $\{(x, y) \mid x \sqsubseteq_X y \text{ and } y \sqsubseteq_X x\}$):

Theorem 6.1. *Let X be a T_0-space, and R a relation on X. Then there exists a T_0-space Y and continuous $\phi : X \to Y$ such that*

(1) *For all $x, y \in X$, $xRy \Rightarrow \phi(x) \sqsubseteq_Y \phi(y)$;*
(2) *Any continuous $\phi' : X \to Y'$ with the same property factors uniquely through ϕ.*

Moreover, the space Y so specified is unique up to homeomorphism (and ϕ is surjective).

Proof. (Outline) Let X' be the set X taken with the R-saturated topology, Y the T_0-ification of X', and $\phi : X' \to Y$ the canonical surjection. Then Y, ϕ satisfy (1).

Suppose that Y', ϕ' also satisfy (1). It has to be shown that, if $x \equiv_{X'} y$, then $\phi'(x) = \phi'(y)$. But this is immediate by the continuity of ϕ' and the T_0 property of Y'. □

In this construction we may without loss of generality require that R be a pre-order, since a relation R may evidently be replaced by its reflexive transitive closure, without affecting the construction. We may remark further that a pre-order over X may be defined by fixing a *cover* of X. Specifically, given the pre-order R, we have the cover $\mathcal{C}_R$ by the R-saturated sets, from which R is recovered by

$$xRy \equiv \forall S \in \mathcal{C}_R.\ x \in S \Rightarrow y \in S$$

Example 6.2. The cover of the real line $\mathbb{R}$ by the sets $(k-1, k+1)$ $(k \in \mathbb{Z})$ specifies the Khalimsky line.

If R is an equivalence relation, the construction reduces to the usual quotient, provided that the quotient is T_0. On the other hand, in the case that X is stably compact and R a preorder closed in $X \times X^{op}$, the construction reduces to that given in Section 4.2.

What has been shown above is that, in approximating stably compact spaces finitely, we have (in general) to use *representations* (X, R) explicitly.

Besides the general machinery, we have in this paper considered only one application (the treatment of arcs, Section 3). In the symmetric case (that is, the approximation of Hausdorff spaces by undirected graphs) by comparison, a fairly detailed study of the properties of continua in relation to those of finite graphs has been carried out in [27, 29]. There also could be mentioned the use of graphs in approximating fractals by Bandt and Keller [3]. In recent work, Smyth and Tsaur [24] have studied fixed points with these techniques, showing that new fixed point results for both graphs and spaces can be obtained. We may expect further applications in this case as well as in the (more general) asymmetric case: the approximation of T_0-spaces by directed graphs.

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