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Transitivity of hereditarily metacompact spaces

Hans-Peter A. Künzi*

Dedicated to Professor S. Naimpally on the occasion of his 70^{th} birthday.

Abstract. We prove that each regular hereditarily metacompact (monotonic) β-space has the property that the third power of any neighbornet belongs to its point-finite quasi-uniformity.

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1. Introduction.

Junnila [6, Corollary 4.13] showed (see also [4, Theorem 6.21]) that in a semistratifiable metacompact space the third power of each neighbornet belongs to the point-finite quasi-uniformity. Similarly, in [7] it was proved that each regular hereditarily metacompact compact space possesses the latter property.

Junnila's result and the techniques used in [7] suggested that it should be possible to generalize the latter result beyond (local) compactness using methods known from the theory of monotonic properties (compare [2]).

In this note we verify this conjecture by presenting a proof which shows that each regular hereditarily metacompact (monotonic) β-space satisfies the condition that the third power of any neighbornet belongs to its point-finite quasi-uniformity.

Recall that a topological space is called *transitive* (see e.g. [4]) provided that its finest compatible quasi-uniformity has a base consisting of transitive entourages. Hence in particular our result implies that each regular hereditarily metacompact (monotonic) β-space is transitive.

For basic facts about quasi-uniformities we refer the reader to [4].

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2. Main result.

Let us first mention some pertinent definitions and recall a few well-known facts. A regular topological space X is said to be a *monotonic β-space* [1] if, for each point $x \in X$, there exists a decreasing sequence $\langle \mathcal{B}_n(x)\rangle_{n\in\omega}$ of open neighborhood bases of X at the point x such that if $B_n \in \mathcal{B}_n(x_n)$ and $B_{n+1} \subseteq B_n$ whenever $n \in \omega$ and if $\bigcap_{n\in\omega} B_n$ is nonempty, then the sequence $\langle x_n\rangle_{n\in\omega}$ has a cluster point. The family $\{\langle \mathcal{B}_n(x)\rangle_{n\in\omega} : x \in X\}$ is called a *monotonic β-system* of X.

The following results are known to hold (in the class of regular (T_1-)spaces): Each β-space is a monotonic β-space. Every monotonic p-space is a monotonic β-space [1, Proposition 1.7]. Furthermore, every submetacompact monotonic p-space is a p-space [2, Theorem 2.8(b)]. Recall also that each submetacompact space is a p-space if and only if it is a $w\Delta$-space [5, Theorem 3.19].

We shall find it convenient to work with the following class of (regular) topological spaces X that is defined in terms of a game $G(X)$ in X, which is a modification of certain games introduced in [3] and was suggested to us by Prof. J. Chaber. The game $G(X)$ is similar to the strong game of Choquet. Player I starts the game by choosing a nonempty open set V_0 and a point $x_0 \in V_0$. After player I has chosen his nonempty open set V_n and $x_n \in V_n$ in his n^{th} move where $n \in \omega$, player II replies with an open set $W_n \subseteq V_n$ containing x_n and player I, in the next move, has to pick V_{n+1} inside W_n. Player II wins if either $\cap_{n\in\omega} W_n = \varnothing$ or the sequence $\langle x_n\rangle_{n\in\omega}$ has a cluster point in X. A *winning strategy for player* II is a function s into the topology of X defined on all finite sequences of moves of player I so that player II always wins when using the function s to determine his next move.

It is readily seen that for each (regular) monotonic β-space X, player II has a winning strategy for the game $G(X)$. Indeed in his n^{th} move he will choose as W_n some (fixed) member of $\mathcal{B}_n(x_n)$ contained in V_n.

A *scattered partition* (see e.g. [8, Definition 2.4]) of a topological space X is a cover $\{L_\alpha : \alpha < \gamma\}$ of X by pairwise disjoint sets such that the set $S_\beta = \bigcup\{L_\alpha : \alpha < \beta\}$ is open for each $\beta \leq \gamma$.

A binary relation N on a topological space X is called a *neighbornet* of X if $N(x) = \{y \in X : (x,y) \in N\}$ is a neighborhood at x whenever $x \in X$. For any interior-preserving open cover $\mathcal{C}$ of a topological space X, we define the neighbornet $D\mathcal{C}$ of X by setting $D\mathcal{C}(x) = \bigcap\{C \in \mathcal{C} : x \in C\}$ whenever $x \in X$. We recall that the filter on $X \times X$ generated by the subbase $\{D\mathcal{C} : \mathcal{C}$ is a point-finite open cover of $X\}$ is called the *point-finite quasi-uniformity* of X (see e.g. [4]).

Lemma 2.1. *Suppose that X is a hereditarily metacompact space and let O be a neighbornet of X such that $O(x)$ is open whenever $x \in X$. Then there is a point-finite open cover $\mathcal{G}(X)$ of X such that for each member $H \in \mathcal{G}(X)$ there is $x_H \in X$ such that $x_H \in H \subseteq O(x_H)$.*

Proof. Choose inductively a possibly transfinite sequence $\langle x_\alpha\rangle_\alpha$ of points in X such that $x_\alpha \in X \setminus \bigcup_{\beta<\alpha} O(x_\beta)$ as long as possible, say whenever $\alpha < \gamma$. Then

$\{O(x_\alpha) \setminus \bigcup_{\beta<\alpha} O(x_\beta) : \alpha < \gamma\}$ is a scattered partition of X. According to [8, Theorem 6.3] a topological space X is hereditarily metacompact if and only if every scattered partition of X has a point-finite open expansion.

Hence there is a point-finite open collection $\{P_\alpha : \alpha < \gamma\}$ of X such that $[O(x_\alpha) \setminus \bigcup_{\beta<\alpha} O(x_\beta)] \subseteq P_\alpha$ whenever $\alpha < \gamma$. It follows that $\bigcup\{P_\alpha \cap O(x_\alpha) : \alpha < \gamma\} = X$ and $x_\alpha \in P_\alpha \cap O(x_\alpha) \subseteq O(x_\alpha)$ whenever $\alpha < \gamma$. Therefore we can set $\mathcal{G}(X) = \{P_\alpha \cap O(x_\alpha) : \alpha < \gamma\}$. □

Theorem 2.2. *Let O be a neighbornet of a regular hereditarily metacompact space X. If player II has a winning strategy in the game $G(X)$ described above, then there exists a point-finite open family $\mathcal{U}$ of X such that $D\mathcal{U} \subseteq O^3$.*

Proof. Without loss of generality we suppose that $O(x)$ is open whenever $x \in X$. Inductively for each $n \in \omega$ we shall define a point-finite open family $\mathcal{U}_n$ of X.

By Lemma 2.1 there is a point-finite open cover $\mathcal{U}_0$ of X which has the property that for each $U_0 \in \mathcal{U}_0$ there is some point $p_{U_0} \in X$ such that $p_{U_0} \in U_0 \subseteq O(p_{U_0})$.

Let $n \in \omega$. Suppose that we have defined the point-finite open family $\mathcal{U}_{k+1}$ as the union of families $\mathcal{G}_{k+1}(U)$ where U runs through a subfamily of $\mathcal{U}_k$ whenever $k < n$. (In the following we distinguish between members in different families $\mathcal{G}_{k+1}(U)$ or in $\mathcal{U}_0$ that denote the same set; in this way each member arises on a well-defined level of the construction and each member V belonging to the level $\mathcal{U}_{k+1}$ has a unique element U in the level $\mathcal{U}_k$ preceding it in the sense that $V \in \mathcal{G}_{k+1}(U)$.)

Furthermore suppose that each U_n where $U_{k+1} \in \mathcal{G}_{k+1}(U_k)$ whenever $k < n$ determines the sequence $(U_0 \setminus \overline{O^{-1}(p_{U_0})}, p_{U_1}, W_0, U_1 \setminus \overline{O^{-1}(p_{U_1})}, p_{U_2}, W_1, \dots, U_{n-1} \setminus \overline{O^{-1}(p_{U_{n-1}})}, p_{U_n}, W_{n-1})$ which describes the moves $k < n$ of a well-defined instance of the game $G(X)$ in the sense that

(1) player I has used $U_k \setminus \overline{O^{-1}(p_{U_k})}$ and some well-defined point $p_{U_{k+1}}$ of $U_k \setminus \overline{O^{-1}(p_{U_k})}$ in his k^{th}-move whenever $k < n$,

and (2) player II has chosen the set W_k according to his winning strategy in his k^{th} move whenever $k < n$.

In particular note that each W_k is determined by the preceding moves of player I and that each $U_k \setminus \overline{O^{-1}(p_{U_k})} \neq \varnothing$ whenever $k < n$. Call a member U_n of $\mathcal{U}_n$ *suitable* (in $\mathcal{U}_n$) if $U_n \not\subseteq \overline{O^{-1}(p_{U_n})}$.

Assume now that U_n is a suitable member of $\mathcal{U}_n$. Suppose that player I continues the beginning of the game $G(X)$ associated with U_n by choosing $U_n \setminus \overline{O^{-1}(p_{U_n})}$ and any $x \in U_n \setminus \overline{O^{-1}(p_{U_n})}$ in his n^{th} move. Then player II finds $W_n(\dots, U_n, x)$ according to his winning strategy such that $x \in W_n(\dots, U_n, x) \subseteq U_n \setminus \overline{O^{-1}(p_{U_n})}$.

By hereditary metacompactness and regularity of X there exists a point-finite open cover $\mathcal{V}_{n+1}(U_n)$ of $U_n \setminus \overline{O^{-1}(p_{U_n})}$ such that the closures of its members are all contained in $U_n \setminus \overline{O^{-1}(p_{U_n})}$. Consider the neighbornet of the subspace $U_n \setminus \overline{O^{-1}(p_{U_n})}$ of X determined by the neighborhoods $W_n(\dots, U_n, x) \cap$

$\bigcap\{E \in \mathcal{V}_{n+1}(U_n) : x \in E\} \cap O(x)$ whenever $x \in U_n \setminus \overline{O^{-1}(p_{U_n})}$. By Lemma 2.1 there exists a point-finite open cover $\mathcal{G}_{n+1}(U_n)$ of $U_n \setminus \overline{O^{-1}(p_{U_n})}$ such that for each $U_{n+1} \in \mathcal{G}_{n+1}(U_n)$ there is some point $p_{U_{n+1}} \in U_n \setminus \overline{O^{-1}(p_{U_n})}$ satisfying $p_{U_{n+1}} \in U_{n+1} \subseteq W_n(\ldots, U_n, p_{U_{n+1}}) \cap \bigcap\{E \in \mathcal{V}_{n+1}(U_n) : p_{U_{n+1}} \in E\} \cap O(p_{U_{n+1}})$.

Set $\mathcal{U}_{n+1} = \bigcup\{\mathcal{G}_{n+1}(U_n) : U_n$ is a suitable member of $\mathcal{U}_n\}$. Note that $\mathcal{U}_{n+1}$ is a point-finite open family of X. Observe also that for each suitable U_n of $\mathcal{U}_n$ the closures of all members of $\mathcal{G}_{n+1}(U_n)$ are contained in $U_n \setminus \overline{O^{-1}(p_{U_n})}$ because $\mathcal{V}_{n+1}(U_n)$ had the latter property.

Furthermore by the construction above it is readily checked that for each member $U_{n+1} \in \mathcal{G}_{n+1}(U_n)$ we have constructed the moves $k < n+1$ of the instance of the game $G(X)$ associated with U_{n+1} by adding to the (unique) sequence of moves associated with U_n the n^{th} moves $(U_n \setminus \overline{O^{-1}(p_{U_n})}, p_{U_{n+1}}, W_n)$ of player I and player II, respectively, where $W_n = W_n(\ldots, U_n, p_{U_{n+1}})$.

Claim 2.3. *There exists a point-finite family $\mathcal{U}$ of open sets of X such that the family $\mathcal{H} = \{U \cap \overline{O^{-1}(p_U)} : U \in \mathcal{U}\}$ covers X.*

We shall show that our claim holds for the family $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$:

Suppose that for some $x \in X$ there are infinitely many sets in $\mathcal{U}$ containing x. Consider the family $\mathcal{S}$ of all sets in $\mathcal{U}$ containing x. Since each family $\mathcal{U}_n$ is point-finite, we conclude by König's Lemma [9] and the definition of the families $\mathcal{U}_{n+1}$ that in $\mathcal{S}$ there exists a sequence $\langle U_n\rangle_{n\in\omega}$ such that for each $n \in \omega$, $U_{n+1} \in \mathcal{G}_{n+1}(U_n)$. We shall show next that such a sequence does not exist.

Note first that the sequence $\langle U_n \setminus \overline{O^{-1}(p_{U_n})}, p_{U_{n+1}}\rangle_{n\in\omega}$ yields the moves of player I in an instance of the game $G(X)$ where player II uses his winning strategy to find the sets $W_n = W_n(\ldots, U_n, p_{U_{n+1}})$ whenever $n \in \omega$.

By the construction of the family $\mathcal{G}_{n+1}(U_n)$, $U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_n \setminus \overline{O^{-1}(p_{U_n})}$ and $p_{U_{n+1}} \in U_{n+1} \subseteq W_n(\ldots, U_n, p_{U_{n+1}}) \subseteq U_n \setminus \overline{O^{-1}(p_{U_n})}$ whenever $n \in \omega$.

Since $x \in \cap_{n\in\omega} U_n$ and thus $x \in \cap_{n\in\omega} W_n(\ldots, U_n, p_{U_{n+1}})$, we conclude that $\langle p_{U_n}\rangle_{n\in\omega}$ has a cluster point z in X. Thus $p_{U_n} \in O(z)$ for infinitely many $n \in \omega$. But also $z \in \overline{U_{n+1}}$ whenever $n \in \omega$, because for each $n \in \omega$ a tail of the sequence $\langle p_{U_n}\rangle_{n\in\omega}$ is contained in U_{n+1}. Since $\overline{U_{n+1}} \cap \overline{O^{-1}(p_{U_n})} = \varnothing$ whenever $n \in \omega$, we see that $z \notin \overline{O^{-1}(p_{U_n})}$ whenever $n \in \omega$ —a contradiction. We conclude that the family $\mathcal{U}$ is point-finite.

Suppose that some point $x \in X$ is not contained in any set $U \cap \overline{O^{-1}(p_U)}$ where $U \in \mathcal{U}$. Since $\mathcal{U}_0$ is a cover of X, there exists $U_0 \in \mathcal{U}_0$ such that $x \in U_0$. Suppose that $n \in \omega$ and sets U_k $(k \leq n)$ have inductively been defined such that $x \in U_{k+1} \in \mathcal{G}_{k+1}(U_k)$ $(k < n)$. By our assumption, we have that $x \in U_n \setminus \overline{O^{-1}(p_{U_n})}$. In particular, U_n is suitable in $\mathcal{U}_n$. Since $\mathcal{G}_{n+1}(U_n)$ covers $U_n \setminus \overline{O^{-1}(p_{U_n})}$, there exists $U_{n+1} \in \mathcal{G}_{n+1}(U_n)$ such that $x \in U_{n+1}$. This concludes the induction. Of course, $x \in \cap_{n\in\omega} U_n$. But as we just noted above such a sequence $\langle U_n\rangle_{n\in\omega}$ cannot exist. Hence $\mathcal{H}$ is a cover of X.

Finally we show that $D\mathcal{U} \subseteq O^3$. Let $x \in X$. By the claim verified above there exists $U \in \mathcal{U}$ such that $x \in U \cap \overline{O^{-1}(p_U)}$. Furthermore, we see that $D\mathcal{U}(x) = \bigcap\{V \in \mathcal{U} : x \in V\} \subseteq U \subseteq O(p_U)$ by the selection of the sets U belonging to $\mathcal{U}$. Since we have that $x \in \overline{O^{-1}(p_U)}$, there exists a point $y \in O(x) \cap O^{-1}(p_U)$. We now conclude that $y \in O(x)$ and $p_U \in O(y)$. It follows that $p_U \in O^2(x)$ and, furthermore, that $O(p_U) \subseteq O^3(x)$. As a consequence, we see that $D\mathcal{U}(x) \subseteq O(p_U) \subseteq O^3(x)$, which confirms the assertion. □

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Hans-Peter A. Künzi
Dept. Math. and Appl. Math., University of Cape Town, Rondebosch 7701, South Africa

E-mail address: kunzi@maths.uct.ac.za