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Extensions of closure spaces

D. Deses*, A. De Groot-Van der Voorde, E. Lowen-Colebunders

Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

Abstract. A closure space X is a set endowed with a closure operator $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$, satisfying the usual topological axioms, except finite additivity. A T_1 closure extension Y of a closure space X induces a structure γ on X satisfying the smallness axioms introduced by H. Herrlich [?], except the one on finite unions of collections. We'll use the word seminearness for a smallness structure of this type, i.e. satisfying the conditions (S1),(S2),(S3) and (S5) from [?]. In this paper we show that every T_1 seminearness structure γ on X can in fact be induced by a T_1 closure extension. This result is quite different from its topological counterpart which was treated by S.A. Naimpally and J.H.M. Whitfield in [?]. Also in the topological setting the existence of (strict) extensions satisfying higher separation conditions such as T_2 and T_3 has been completely characterized by means of concreteness, separatedness and regularity [?]. In the closure setting these conditions will appear to be too weak to ensure the existence of suitable (strict) extensions. In this paper we introduce stronger alternatives in order to present internal characterizations of the existence of (strict) T_2 or strict regular closure extensions.

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1. Introduction.

The structures we will be dealing with, can be defined in various equivalent ways, from which we shall use frequently two particular descriptions, namely by small collections and by uniform covers.

*The first author is 'aspirant' of the F.W.O.-Vlaanderen.

1.1. Let X be a set. A nonempty collection $\mathcal{A} \subset \mathcal{P}(X)$, not containing $\varnothing$ is said to be a *stack* if $A \in \mathcal{A}$ whenever there exists $B \in \mathcal{A}$ with $B \subset A$. If $\mathcal{A} \subset \mathcal{P}(X)$ is an arbitrary collection of nonempty subsets then we put

$$stack\ \mathcal{A} = \{A \subset X | \exists B \in \mathcal{A} : B \subset A\}$$

and

$$sec\ \mathcal{A} = \{A \subset X | \forall B \in \mathcal{A} : A \cap B \neq \varnothing\}$$

We write $\dot{x}$ for $stack\ \{\{x\}\}$. In [?] H. Herrlich considered the following smallness axioms, which can be expressed in terms of stacks in the following way. For $\gamma \subset \mathcal{P}^2(X)$, a collection of stacks, consider the conditions

(S1) If $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A} \in \gamma$ then $\mathcal{B} \in \gamma$
(S2) $\forall x \in X : \dot{x} \in \gamma$
(S3) $\gamma \neq \mathcal{P}^2(X)$
(S4) If $(\mathcal{A} \cup \mathcal{B}) \in \gamma$ then $\mathcal{A} \in \gamma$ or $\mathcal{B} \in \gamma$
(S5) If $sec\ \{cl\ A | A \in \mathcal{A}\} \in \gamma$ then $sec\ \mathcal{A} \in \gamma$ where $cl\ A = \{x \in X | sec\ \{A, \{x\}\} \in \gamma\}$

A structure γ satisfying (S1),(S2),(S3) is called a prenearness structure, if (S4) is added γ is a merotopic structure and if γ satisfies all five of the conditions then it is a nearness structure. We will not be dealing with axiom (S4) but we will assume that γ satisfies (S1), (S2), (S3) and (S5). Then γ is called a *seminearness structure* and (X, γ) is a *seminearness space*. As in [?] a function $f : (X, \gamma) \to (X', \gamma')$ between seminearness spaces is said to be *uniformly continuous* if it preserves smallness in the sense that

$$\mathcal{A} \in \gamma \Rightarrow stack\ \{f(A) | A \in \mathcal{A}\} \in \gamma'$$

Seminearness spaces and uniformly continuous maps form a topological construct in the sense of [?]. We refer to the original papers [?], [?] for a systematic study of prenearness and nearness spaces. On the latter a selfcontained textbook "Uniforme Raüme" appeared [?]. Another textbook by G. Preuss [?] also contains an introduction to nearness spaces and to some of the more general structures such as prenearness and merotopic structures. In [?] however the latter are called seminearness spaces, so that the terminology used in [?] differs from the one we use here.

1.2. In [?] an equivalent way of describing the structure was presented in terms of uniform covers.

If X is a set then the following conditions on $\mu \subset \mathcal{P}^2(X)$ are considered

(U1) If $\mathcal{U} \prec \mathcal{V}$ and $\mathcal{U} \in \mu$ then $\mathcal{V} \in \mu$ (where $\prec$ denotes the classical refinement relation)
(U2) If $\mathcal{U} \in \mu$ then $\mathcal{U}$ is a cover of X
(U3) $\varnothing \neq \mu \neq \mathcal{P}^2(X)$
(U4) If $\mathcal{U} \in \mu$ and $\mathcal{V} \in \mu$ then $\{U \cap V | U \in \mathcal{U}, V \in \mathcal{V}\} \in \mu$
(U5) If $\mathcal{U} \in \mu$ then $\{int_\mu U | U \in \mathcal{U}\} \in \mu$ where $int_\mu U = \{x \in X | \{U, X - \{x\}\} \in \mu\}$

The covers in μ are called uniform covers. In our setting we will not be dealing with (U4), so our covering structures μ satisfy (U1), (U2), (U3) and

(U5). (X, μ) then forms an equivalent way for the description of a seminearness space and the translation between (X, γ) and (X, μ) is as usual:

$$\mathcal{U} \in \mu \iff \forall \mathcal{A} \in \gamma : \mathcal{U} \cap \mathcal{A} \neq \varnothing$$

$$\mathcal{A} \in \gamma \iff \forall \mathcal{U} \in \mu : \mathcal{U} \cap \mathcal{A} \neq \varnothing$$

1.3. Some of the examples we will construct in the last section of the paper, satisfy even stronger conditions than the seminearness axioms. A uniform space, described in terms of covers satisfies (U1), (U2), (U3), (U4) and the condition (U5'), saying that every uniform cover has a uniform star refinement, which is in fact stronger than (U5). If we leave out (U4), as we did before, and retain (U1), (U2), (U3) and (U5') then we still have a (covering) seminearness space. In this case we will say that the seminearness space is a uniform seminearness space . A special case of this situation is the following. Let X be a set and let

$$\{\mathcal{U}_i | i \in I\}$$

be any collection of partitions of X then

$$\mathcal{U} \in \mu \iff \exists i \in I : \mathcal{U}_i \prec \mathcal{U}$$

defines a (covering) uniform seminearness structure on X. It is said to be zero dimensional since it is generated by a collection of partitions. These structures are investigated in more detail in [**?**].

1.4. A closure space $(X, \mathcal{C})$ is a pair, where X is a set and $\mathcal{C}$ is a subset of the power set $\mathcal{P}(X)$ satisfying the conditions that X belongs to $\mathcal{C}$ and that $\mathcal{C}$ is closed for arbitrary unions. The sets in $\mathcal{C}$ are called open sets. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ between closure spaces $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ is said to be continuous if $f^{-1}(D) \in \mathcal{C}$ whenever $D \in \mathcal{D}$. **Cl** is the construct with closure spaces as objects and continuous maps as morphisms. Some isomorphic descriptions of **Cl** are often used f.i. by giving the collection of all closed sets (the so called Moore family [**?**]) where, as usual, the closed sets are the complements of the open ones and continuity is defined accordingly. Another isomorphic description is obtained by means of a closure operator [**?**]. The closure operation $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ associated with a closure space $(X, \mathcal{C})$ is defined in the usual way by $x \in cl\ A \iff (\forall C \in \mathcal{C} : x \in C \Rightarrow C \cap A \neq \varnothing)$ where $A \subset X$ and $x \in X$. This closure need not be finitely additive, but it does satisfy the conditions $cl\ \varnothing = \varnothing, (A \subset B \Rightarrow cl\ A \subset cl\ B), A \subset cl\ A$ and $cl(cl\ A) = cl\ A$ whenever A and B are subsets of X. Continuity is then characterized in the usual way. Finally closure spaces can also be equivalently described by means of neighborhood collections of the points. These neighborhood collections satisfy the usual axioms, except for the fact that the collections need not to be filters. So in a closure space the neighborhood collection $\mathcal{V}(x)$ of a point x is a stack, where every $V \in \mathcal{V}(x)$ contains x and $\mathcal{V}(x)$ satisfies the open kernel condition. In the sequel we will just write X for a closure space and we'll choose the most convenient form for its explicit structure.

Motivations for considering closure spaces can be found in several applications. We refer to [?] and [?] for applications in geometry, to [?] for applications in lattice theory, to [?], [?] and [?] for the use of closures in the development of representations of physical systems, to [?] for the use in social sciences and to [?] for applications in the context of knowledge representation. The introduction of [?] contains some more details on motivation.

A closure space satisfies the R_0 *symmetry axiom* if $x \in cl\ \{y\} \iff y \in cl\ \{x\}$, $\forall x, y \in X$ and it satisfies T_1 if $\{x\}$ is closed for every $x \in X$. If (X, γ) is a seminearness space then the closure defined in paragraph 1.1 by

$$cl\ A = \{x \in X | sec\ \{A, \{x\}\} \in \gamma\}$$

is an R_0 closure in our sense. This closure is the *underlying closure* of (X, γ) and we also say that it is compatible with (X, γ). Whenever we consider neighborhood collections $\mathcal{V}_\gamma(x)$, convergence or open sets for a seminearness space (X, γ), we are in fact referring to the underlying closure. As for nearness spaces we have in this more general context that the neighborhood collections $\mathcal{V}_\gamma(x)$ are minimal small stacks (where minimality refers to the inclusion order). Next we further illustrate the relation between R_0 closure spaces and seminearness spaces.

1.5. Let Y be an R_0 closure space and define a seminearness structure by

$$\mathcal{A} \in \gamma \iff \exists y \in Y : \mathcal{V}(y) \subset \mathcal{A}$$

Remark that the underlying closure of the seminearness γ coincides with the given closure on Y. The construct of R_0 closure spaces is bicoreflectively embedded in the construct of seminearness spaces, cfr. [?], [?].

1.6. Let Y be an R_0 closure space and let X be a subset of Y. The closure structure of Y induces a seminearness structure on X as follows

$$\mathcal{A} \in \gamma \iff \exists y \in Y : \mathcal{V}(y) \subset stack_Y \mathcal{A}$$

The underlying closure of γ on X coincides with the closure structure induced by Y on X.

If X is dense in Y ($cl_Y X = Y$) then Y is said to be a *closure extension* of X and we say that (X, γ) is *induced by the extension* Y of X. X is said to be *strictly dense* in Y if $\{cl_Y B | B \subset X\}$ is a base for the closed subsets of Y, in the sense that every closed set of Y can be obtained by intersecting sets from the base. In that case Y is said to be a *strict extension* of X and (X, γ) is said to be *induced by a strict extension.*

The meaning of 1.6 is that, given an R_0 closure extension Y of a closure space X, a seminearness structure γ is induced on X which is compatible with the given closure on X.

The first question we will be dealing with in this paper is, whether every seminearness γ compatible with X as a closure space, can be induced by some R_0 closure extension Y of X.

The parallel question in the setting of topological spaces is whether every compatible nearness space can be induced by some R_0 topological extension. This question was answered negatively by S.A. Naimpally and J.H.M. Whitfield in [**?**].

A thorough study on extensions of topological spaces was later carried out by H.L. Bentley and H. Herrlich in [**?**], in particular giving internal characterizations for nearness spaces to be induced by T_1, T_2 or T_3 (strict) extensions.

In this paper we'll deal with the closure counterparts of such questions.

2. Extensions.

In this section, starting from a seminearness space (X, γ) we construct two types of enlargements, one type are the so called "loose" enlargements and the other type is a strict one.

2.1. **Construction of a loose enlargement.** Let (X, γ) be a seminearness space and let $\{y_{\mathcal{A}} | \mathcal{A} \in \alpha\}$ be a collection of points, not belonging to X and in one to one correspondence to a collection α of nonconvergent small stacks with an open base. Let $X' = X \cup \{y_{\mathcal{A}} | \mathcal{A} \in \alpha\}$. On X' we define a closure structure cl' by determining the neighborhood collections of the points as follows:

$$\mathcal{V}'(x) = stack_{X'} \mathcal{V}_\gamma(x) \text{ for } x \in X$$

$$\mathcal{V}'(y_{\mathcal{A}}) = stack_{X'} \mathcal{A} \cap \dot{y}_{\mathcal{A}} \text{ for } \mathcal{A} \in \alpha$$

Clearly (X', cl') is an R_0 closure space of which X is a dense subset. Moreover if $\mathcal{D}$ is a stack on X and $stack_{X'}\mathcal{D}$ converges in X', then $\mathcal{D}$ is small in (X, γ). It is clear that in order to obtain an extension of (X, γ) in the sense of 1.6 the condition $\mathcal{D} \in \gamma \Rightarrow \exists \mathcal{A} \in \alpha : \mathcal{A} \subset \mathcal{D}$ has to be fulfilled.

The following proposition is relevant in this respect since it shows that in fact openbased stacks determine the structure.

Proposition 2.1. *If (X, γ) is a seminearness space then for every $\mathcal{A} \in \gamma$ there exists a $\mathcal{B} \in \gamma$ such that $\mathcal{B}$ has an open base and such that $\mathcal{B} \subset \mathcal{A}$.*

Proof. For $\mathcal{A} \in \gamma$ let $\mathcal{B} = stack\ \{B \subset X | B \text{ open}, B \in \mathcal{A}\}$. If $\mathcal{U} \in \mu$ then so is $\{int\ U | U \in \mathcal{U}\}$. So finally $\mathcal{B} \cap \mathcal{U} \neq \varnothing$. □

Proposition 2.2. *Let (X, γ) be a seminearness space and let*

$$\alpha = \{\mathcal{A} \in \mathcal{P}^2(X) | \mathcal{A} \textit{ is a small openbased nonconvergent stack}\}$$

Then (X', γ') is an R_0 closure extension of (X, γ).

We conclude from this fact that every seminearness space can be induced by an R_0 closure extension.

Remark that by exactly the same construction one has that every T_1 seminearness space is induced by some T_1 extension. Remark also that these results deviate from their well known topological counterparts, cfr. [**?**], [**?**], [**?**].

In general, given a T_1 space (X, γ) there can be many different T_1 extensions inducing γ. On the other hand, as we will see, strict T_1 extensions need not exist. However, if there exist T_1 strict extensions, then they are essentially unique. The reason for this is explained in the next results on minimal small stacks, where again minimality refers to the inclusion order on stacks.

The following result is quite parallel to its topological counterpart developed in [**?**].

Proposition 2.3.

(1) *If (X, γ) is a seminearness space induced by an R_0 extension Y of X, then every minimal small stack is a trace $\mathcal{V}(y)|_X$ for some $y \in Y$.*

(2) *If (X, γ) is a seminearness space induced by a strict R_0 extension Y of X then*

$$\{\mathcal{V}(y)|_X | y \in Y\}$$

is the collection of all minimal small stacks.

Proof.

(1) If $\mathcal{M}$ is minimal small and $stack_Y \mathcal{M}$ converges to $y \in Y$ then $\mathcal{V}(y)|_X \subset \mathcal{M}$ and hence $\mathcal{V}(y)|_X = \mathcal{M}$.

(2) Let $y \in Y$ be some point of a strict extension Y, and suppose that $\mathcal{A} \subset \mathcal{V}(y)|_X$ and that $\mathcal{A}$ is small. Suppose that $stack_Y \mathcal{A}$ converges to z, i.e. $\mathcal{V}(z)|_X \subset \mathcal{A}$. It follows that $z \in cl_Y\{y\}$. Indeed otherwise there would exist a subset $B \subset X$ such that $y \in cl_Y\ B$ and $z \notin cl_Y\ B$. And this is impossible. Hence we can conclude that $\mathcal{V}(y) = \mathcal{V}(z)$ and so $\mathcal{A} = \mathcal{V}(y)|_X$.

$\square$

Corollary 2.4. *If (X, γ) is a seminearness space and Y is a strict T_1 extension then the points of Y are in one to one correspondence to the minimal small stacks in γ.*

Also the following construction is quite similar to its topological counterpart [**?**],[**?**].

2.2. **Construction of a strict enlargement.** Let (X, γ) be a seminearness space and let

$$\hat{X} = X \cup \{y_{\mathcal{M}} | \mathcal{M} \text{ nonconvergent minimal small}\}$$

where again different points $y_{\mathcal{M}}$ are chosen to be outside of X and in one to one correspondence with the minimal small nonconvergent stacks.

For $A \subset X$ put

$$O(A) = int\ A \cup \{y_{\mathcal{M}} | A \in \mathcal{M}\}$$

and let $\hat{cl}$ on $\hat{X}$ be the closure having as an open base $\{O(A) | A \subset X\}$.

Remark that $\{\hat{cl}\ K | K \subset X\}$ is a base for the closed sets where $\hat{cl}\ K = cl\ K \cup \{y_{\mathcal{M}} | K \in sec\ \mathcal{M}\}$.

Clearly $(\hat{X}, \hat{cl})$ is an R_0 closure space and it contains X as a strictly dense subset. Moreover if $\mathcal{D}$ is a stack on X and $stack_{\hat{X}} \mathcal{D}$ converges in $\hat{X}$ then: Either $\mathcal{V}_{\hat{X}}(y_{\mathcal{M}}) \subset stack\ \mathcal{D}$ for $\mathcal{M}$ minimal small and not convergent, then we have $\mathcal{M} \subset \mathcal{D}$. Or $\mathcal{V}_{\hat{X}}(x) \subset$ *stack* $\mathcal{D}$ and then $\mathcal{V}_X(x) \subset \mathcal{D}$. So in any case $\mathcal{D} \in \gamma$.

In order to obtain an extension of (X, γ) we need to impose the following condition (cfr. [**?**]).

Definition 2.5. A seminearness space is concrete if the minimal small stacks determine the structure in the following sense

$$\forall \mathcal{A} \in \gamma : \exists \mathcal{M} \text{ minimal small } \mathcal{M} \subset \mathcal{A}$$

Proposition 2.6. (X, γ) *is induced by a strict* R_0 *extension if and only if it is a concrete seminearness space.*

Proof. If (X, γ) is a concrete seminearness space we make the construction developed in paragraph 2.2 and we prove that $(\hat{X}, \hat{cl})$ is an extension. So it remains to show that if $\mathcal{D}$ is a small stack on X it converges in $(\hat{X}, \hat{cl})$. Choose $\mathcal{M} \subset \mathcal{D}$ minimal small.
Either $\mathcal{M} = \mathcal{V}(x)$ for some $x \in X$ and then for $A \subset X$ with $x \in int\ A$ we have $int\ A \in \mathcal{D}$ and hence $O(A) \in \mathcal{D}$. So we have $\mathcal{V}_{\hat{X}}(x) \subset \mathcal{D}$.
Or $\mathcal{M}$ does not converge. Then we prove that

$$stack_{\hat{X}}\mathcal{D} \supset \mathcal{V}_{\hat{X}}(y_{\mathcal{M}})$$

Let $A \subset X$ such that $y_{\mathcal{M}} \in O(A)$, i.e. $A \in \mathcal{M}$. In view of proposition 2.1 the stack $\mathcal{M}$ has an open base. Then also $int\ A \in \mathcal{M}$ and finally $int\ A \in \mathcal{D}$. So again we can conclude that $O(A) \in stack_{\hat{X}}\mathcal{D}$.

Conversely, suppose that (X, γ) has a strict R_0 extension (Y, cl_Y). Let $\mathcal{D}$ be small in (X, γ) then $stack_Y\mathcal{D}$ converges to some $y \in Y$. Then clearly $\mathcal{D} \supset \mathcal{V}_Y(y)|_X$ and in view of proposition 2.3 we have that $\mathcal{V}_Y(y)|_X$ is a minimal small stack. Hence (X, γ) is concrete. $\square$

Remark that using exactly the same construction one has that (X, γ) is induced by a strict T_1 closure extension if and only if it is T_1 and concrete. Remark that if Y is a strict T_1 extension of (X, γ) then Y is unique up to an isomorphism leaving X pointwise fixed. It can easily be seen that the function $\phi : Y \to \hat{X}$ mapping $y \in Y$ to $y_{\mathcal{M}}$ with $\mathcal{M} = \mathcal{V}_Y(y)|_X$ if $y \notin X$ and mapping $x \in X$ to x, is bijective and satisfies $\phi(cl_Y B) = \hat{cl}\ B$, for every $B \subset X$. Therefore we also have $\phi(cl_Y Z) = \hat{cl}(\phi(Z))$ for every $Z \subset Y$.

The previous results on R_0 and T_1 strict extensions are completely analogous to their topological counterparts. In the next section, where higher separation is considered, the parallelism with the topological situation does not go through.

3. Separation and extensions.

In this section we introduce higher separation conditions for seminearness spaces. The notion "separatedness" was introduced in [**?**] in the setting of prenearness spaces and it proved to be very useful in the study of topological extensions. However, in our setting, in order to produce Hausdorff closure extensions, "separatedness" will no longer be strong enough. We briefly recall some definitions and results.

If (X,γ) is a seminearness space then a stack $\mathcal{A}$ is said to be *near* if *sec* $\mathcal{A}$ is small. For instance, if $\bigcap_{A\in\mathcal{A}} cl_\gamma A \neq \varnothing$ then $\mathcal{A}$ is near. A stack $\mathcal{A}$ is said to be *concentrated* if it is small and near. For example, the neighborhood collections in (X,γ) are concentrated. Small filters are also always concentrated.

Definition 3.1. [**?**] A seminearness space (X,γ) is separated if for every concentrated stack $\mathcal{A}$ also

$$\mathcal{B} = \{B \subset X | stack_X\{B\} \cup \mathcal{A} \text{ near}\}$$

is near.

The proof of the following proposition is similar to the one of proposition 10.5 in [**?**] and can be found in [**?**].

Proposition 3.2. *For a seminearness (X,γ) the following are equivalent*

(1) (X,γ) *is separated*

(2) *Every concentrated stack contains a unique minimal small stack*

Proposition 3.3. *If (X,γ) is separated, $\mathcal{M}$ is a minimal small concentrated stack and $\mathcal{M} \neq \mathcal{V}(x)$ then*

$$\exists A \in \mathcal{V}(x) : \exists M \in \mathcal{M} : M \cap A = \varnothing$$

Proof. If on the other hand every $A \in \mathcal{V}(x)$ intersects every $M \in \mathcal{M}$ then $\mathcal{V}(x) \cup \mathcal{M}$ would be concentrated and then $\mathcal{V}(x) = \mathcal{M}$ in view of the previous proposition. □

Corollary 3.4. *If (X,γ) is separated and T_1 then in the underlying closure, distinct points have disjoint neighborhoods. In particular a closure space (considered as a seminearness space) is separated and T_1 if and only if distinct points have disjoint neighborhoods. We use the label "Hausdorff" or T_2 for this property.*

The following conditions (i) and (ii) clearly are strengthening those formulated in proposition 3.3.

Proposition 3.5. *For a seminearness space (X,γ) the following are equivalent*

(i) *for $\mathcal{M}$ and $\mathcal{N}$ minimal small concentrated stacks, $\mathcal{M} \neq \mathcal{N}$ then*

$$\exists M \in \mathcal{M} : \exists N \in \mathcal{N} : N \cap M = \varnothing$$

(ii) *for $\mathcal{M}$ and $\mathcal{N}$ minimal small stacks, $\mathcal{M} \neq \mathcal{N}$ then*

$$\exists M \in \mathcal{M} : \exists N \in \mathcal{N} : N \cap M = \varnothing$$

Proof. That (i) implies (ii) follows from the observation that when $\mathcal{N}$ is small but not concentrated, then *sec* $\mathcal{N}$ is not small. So if $\mathcal{M}$ is small we have $\mathcal{M} \not\subset sec\, \mathcal{N}$. Therefore $\mathcal{M}$ and $\mathcal{N}$ contain disjoint sets. □

Definition 3.6. A seminearness space (X,γ) satisfies (S) if it fulfills one (and hence both) of the conditions formulated in proposition 3.5

Remark that if (X,γ) satisfies (S) and $\mathcal{M}$ and $\mathcal{N}$ are different minimal small stacks, $M \in \mathcal{M}$ and $N \in \mathcal{N}$ satisfying (ii) can be taken to be disjoint and open. Hence in that case we have

$$\forall x \in X : M \notin \mathcal{V}(x) \text{ or } N \notin \mathcal{V}(x)$$

Next we generalize these ideas in order to introduce an even stronger separation condition. Let (X,γ) be a seminearness space and let

$$\Sigma = \{\dot{x} | x \in X\} \cup \{\mathcal{M} | \text{ minimal small nonconvergent}\}$$

Definition 3.7. Subsets D and B are said to be γ-disjoint if

$$\forall \mathcal{P} \in \Sigma : D \notin \mathcal{P} \text{ or } B \notin \mathcal{P}$$

Clearly D and B are γ-disjoint if and only if

(i) $D \cap B = \varnothing$
(ii) for every $\mathcal{M}$ minimal small nonconvergent stack, $D \notin \mathcal{M}$ or $B \notin \mathcal{M}$.

Definition 3.8. A seminearness space (X,γ) is said to satisfy (T) if minimal small stacks $\mathcal{M} \neq \mathcal{N}$ contain sets $M \in \mathcal{M}$ and $N \in \mathcal{N}$ that are γ-disjoint.

Conditions (S) and (T) will play an important role in the investigation of Hausdorff closure extensions. First we discuss the relation between the various separation conditions.

Proposition 3.9.

(1) *In a seminearness space we have*

$$\text{(T)} \Rightarrow \text{(S)}$$

(2) *In a concrete seminearness space we have*

$$\text{(T)} \Rightarrow \text{(S)} \Rightarrow \textit{separated}$$

Proof.

(1) Let $\mathcal{M}$ and $\mathcal{N}$ be (concentrated) minimal small, choose γ-disjoint sets $M \in \mathcal{M}$ and $N \in \mathcal{N}$. Then we have $M \cap N = \varnothing$.
(2) (X,γ) is concrete and satisfies (S). Let $\mathcal{A}$ be concentrated and let $\mathcal{M}$ be a minimal small stack, $\mathcal{M} \subset \mathcal{A}$. Consider *sec* $\mathcal{A}$ which is small and a minimal small stack $\mathcal{N} \subset$ *sec* $\mathcal{A}$. It follows that $\mathcal{M} = \mathcal{N}$. Finally by proposition 3.2 the space (X,γ) is separated.

□

From the proof of (2) we immediately have the following.

Corollary 3.10. *If (X,γ) is concrete and satisfies (S) then every minimal small concentrated stack $\mathcal{M}$ satisfies*

$$\mathcal{M} \subset \mathit{sec}\ \mathcal{M}$$

i.e. $\mathcal{M}$ is a linked system in the sense of [**?**].

No other implications between the conditions (T), (S) and "separated" are true in general (except those obtained by transitivity). We refer to section **??** for the summarizing diagrams. Example **??** provides a separated concrete seminearness space which does not satisfy (S). Example **??** is a concrete seminearness space satisfying (S) but not (T). Example 3 in [**?**] satisfies (T) but it is not separated (and not concrete). Remark that this example moreover is a nearness space.

In the case of a nearness space however some other implications become true.

Proposition 3.11.

(1) *For a nearness space we have*

$$\textit{separated} \Rightarrow \textit{(S)} \Leftrightarrow \textit{(T)}$$

(2) *For a concrete nearness space we have*

$$\textit{separated} \Leftrightarrow \textit{(S)} \Leftrightarrow \textit{(T)}$$

Proof.

(1) Let (X, γ) be a separated nearness space. If $\mathcal{M}$ and $\mathcal{N}$ are minimal small concentrated stacks then $\mathcal{M}$ and $\mathcal{N}$ are filters [**?**], [**?**]. If every $M \in \mathcal{M}$ would intersect every $N \in \mathcal{N}$ then *stack* $\{M \cap N | M \in \mathcal{M}, N \in \mathcal{N}\}$ would be a filter too and so it would be concentrated. Hence $\mathcal{M} = \mathcal{N}$. So (X, γ) satisfies (S).

Next suppose (X, γ) is a nearness space satisfying (S). Let $\mathcal{M}$ and $\mathcal{N}$ be minimal small. Since $\mathcal{M}$ and $\mathcal{N}$ are filters, both are concentrated and so (S) implies that $\exists M \in \mathcal{M}, \exists N \in \mathcal{N} : M \cap N = \varnothing$. Now every minimal small $\mathcal{P}$ is a filter too, so M and N can not be both in $\mathcal{P}$.

(2) Follows immediately from (1) in combination with proposition 3.9 (2).

□

That (S) $\not\Rightarrow$ separated, even in the nearness case follows from example 3 in [**?**].

Next we discuss the impact of the separation conditions on extensions.

Proposition 3.12. *A seminearness space is induced by a Hausdorff closure extension if and only if it is separated, T_1 and satisfies (S).*

Proof. Suppose (Y, cl_Y) is a Hausdorff extension of (X, γ) then if $\mathcal{M}$ and $\mathcal{N}$ are different minimal small stacks, by proposition 2.3 there are y and z in Y such that $\mathcal{M} = \mathcal{V}_Y(y)|_X$ and $\mathcal{N} = \mathcal{V}_Y(z)|_X$. It follows that $y \neq z$ and then disjoint sets can be chosen using the Hausdorff property of (Y, cl_Y). So (X, γ) satisfies (S). That (X, γ) is separated and T_1 follows from the fact that these properties are hereditary.

Conversely, suppose (X, γ) is separated, T_1 and satisfies (S). Consider

$$\begin{aligned} \alpha &= \{\mathcal{M} | \mathcal{M} \text{ minimal small, concentrated, not convergent}\} \\ &\cup \{\mathcal{A} | \mathcal{A} \text{ small, openbased, not concentrated, not convergent}\} \end{aligned}$$

and construct the loose enlargement on

$$X' = X \cup \{y_{\mathcal{P}} | \mathcal{P} \in \alpha\}$$

In view of paragraph 2.1, in order to conclude that (X', cl') is an extension, it suffices to prove that if $\mathcal{D} \in \gamma$ is openbased and not convergent in X then $stack_{X'}\mathcal{D}$ converges to some point in X'. Either $\mathcal{D}$ is concentrated and then $\mathcal{D} \supset \mathcal{M}$ for a unique minimal small concentrated stack $\mathcal{M}$. In this case $stack_{X'}\mathcal{D} \supset stack_{X'}\mathcal{M} \cap \dot{y}_{\mathcal{M}}$ for $y_{\mathcal{M}} \in X'$. Or $\mathcal{D}$ is not concentrated and then $stack_{X'}\mathcal{D}$ converges to $y_{\mathcal{D}} \in X'$.

Finally we prove that this loose extension is a Hausdorff closure space. Consider two different points in X'. If each of them corresponds to a concentrated minimal small stack, then (S) implies that disjoint neighborhoods can be found. If at least one of the points corresponds to some openbased small stack $\mathcal{A}$ which is not concentrated and not convergent, then *sec* $\mathcal{A}$ is not small and then the argument developed in the proof of proposition 3.5 (ii) can be used to obtain disjoint neighborhoods.

□

Proposition 3.13. *A seminearness space (X, γ) is induced by a strict Hausdorff closure extension if and only if (X, γ) is concrete, T_1 and satisfies (T).*

Proof. Suppose (X, γ) is induced by a strict Hausdorff closure extension (Y, cl_Y). Then by proposition 2.6 we already know that (X, γ) is T_1 and concrete. In paragraph 2.2 we remarked that (Y, cl_Y) is unique up to isomorphism. So we have that $(\hat{X}, \hat{cl})$ is Hausdorff. In order to prove (T), let $\mathcal{M} \neq \mathcal{N}$ be minimal small stacks. By corollary 2.4 they correspond to different points y and z in $\hat{X}$. Consider disjoint basic neighborhoods $O(A)$ and $O(B)$ of y and z, respectively where $A, B \subset X$. Then clearly *int* $A \in \mathcal{M}$ and *int* $B \in \mathcal{N}$, so *int* A and *int* B are γ-disjoint.

Conversely, suppose (X, γ) is T_1, concrete and satisfies (T). Then we already know that $(\hat{X}, \hat{cl})$ is a strict T_1 closure extension. In order to prove that $(\hat{X}, \hat{cl})$ is Hausdorff let y and z be different points. These points correspond to different minimal small stacks in (X, γ) which therefore contain γ-disjoint sets A and B. It follows that $O(A)$ and $O(B)$ are disjoint and belong to the respective neighborhood collections $\mathcal{V}_Y(y)$ and $\mathcal{V}_Y(z)$. □

Remark that again our situation differs fundamentally from its topological counterpart. The existence of a topological Hausdorff extension inducing (X, γ) implies the existence of a strict topological Hausdorff extension [**?**]. Whereas here for the existence of a Hausdorff closure extension only separated and (S) are needed on (X, γ), and for the existence of a strict Hausdorff closure extension concreteness and (T) are involved and as announced in 3.10 these conditions are not equivalent. In section **??** examples are listed showing that even in the concrete case (S) plus separated does not imply (T).

4. Regularity and extensions.

Regularity was introduced in the context of prenearness spaces in [**?**]. So we can apply the definition to our setting of seminearness spaces.

If A and B are subsets in (X, γ) one puts

$$B <_\mu A \iff \{A, X - B\} \in \mu$$

where μ is the covering structure associated with γ. For a stack $\mathcal{A}$ one puts

$$\mathcal{A}_{<_\mu} = \{A | \exists B \in \mathcal{A} : B <_\mu A\}$$

and for covers $\mathcal{U}$ and $\mathcal{V}$ one writes

$$\mathcal{V} <_\mu \mathcal{U} \iff \forall V \in \mathcal{V} : \exists U \in \mathcal{U} : V <_\mu U$$

Using this notation one has the equivalence of the following statements

(i) $\forall \mathcal{A} \in \gamma$ also $\mathcal{A}_{<_\mu} \in \gamma$
(ii) $\forall \mathcal{U} \in \mu : \exists \mathcal{V} \in \mu : \mathcal{V} <_\mu \mathcal{U}$

A seminearness space is said to be regular if it satisfies the previous equivalent statements [**?**].

Next we introduce a stronger version of regularity by strengthening "disjointness" as we did before. Using γ-disjointness instead of disjointness we again obtain equivalent statements.

Proposition 4.1. *Let (X, γ) be a seminearness space and μ be the associated covering structure. Let A and B be subsets of X. The following are equivalent:*

(i) $\{A\} \cup \{D | D$ *and* B γ*-disjoint*$\} \in \mu$
(ii) $\forall \mathcal{A}$ *small, if for every $D \in \mathcal{A}$ the sets D and B are not γ-disjoint, then $A \in \mathcal{A}$*

We write $B <<_\mu A$ if one and then both conditions stated in the previous proposition hold. In view of the fact that

$$\{A\} \cup \{D | D \text{ and } B\ \gamma\text{-disjoint}\} \prec \{A, X - B\}$$

we have that

$$B <<_\mu A \Rightarrow B <_\mu A$$

For a stack $\mathcal{A}$ let

$$\mathcal{A}_{<<_\mu} = \{A | \exists B \in \mathcal{A} : B <<_\mu A\}$$

For covers $\mathcal{U}$ and $\mathcal{V}$ we write

$$\mathcal{V} <<_\mu \mathcal{U} \iff \forall V \in \mathcal{V} : \exists U \in \mathcal{U} : V <<_\mu U$$

Using this notation we obtain the following equivalent statements. The proof of the equivalence is quite similar to the equivalence based on $<_\mu$ instead of $<<_\mu$.

Proposition 4.2. *The following are equivalent:*

(i) $\forall \mathcal{A} \in \gamma$ *we have* $\mathcal{A}_{<<_\mu} \in \gamma$
(ii) $\forall \mathcal{U} \in \mu : \exists \mathcal{V} \in \mu : \mathcal{V} <<_\mu \mathcal{U}$

Definition 4.3. A seminearness space satisfies (R) if it fulfills one (and then both) of the previous statements.

Clearly (R) implies regularity. Moreover as for nearness spaces every uniform seminearness space is regular. However not every uniform seminearness space satisfies (R), example **??** serves as a counterexample.

In general we have the following implications.

Proposition 4.4. *If (X, γ) is a seminearness space then we have*

(1) *regular implies separated and regular implies (S)*
(2) *(R) implies regularity and (R) implies (T)*

Proof.

(1) That a regular seminearness space is separated is proved analogously to the nearness case. A regular seminearness space also satisfies (S). Indeed: let $\mathcal{M}$ and $\mathcal{N}$ be small stacks and suppose $\forall M \in \mathcal{M}, \forall N \in \mathcal{N} : M \cap N \neq \varnothing$. Consider the stack $\mathcal{M} \cap \mathcal{N}$ and let $\mathcal{U} \in \mu$. Take $\mathcal{V} \in \mu$ such that $\mathcal{V} <_\mu \mathcal{U}$. Further let $V \in \mathcal{V} \cap \mathcal{M}$ and let $U \in \mathcal{U}$ be such that $V <_\mu U$. Now since $\mathcal{N}$ is small we have
$$\mathcal{N} \cap \{U, X - V\} \neq \varnothing$$
Clearly $X - V \notin \mathcal{N}$ and so finally we have $U \in \mathcal{M} \cap \mathcal{N}$. So we can conclude that $\mathcal{M} \cap \mathcal{N}$ is small and in case $\mathcal{M}$ and $\mathcal{N}$ are minimal small this implies $\mathcal{M} = \mathcal{N}$.

(2) (R) implies (T). Let $\mathcal{M}$ and $\mathcal{N}$ be minimal small stacks. Suppose $\mathcal{M} \neq \mathcal{N}$ and assume that $\mathcal{M}$ and $\mathcal{N}$ do not contain γ-disjoint sets. We prove that $\mathcal{M} \cap \mathcal{N}$ is small. Let $\mathcal{U} \in \mu$ and consider $\mathcal{V} \in \mu$ such that $\mathcal{V} <<_\mu \mathcal{U}$. Since $\mathcal{M}$ is small we can take $V \in \mathcal{V} \cap \mathcal{M}$ and then $U \in \mathcal{U}$ such that $V <<_\mu U$. Consider the uniform cover
$$\mathcal{W} = \{U\} \cup \{D | D \text{ and } V \ \gamma\text{-disjoint}\}$$
Then $\mathcal{N} \cap \mathcal{W} \neq \varnothing$. Now by assumption on $\mathcal{M}$ and $\mathcal{N}$, U must belong to $\mathcal{N}$. So finally $U \in \mathcal{M} \cap \mathcal{N}$ and the rest follows as in the previous part.

□

In fact no other implications (except for those obtained by transitivity) hold. Counterexamples for the nonvalid ones can be found in section **??**.

Proposition 4.5. *Let (X, γ) be a seminearness space. If $\mathcal{M} \subset sec\ \mathcal{M}$ for all minimal small stacks $\mathcal{M}$ that do not converge, then we have*

(i) *D and B are γ-disjoint if and only if D and B are disjoint*
(ii) *$B <<_\mu A$ if and only if $B <_\mu A$*
(iii) *(X, γ) has (R) if and only if it is regular*

Proof. We only need to show (i). Clearly if $\mathcal{M} \subset sec\ \mathcal{M}$ then disjoint sets D and B can not both belong to $\mathcal{M}$ □

In every nearness space minimal small stacks are filters and so the statements in the previous proposition hold, in particular for nearness spaces, regularity is equivalent to (R). Applying corollary 3.10 we obtain that in every concrete seminearness space in which all minimal small stacks are concentrated and in which (S) holds, also $\mathcal{M} \subset sec\,\mathcal{M}$ holds for every minimal small stack. It follows that every regular concrete seminearness space in which all minimal small stacks are concentrated satisfies (R). In particular a closure (seminearness) space is regular if and only if it satisfies (R). It can be easily seen that analogously to the topological case, a closure (seminearness) space X is regular if and only if

$$\forall x \in X : \mathcal{V}(x) \text{ has a closed base}$$

However even on a nearness space the condition that "$cl_\gamma \mathcal{A}$ is small whenever $\mathcal{A}$ is small" (called weakly regular in [**?**]) is strictly weaker than regularity.

Next we investigate the regularity of the strict extension of a concrete seminearness space.

Proposition 4.6. *Let (X, γ) be a concrete T_1 seminearness space, then it is induced by a strict regular extension if and only if (X, γ) satisfies (R).*

Proof. Suppose (X, γ) satisfies (R). We prove that the strict T_1 extension $(\hat{X}, \hat{cl})$ is regular. Let $y_\mathcal{M} \in \hat{X}$ where $\mathcal{M}$ is a minimal small stack that is not convergent (or alternatively let $x \in X$). Let $O(A)$ be a basic open set containing $y_\mathcal{M}$ (or x) where A is some subset of X. So $A \in \mathcal{M}$ (or $A \in \mathcal{V}(x)$). Since $\mathcal{M}_{<<_\mu} = \mathcal{M}$ ($\mathcal{V}(x)_{<<_\mu} = \mathcal{V}(x)$) we have $A \in \mathcal{M}_{<<_\mu}$ ($A \in \mathcal{V}(x)_{<<_\mu}$) and so we can find $B \in \mathcal{M}$ ($B \in \mathcal{V}(x)$) such that $B <<_\mu A$. Now we prove that

$$\hat{cl}\ O(B) \subset O(A)$$

Let $y_\mathcal{N} \in \hat{cl}\ O(B)$ where $\mathcal{N}$ is minimal small, not convergent. Then for every $D \in \mathcal{N}$ we have

$$O(D) \cap O(B) \neq \varnothing$$

It follows that for every $D \in \mathcal{N}$ the sets D and B are not γ-disjoint. Hence $A \in \mathcal{N}$ and then $y_\mathcal{N} \in O(A)$.

Moreover if $x \in \hat{cl}\ O(B)$ then for every $D \in \mathcal{V}(x)$ we have $O(D) \cap O(B) \neq \varnothing$. Again it follows that for every $D \in \mathcal{V}(x)$ the sets D and B are not γ-disjoint and therefore $A \in \mathcal{V}(x)$ and $x \in O(A)$.

Conversely suppose that (X, γ) is induced by a strict regular T_1 extension. By proposition 2.6 this means that $(\hat{X}, \hat{cl})$ is regular. We prove that (X, γ) satisfies (R).

Let $\mathcal{A}$ be small. Since (X, γ) is concrete we can take a minimal small stack $\mathcal{M}$ such that $\mathcal{M} \subset \mathcal{A}$. We prove that $\mathcal{M} \subset \mathcal{M}_{<<_\mu}$.

Let $A \in \mathcal{M}$. If $\mathcal{M}$ is not convergent we have $y_\mathcal{M} \in O(A)$ (or if $\mathcal{M} = \mathcal{V}(x)$ we have $x \in O(A)$). Since $O(A)$ is open in $(\hat{X}, \hat{cl})$ it contains a closed neighborhood and so we can find $B \subset X$ such that $y_\mathcal{M} \in O(B)$ (or $x \in O(B)$) and such that

$$O(B) \subset \hat{cl}\ O(B) \subset O(A)$$

But then we have $B \in \mathcal{M}$ (or $B \in \mathcal{V}(x)$) and it is clear that B can be assumed to be open. Now $B <<_\mu A$. Indeed let $\mathcal{D}$ be small such that for every $D \in \mathcal{D}$ the sets D and B are not γ-disjoint. Let $\mathcal{N}$ be minimal small with $\mathcal{N} \subset \mathcal{D}$. Then also for every $N \in \mathcal{N}$ the sets N and B (which can be considered to be open) are not γ-disjoint. It follows that

$$O(N) \cap O(B) \neq \varnothing, \forall N \in \mathcal{N}$$

and finally that $y_{\mathcal{N}} \in \hat{cl}\ O(B)$ if $\mathcal{N}$ is not convergent, or that $x \in \hat{cl}\ O(B)$ if $\mathcal{N} = \mathcal{V}(x)$. So we have $y_{\mathcal{N}} \in O(A)$ or $x \in O(A)$, respectively and we can conclude that $A \in \mathcal{N}$. $\square$

It is known that for a topological extension, regularity of the extension implies strictness. In our setting of closure extensions this is no longer true. A seminearness space can be induced by a regular T_1 extension without having a strict regular T_1 extension. In examples **??** and **??** we'll prove that the structure is induced by a regular T_1 extension. However neither **??** nor **??** satisfies (R).

5. Summarizing diagrams and examples.

5.1. For closure spaces and for concrete nearness spaces we have that several notions coincide, namely that "T $\iff$ S $\iff$ separated" and that "R $\iff$ regularity".

5.2. For arbitrary nearness spaces we have the implications of figure **??**.

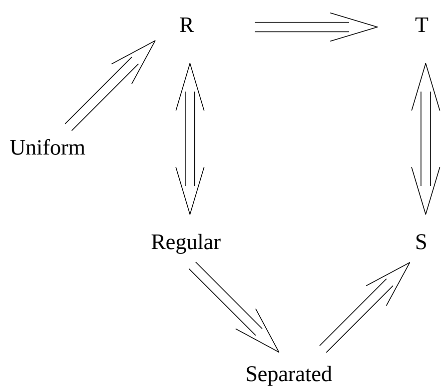

Figure 1. Implications for nearness spaces

The only valid implications are those indicated and those obtained by transitivity.

5.3. As an example of a nonconcrete nearness space satisfying (S) but that is not separated, one can consider example 3 in [**?**]. That this example indeed satisfies condition (S) follows from the fact that the only concentrated minimal small stacks are the pointfilters.

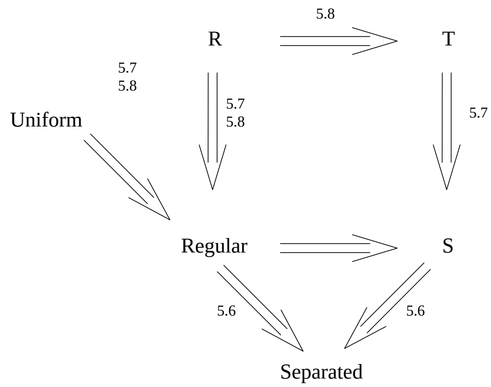

Figure 2. Implications for concrete seminearness spaces

5.4. For concrete seminearness spaces we proved the implications in figure **??**.

Again no other implications hold except for those obtained by transitivity. The numbers refer to the examples presented below and these are counterexamples for the reversed arrows.

5.5. For general seminearness spaces the diagram is as in figure **??**.

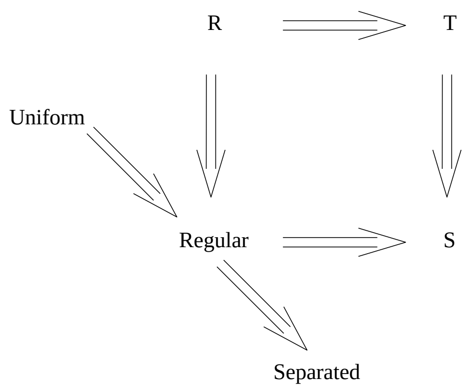

Figure 3. Implications for seminearness spaces

As a counterexample showing that (S) does not imply separated in the nonconcrete case, we again refer to example 3 in [**?**].

5.6. *A concrete and separated T_1 seminearness space which does not satisfy (S). Therefore it is nonregular, nonuniform and satisfies neither (R) nor (T).*

Let $X = \mathbb{R}^2$ and define

$$\mathcal{M} = stack\ \{pr_1^{-1}(c) | c \in \mathbb{R}\}$$
$$\mathcal{N} = stack\ \{pr_2^{-1}(c) | c \in \mathbb{R}\}$$

For a stack $\mathcal{A}$ we define

$$\mathcal{A} \in \gamma \iff \mathcal{M} \subset \mathcal{A} \text{ or } \mathcal{N} \subset \mathcal{A} \text{ or } \dot{x} \subset \mathcal{A} \text{ for some } x \in X$$

The underlying closure is the discrete one. Clearly every small stack contains a unique minimal small stack, but $\mathcal{M}$ and $\mathcal{N}$ do not contain disjoint sets and so (X, γ) does not satisfy (S).

5.7. *A concrete T_1 seminearness space which is uniform (and even zerodimensional) and so it is regular and satisfies (S) and is separated. However it satisfies neither (T) nor (R).*

Let A, B, C be three pairwise disjoint sets with more than one point and $X = A \cup B \cup C$. In order to define γ on X consider the following stacks

$$\mathcal{M} = \textit{stack } \{A, B\}$$

$$\mathcal{N} = \textit{stack } \{B, C\}$$

$$\mathcal{P} = \textit{stack } \{A, C\}$$

Define a stack $\mathcal{A}$ to be small if and only if

$$\mathcal{M} \subset \mathcal{A} \text{ or } \mathcal{N} \subset \mathcal{A} \text{ or } \mathcal{P} \subset \mathcal{A} \text{ or } \dot{x} \subset \mathcal{A} \text{ for some } x \in X$$

Then the pointfilters $\dot{x}$ for $x \in X$ are the only concentrated minimal stacks, and the other minimal stacks $\mathcal{M}, \mathcal{N}, \mathcal{P}$ are not concentrated.

(X, γ) does not satisfy (T) since for instance for $\mathcal{M}$ and $\mathcal{N}$ neither of the disjoint sets A and B, A and C or B and C are γ-disjoint. It follows that (X, γ) does not satisfy (R).

However (X, γ) is uniform since its collection μ of uniform covers is generated by the following collection $\mu' = \{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3\}$ of partitions

$$\mathcal{U}_1 = \{A, B\} \cup \{\{x\} | x \in C\}$$

$$\mathcal{U}_2 = \{B, C\} \cup \{\{x\} | x \in A\}$$

$$\mathcal{U}_3 = \{A, C\} \cup \{\{x\} | x \in B\}$$

It follows that (X, γ) is regular, also separated and satisfies (S). So the strict extension $(\hat{X}, \hat{cl})$ is a T_1 extension that is not Hausdorff and not regular. Remark however that the loose extension constructed by adding different points for the minimal small stacks that do not converge, is regular and T_1.

5.8. *A concrete T_1 seminearness space which is uniform (and even zerodimensional) and therefore is regular. It satisfies (T) and hence also (S) and it is separated. However it does not satisfy (R).*

Let A, A', P, P', Q and Q' be pairwise disjoint sets with more than one point and let

$$X = A \cup A' \cup P \cup P' \cup Q \cup Q'$$

In order to define γ on X consider the following stacks

$$\begin{aligned}\mathcal{A} &= \textit{stack } \{A, A'\}\\ \mathcal{P} &= \textit{stack } \{P, P', A\}\\ \mathcal{Q} &= \textit{stack } \{Q, Q', A\}\\ \mathcal{B} &= \textit{stack } \{P, Q\}\end{aligned}$$

Define a stack $\mathcal{S}$ to be small if and only if

$$\mathcal{A} \subset \mathcal{S}, \mathcal{P} \subset \mathcal{S}, \mathcal{Q} \subset \mathcal{S}, \mathcal{B} \subset \mathcal{S} \text{ or } \dot{x} \subset \mathcal{S} \text{ for some } x \in X$$

Clearly (X, γ) is concrete and T_1. Use the fact that A', P', Q' are sets belonging to just one nonconvergent minimal small stack to see that (X, γ) satisfies (T).

Again (X, γ) is uniform and in fact (X, μ) is generated by a collection of partitions. So (X, γ) is regular.

However (R) is not satisfied. Let's concentrate on $\mathcal{A}$ and consider $A \in \mathcal{A}$. The sets P and A are not γ-disjoint since they both belong to $\mathcal{P}$. Also Q and A are not γ-disjoint. It follows that for

$$\mathcal{U} = \{A\} \cup \{D | D \text{ and } A \ \gamma\text{-disjoint}\}$$

we have $\mathcal{U} \cap \mathcal{B} = \varnothing$. So $\mathcal{U} \notin \mu$ and therefore $A <\not<_\mu A$. Clearly this implies that $\mathcal{A}_{<<_\mu} \notin \gamma$. It follows that the strict extension $(\hat{X}, \hat{cl})$ of (X, γ) is a Hausdorff closure space that is not regular. Remark however that the loose extension constructed by adding different points for the minimal small stacks that do not converge, is a regular T_1 extension.

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D. Deses, A. De Groot-Van der Voorde, E. Lowen-Colebunders
Department of Mathematics, Vrije Universiteit Brussel, 1050 Brussels, Belgium

E-mail address: diddesen@vub.ac.be, avdvoord@vub.ac.be, evacoleb@vub.ac.be