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# Quasi-pseudometric properties of the Nikodym-Saks space

Jesús Ferrer*

Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

Abstract. For a non-negative finite countably additive measure $\mu$ defined on the $\sigma$-field $\Sigma$ of subsets of $\Omega$, it is well known that a certain quotient of $\Sigma$ can be turned into a complete metric space $\Sigma(\Omega)$, known as the Nikodym-Saks space, which yields such important results in Measure Theory and Functional Analysis as Vitali-Hahn-Saks and Nikodym's theorems. Here we study some topological properties of $\Sigma(\Omega)$ regarded as a quasi-pseudometric space.

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## 1. Introduction.

All throughout this paper we shall assume that the measure space $(\Omega, \Sigma, \mu)$ corresponds to a non-negative finite countably additive measure $\mu$ defined in the $\sigma$-algebra $\Sigma$ of subsets of $\Omega$, $0 < \mu(\Omega) < \infty$.

Following the terminology of [5, p.156], by $\Sigma(\Omega)$ we denote the quotient space obtained after identifying the measurable sets $A, B$ such that their symmetric difference $A \bigtriangledown B$ has zero measure. For the sake of convenience, we shall not use any special symbol to distinguish between the elements of $\Sigma$ and the equivalence classes in $\Sigma(\Omega)$.

It is shown in [5, p.156], and also in [3, p.86] and [7, p.208], that the function $\mu(A \bigtriangledown B)$ defines a metric in $\Sigma(\Omega)$ such that it becomes a complete metric space. This property allows one to apply Baire category arguments to obtain important results in convergence of measures such as the theorems of Vitali-Hahn-Saks and Nikodym.

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Besides, the set operations in $\Sigma(\Omega)$ are well defined and are continuous respect to the metric considered, so that $(\Sigma(\Omega), \bigtriangledown, \cap)$ may also be regarded as a topological ring. Generalizations of this can be found in [4].

The purpose of this paper is to notice that the metric space $\Sigma(\Omega)$ admits a quasi-pseudometric structure which determines in the standard way both the topology and the order given by set-inclusion. We shall as well revisit some topological properties of $\Sigma(\Omega)$, such as completeness, compactness and connectedness from a quasi-pseudometric perspective.

## 2. Nikodym-Saks' complete quasi-pseudometric ordered space.

Given $A, B \in \Sigma(\Omega)$, we define

$$q(A, B) \ := \ \mu(B \setminus A).$$

It is immediate to verify that $q$ is a quasi-pseudometric in $\Sigma(\Omega)$. By $q^{-1}$ and $q^*$ we denote the conjugate quasi-pseudometric and the metric associated to $q$, respectively, that is

$$q^{-1}(A, B) = q(B, A) \ = \ \mu(A \setminus B),$$

$$q^*(A, B) = \ q(A, B) \ + \ q^{-1}(A, B) \ = \ \mu(B \setminus A) \ + \ \mu(A \setminus B) \ = \ \mu(A \bigtriangledown B).$$

It is also quite simple to see that the set operations $\cup$ and $\cap$ are continuous in the quasi-pseudometric space $(\Sigma(\Omega), q)$. Also, the mapping $A \longrightarrow \Omega \setminus A$ is a quasi-uniform isomorphism from $(\Sigma(\Omega), q)$ onto $(\Sigma(\Omega), q^{-1})$. In the same manner, one may easily see that $\mu : \ (\Sigma(\Omega), q) \longrightarrow \mathbb{R}$ is quasi-uniformly upper semicontinuous, i.e., given $\varepsilon > 0$, there is $\delta > 0$ such that, whenever $q(A, B) < \delta$, we have $\mu(B) - \mu(A) < \varepsilon$.

Noticing that set-inclusion $\subseteq$ is an ordering compatible with the equivalence relation defined in $\Sigma$, we may regard $(\Sigma(\Omega), \subseteq)$ as an ordered space. Again, following the terminology of [6], we have the following result.

**Proposition 2.1.** $(\Sigma(\Omega), q^*, \subseteq)$ *is a metric ordered space determined by the quasi-pseudometric* $q$.

*Proof.* It all reduces to see that the graph of the order relation $\subseteq$ coincides with $\cap_{\varepsilon>0} V_\varepsilon^{-1}$, where

$$V_\varepsilon^{-1} \ = \ \{(A, B) \in \Sigma(\Omega) \times \Sigma(\Omega) \ : \ \ q^{-1}(A, B) < \varepsilon\}.$$

This is simple, since $(A, B) \in \cap V_{\varepsilon>0}^{-1}$ if and only if $q^{-1}(A, B) = 0$, which is equivalent to $q(B, A) = \mu(A \setminus B) = 0$. That is, $A \setminus B = \varnothing$ and so $A \subseteq B$. $\square$

Notice that the order defined by set-inclusion coincides with the so called "specialization order" defined by the quasi-pseudometric $q$, i.e.,

$$A \subseteq B \ \Leftrightarrow \ q(B, A) = 0 \ \Leftrightarrow \ B \in \overline{\{A\}},$$

that is, "it takes no effort to move from B to A, so A must be lower".

We introduce a couple of definitions by means of which we shall show that Nikodym-Saks' space is complete from a quasi-pseudometric perspective.

**Definition 2.2.** In a quasi-pseudometric space $(X, q)$, we say that a subset $A$ is **quasi-bounded** provided there is an element $x_0 \in A$ such that the set of reals $\{q(x, x_0) : x \in A\}$ is bounded.

It is plain that the associated pseudometric space $(X, q^*)$ is bounded if and only if the quasi-pseudometric spaces $(X, q)$ and $(X, q^{-1})$ are both quasi-bounded.

**Definition 2.3.** By a quasi-pseudometric ordered space we mean a triple $(X, q, \leq)$ such that the quasi-pseudometric $q$ determines the topological ordered space $(X, q^*, \leq)$ in the sense given in [6]. Thus, we say that the quasi-pseudometric ordered space $(X, q, \leq)$ is **orderly quasi-complete** whenever every quasi-bounded sequence $(x_n)_{n=1}^{\infty}$ satisfies the following two conditions:

1) $(x_n)_{n=1}^{\infty}$ admits a supremum (least upper bound) and an infimum (greatest lower bound) in $(X, \leq)$.

2) For each $n$, if $y_n := inf\{x_j : j \geq n\}$, then

$$q(y_n, x_n) \ \leq \ \sum_{j=n}^{\infty} q(x_{j+1}, x_j).$$

Following the terminology introduced in [10], if $(X, q)$ is a quasi-pseudometric space, a sequence $(x_n)_{n=1}^{\infty}$ in $X$ is said to be *right-k-Cauchy* whenever, given $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that, for $n \geq m \geq k$, we have $q(x_n, x_m) < \varepsilon$. We say that $(X, q)$ is *right-k-sequentially complete* provided every right-k-Cauchy sequence converges.

**Proposition 2.4.** *If $(X, q, \leq)$ is orderly quasi-complete, then $(X, q)$ is right-k-sequentially complete.*

*Proof.* Let $(x_n)_{n=1}^{\infty}$ be a right-k-Cauchy sequence in $(X, q)$. We define inductively an increasing sequence $(k_j)_{j=1}^{\infty}$ of positive integers such that, for each $j$, if $n \geq m \geq k_j$, then $q(x_n, x_m) < 2^{-j}$. Now, since $(x_{k_j})_{j=1}^{\infty}$ is quasi-bounded, if, for each $j$, $y_j := inf\{x_{k_i} : i \geq j\}$, and $y := sup\{y_j : j \geq 1\} = \underline{lim}_j x_{k_j}$, then, for each $j$, using condition 2 of the former definition,

$$q(y, x_{k_j}) \leq q(y, y_j) + q(y_j, x_{k_j}) = q(y_j, x_{k_j})$$

$$\leq \ \sum_{i=j}^{\infty} q(x_{k_{i+1}}, x_{k_i}) < \sum_{i=j}^{\infty} 2^{-i} = 2^{-j+1}.$$

Finally, for $\varepsilon > 0$, let $j_0$ be such that $2^{-j_0+1} < \varepsilon/2$. Then, for $n \geq k_{j_0}$, we take $j_1 \geq j_0$ with $k_{j_1} \geq n$, and so

$$q(y, x_n) \leq q(y, x_{k_{j_1}}) + q(x_{k_{j_1}}, x_n) \leq 2^{-j_1+1} + 2^{-j_0} < \varepsilon.$$

□

**Corollary 2.5.** *Nikodym-Saks' quasi-pseudometric space $(\Sigma(\Omega), q)$ is right-k-sequentially complete.*

*Proof.* After Proposition 2.2, it all reduces to see that $(\Sigma(\Omega), q, \subseteq)$ is orderly quasi-complete. For any sequence $(A_n)_{n=1}^\infty$ in $\Sigma(\Omega)$, it is clear that $\cup_{n=1}^\infty A_n$ and $\cap_{n=1}^\infty A_n$ are in $\Sigma(\Omega)$ and they correspond to $\sup_n A_n$, $\inf_n A_n$, respectively. Now, for each $n$, let $B_n := \cap_{j=n}^\infty A_j$, then

$$q(B, A_n) = \mu(A_n \backslash B) = \mu(\cup_{j=n}^\infty (A_n \backslash A_j)) \leq \mu(\cup_{j=n}^\infty (A_j \backslash A_{j+1}))$$
$$= \sum_{j=n}^\infty \mu(A_j \backslash A_{j+1}) = \sum_{j=n}^\infty q(A_{j+1}, A_j).$$

□

Again following [6], a quasi-pseudometric space $(X, q)$ is *bicomplete* when its associated pseudometric space $(X, q^*)$ is complete. The completeness of Nikodym-Saks' space can now be reobtained by means of quasi-pseudometrics.

**Corollary 2.6.** *Nikodym-Saks' quasi-pseudometric space* $(\Sigma(\Omega), q)$ *is bicomplete.*

*Proof.* Let $(A_n)_{n=1}^\infty$ be a Cauchy sequence in $(\Sigma(\Omega), q^*)$. For each $j \in \mathbb{N}$, there is $k_j \in \mathbb{N}$ such that, if $n, m \geq k_j$, then $q^*(A_n, A_m) < 2^{-j}$. So, if $n, m \geq k_j$, we have

$$q(A_n, A_m) < 2^{-j}, \quad q^{-1}(A_n, A_m) < 2^{-j},$$

thus obtaining, after what we did previously, that, if $B := \underline{lim}_j A_{k_j}$ and $C := \overline{lim}_j A_{k_j}$, then $(A_n)_{n=1}^\infty$ $q$-converges to $B$. Now, since $(A_n)_{n=1}^\infty$ is $q^{-1}$-right-k-Cauchy, it follows that $(\Omega \backslash A_n)_{n=1}^\infty$ is $q$-right-k-Cauchy and, given that, for each $j$,

$$q(\Omega \backslash A_n, \Omega \backslash A_m) = q^{-1}(A_n, A_m) < 2^{-j}, \quad n, m \geq k_j,$$

we have that $(\Omega \backslash A_n)_{n=1}^\infty$ $q$-converges to $\underline{lim}_j(\Omega \backslash A_{k_j}) = \Omega \backslash C$. Hence $(A_n)_{n=1}^\infty$ $q^{-1}$-converges to $C$. Hence, since $B \subseteq C$, and taking limits in

$$\mu(C \setminus B) = q(B, C) \leq q(B, A_n) + q(A_n, C) = q(B, A_n) + q^{-1}(C, A_n),$$

it follows that $\mu(B \setminus C) = \mu(C \setminus B) = 0$. That is, $B = C$ in $\Sigma(\Omega)$, and so $(A_n)_{n=1}^\infty$ converges in $(\Sigma(\Omega), q^*)$. □

Again after [6, p.84], we recall that a quasi-uniformity $\mathbf{U}$ in a space $X$ is convex with respect to the order $\leq$ whenever, given $U \in \mathbf{U}$, there is $V \in \mathbf{U}$ such that $V \subseteq U$, and, for each $x \in X$, $V(x) = \{y \in X : (x, y) \in V\}$ is convex respect to $\leq$, i.e., $a \leq c \leq b$, $a, b \in V(x)$, imply $c \in V(x)$.

After Proposition 4.19 of [6, p.84], in light of our previous result, we know that $(\Sigma(\Omega), q^*)$ is a convex metric space in the sense before defined. Nevertheless, we cannot conclude, as it happens in many metric convex spaces, that every ball $V_\varepsilon^*(A) = \{X \in \Sigma(\Omega) : q^*(A, X) < \varepsilon\}$ has to be a convex set, as our next result proves.

**Proposition 2.7.** *Let* $\Omega = [0, 1]$ *and let* $\lambda$ *represent the Lebesgue measure. Then, for each* $0 < \varepsilon < 1/2$, *there is a measurable set* $A$ *such that the ball* $V_\varepsilon^*(A)$ *is not convex with respect to set-inclusion.*

*Proof.* Let $\varepsilon/2 < a < 1 - \frac{3\varepsilon}{2}$. We consider the following measurable sets

$$A = [a, a + \frac{11\varepsilon}{8}], \quad X = [a, a + \frac{\varepsilon}{8}] \cup [a + \frac{\epsilon}{4}, a + \frac{\varepsilon}{2}] \cup [a + \varepsilon, a + \frac{11\varepsilon}{8}],$$

$$Y = [a - \frac{\varepsilon}{2}, a + \frac{3\varepsilon}{2}], \quad Z = [a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2}] \cup [a + \varepsilon, a + \frac{11\varepsilon}{8}]$$

$$q^*(A, X) = \lambda(X \setminus A) + \lambda(A \setminus X) = \lambda(A \setminus X) = \frac{\varepsilon}{8} + \frac{\varepsilon}{2} = \frac{5\varepsilon}{8} < \varepsilon,$$

$$q^*(A, Y) = \lambda(Y \setminus A) + \lambda(A \setminus Y) = \lambda(Y \setminus A) = \frac{\varepsilon}{2} + \frac{\varepsilon}{8} = \frac{5\varepsilon}{8} < \varepsilon,$$

$$q^*(A, Z) = \lambda(Z \setminus A) + \lambda(A \setminus Z) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we have $X \subseteq Z \subseteq Y$, $X, Y \in V_\varepsilon^*(A)$, but $Z \notin V_\varepsilon^*(A)$. □

The following result will be needed afterwards. Let us recall first that a subset $F$ of an ordered set $(X, \leq)$ is said to be *inductive* whenever every totally ordered subset of $F$ has an upper bound in $X$.

**Proposition 2.8.** *Every non-empty closed subset of* $(\Sigma(\Omega), q^*)$ *is inductive with respect to set-inclusion.*

*Proof.* Let $\mathbf{F}$ be a non-empty closed set in $(\Sigma(\Omega), q^*)$. Let $(F_i)_{i\in I}$ be a totally ordered subset of $\mathbf{F}$. Since $\mu$ is finite, there exists $\rho = sup_{i\in I}\mu(F_i)$. We now start an inductive process by taking $i_1 \in I$ such that $\mu(F_{i_1}) > \rho - \frac{1}{2}$. Assuming already found $F_{i_1} \subseteq F_{i_2} \subseteq ... \subseteq F_{i_n}$ in $\mathbf{F}$ such that, for $j = 1, 2, ..., n$, $\mu(F_{i_j}) > \rho - \frac{1}{2^j}$, we proceed to find $F_{i_{n+1}}$ with the same properties. If $\mu(F_{i_n}) > \rho - \frac{1}{2^{n+1}}$, then we set $F_{i_{n+1}} := F_{i_n}$; if $\mu(F_{i_n}) \leq \rho - \frac{1}{2^{n+1}}$, we find $F_{i_{n+1}}$ in $\mathbf{F}$ such that $\mu(F_{i_{n+1}}) > \rho - \frac{1}{2^{n+1}}$, then $F_{i_n} \subseteq F_{i_{n+1}}$, otherwise, since we are dealing with totally ordered elements, we would have $F_{i_{n+1}} \subseteq F_{i_n}$, and, $\mu(F_{i_{n+1}}) \leq \mu(F_{i_n}) \leq \rho - \frac{1}{2^{n+1}}$, which is a contradiction. We have thus constructed an increasing sequence $(F_{i_n})_{n=1}^\infty$ in $\mathbf{F}$, with $\mu(F_{i_n}) > \rho - \frac{1}{2^n}$, $n \in \mathbb{N}$. We set $F := \cup_{n=1}^\infty F_{i_n}$. Then, $F \in \Sigma(\Omega)$, and, for each $n$,

$$q^*(F, F_{i_n}) = \mu(F \backslash F_{i_n}) = \mu(\cup_{j=1}^\infty (F_{i_j} \backslash F_{i_n})) \leq \mu(\cup_{j=1}^n (F_{i_j} \backslash F_{i_n})) + \mu(\cup_{j=n+1}^\infty (F_{i_j} \backslash F_{i_n}))$$

$$= \mu(\cup_{j=n+1}^\infty (F_{i_j} \setminus F_{i_n})) \leq \mu(\cup_{j=n+1}^\infty (F_{i_j} \setminus F_{i_{j-1}})) = \sum_{j=n+1}^\infty \mu(F_{i_j} \setminus F_{i_{j-1}})$$

$$= \sum_{j=n+1}^\infty (\mu(F_{i_j}) - \mu(F_{i_{j-1}})) \leq \sum_{j=n+1}^\infty (\rho - (\rho - \frac{1}{2^{j-1}})) = \sum_{j=n+1}^\infty \frac{1}{2^{j-1}} = \frac{1}{2^{n-1}}.$$

Hence, $(F_{i_n})_{n=1}^\infty$ converges to $F$ in $(\Sigma(\Omega), q^*)$. Since $\mathbf{F}$ is closed, it follows that $F \in \mathbf{F}$. We show finally that $F$ is an upper bound for the chain $(F_i)_{i\in I}$. Give $i \in I$, we consider two possibilities:

If there is $n_0 \in \mathbb{N}$ such that $F_i \subseteq F_{i_{n_0}}$, then it is clear that $F_i \subseteq F_{i_{n_0}} \subseteq F$.

On the contrary, if $F_i$ is not contained in $F_{i_n}$, $n \in \mathbb{N}$, then again the total ordering guarantees that $F_{i_n} \subseteq F_i$, $n \in \mathbb{N}$, and so $F \subseteq F_i$. Then, since $\rho \geq \mu(F_i) \geq \mu(F) = lim_n \mu(F_{i_n}) \geq \rho$, we have

$$\mu(F_i \bigtriangledown F) = \mu(F_i \setminus F) \ = \ \mu(F_i) - \mu(F) \ = \ \rho - \rho = \ 0.$$

That is, $F_i = F$. □

## 3. Connectedness and compactness in Nikodym-Saks' space.

In this section we study the topological properties of connectedness and compactness in the space $(\Sigma(\Omega), q^*)$, observing that such properties are directly related with the degree of atomicity of the measure $\mu$. We shall introduce again some notation. For $A \in \Sigma(\Omega)$, by $\Sigma(A)$ we denote the collection of elements of $\Sigma(\Omega)$ contained in $A$, we shall also refer to this collection as a *lower interval*; similarly $\Sigma_+(A)$ will stand for all the measurable supersets of $A$ and we will refer to this as an *upper interval*. Let us recall that $E \in \Sigma(\Omega)$ is called an *atom* when it has positive measure and the lower interval $\Sigma(E)$ is reduced to $\{\varnothing, E\}$. When a measure $\mu$ does not have any atom then it is said to be *non-atomic*. It is convenient to recall that a measure can only admit at most a countable amount of disjoint atoms, when $\Omega$ admits a countable partition formed by atoms, then $\mu$ is said to be *purely atomic*.

Before characterizing the connectedness of $(\Sigma(\Omega), q^*)$ in terms of atoms, let us notice that the quasi-pseudometric spaces $(\Sigma(\Omega), q)$ and $(\Sigma(\Omega), q^{-1})$ are always connected: Let us suppose that $(\Sigma(\Omega), q)$ admits two disjoint open sets $\mathbf{A}, \mathbf{B}$ covering $\Sigma(\Omega)$. Then, one of them, say $\mathbf{A}$, contains $\Omega$, so there is $\delta > 0$ such that $V_\delta(\Omega) \subseteq \mathbf{A}$. But, $V_\delta(\Omega) = \Sigma(\Omega)$. Hence, $\mathbf{B} = \varnothing$.

**Proposition 3.1.** *The following are equivalent.*

*(i) No upper interval $\Sigma_+(A)$, $A \neq \varnothing$, is $q^{-1}$-open.*
*(ii) $\mu$ is non-atomic.*
*(iii) $(\Sigma(\Omega), q^*)$ is connected.*
*(iv) For each $A \in \Sigma(\Omega)$ and $\alpha \in [0, \mu(A)]$, there is $B \in \Sigma(A)$ such that*

$$\mu(B) = \alpha.$$

*Proof.* $(i) \Rightarrow (ii)$. Assume that $E$ is an atom. We show that the upper inteval $\Sigma_+(E)$ is $q^{-1}$-open. Given $A \in \Sigma_+(E)$, let $\delta = \mu(E) > 0$. If $X \in V_\delta^{-1}(A)$, then $q^{-1}(A, X) < \delta$ implies $\mu(A \setminus X) < \delta$. But,

$$\mu(E \setminus X) \ = \ \mu(E \setminus X \setminus A) \ + \ \mu(E \cap A \setminus X) \leq \ \mu(E \setminus A) + \mu(A \setminus X) \ < \ \delta.$$

Hence, $\mu(E \setminus X) = 0$, otherwise, since $E$ is an atom, we would have $\mu(E \setminus X) = \mu(E) = \delta$. Thus, $X \in \Sigma_+(E)$.

$(ii) \Rightarrow (iii)$. Let us assume that $(\Sigma(\Omega), q^*)$ is disconnected. So, let $\mathbf{A}, \mathbf{B}$ be two non-empty disjoint closed sets covering $\Sigma(\Omega)$. We may suppose without restriction that $\Omega \in \mathbf{A}$. Applying Proposition 4 to the closed set $\mathbf{B}$ and after Zorn's lemma, there is a maximal element $M$ in $(\mathbf{B}, \subseteq)$. Since $\mathbf{B}$ is also open, there is $\delta > 0$ such that $V_\delta^*(M) \subseteq \mathbf{B}$. Now, since $\Omega \setminus M$ has non-zero measure (otherwise, $\Omega = M$ would also be in $\mathbf{B}$), the fact that $\mu$ is atom-free guarantees, see [1, p.24], that

$$inf\{\mu(E) \;:\; \varnothing \neq E \in \Sigma(\Omega \setminus M)\} \;=\; 0.$$

Thus, we may find $E \in \Sigma(\Omega \setminus M)$ such that $0 < \mu(E) < \delta$. Let $A := M \cup E$. Then, $q^*(M, A) = \mu(E) < \delta$, and so $A \in \mathbf{B}$ with $M \subseteq A$, $M \neq A$, contradicting the maximality of $M$.
$(iii) \Rightarrow (iv)$. The continuous mapping $X \longrightarrow X \cap A$ maps $\Sigma(\Omega)$ onto $\Sigma(A)$. Thus, if $\Sigma(\Omega)$ is connected, so is $\Sigma(A)$. Now, since $\mu$ is continuous, it follows that $\mu(\Sigma(A)) = [0, \mu(A)]$.
$(iv) \Rightarrow (i)$. Let us assume there is $E \in \Sigma(\Omega)$, $E \neq \varnothing$, such that the upper interval $\Sigma_+(E)$ is $q^{-1}$-open. Clearly, since $\mu(E) > 0$, there is $0 < \delta < \mu(E)$ such that $V_\delta^{-1}(E) \subseteq \Sigma_+(E)$. By hypothesis, there is $A \in \Sigma(E)$ for which $\mu(A) = \delta/2$. Let $X := E \setminus A$. Then, $\mu(E \setminus X) = \mu(A) = \delta/2 < \delta$ implies that $X \in V_\delta^{-1}(E)$ and consequently $E \subseteq X$, a contradiction, since $\mu(E \setminus X) \neq 0$. □

By recalling that a finite countably additive measure $\lambda$ in $(\Omega, \Sigma)$ is $\mu$-*continuous* whenever $\lim_{\mu(X)\to 0} \lambda(X) = 0$, we have that in this case the identity mapping is well defined and continuous from $(\Sigma(\Omega), q_\mu^*)$ into $(\Sigma(\Omega), q_\lambda^*)$. Therefore the following result is straightforward.

**Corollary 3.2.** *If $\lambda$ is a finite countably additive $\mu$-continuous measure and $\mu$ is non-atomic, then so is $\lambda$.*

We study in the following the compactness of Nikodym-Saks' space. As we did in the connectedness part, it is curious to notice that the quasi-pseudometric spaces $(\Sigma(\Omega), q)$ and $(\Sigma(\Omega), q^{-1})$ are always compact; just recall that, for instance, any $q$-open cover of $\Sigma(\Omega)$ must have a member containing $\Omega$, consequently, this open set has to be $\Sigma(\Omega)$.

We show next that, in some sense, the compactness of $(\Sigma(\Omega), q^*)$ does not get along with the connectedness. As a matter of fact, we show that compactness is equivalent to $\mu$ being purely atomic. We need, in order to do so, to introduce some more notation.

Given a sequence $(A_n)_{n=1}^\infty$ in $\Sigma(\Omega)$, for each $n$, if $(i_1, i_2, ..., i_n) \in \{0, 1\}^n$, we define the following sets

$$A_{(i_1,i_2,\ldots,i_n)} := (\cap\{A_j :\ 1 \leq j \leq n, i_j = 1\}) \cap (\cap\{\Omega \setminus A_j :\ 1 \leq j \leq n, i_j = 0\}).$$

It is plain that, for each $k = 1, 2, ..., n$, the collections

$$\{A_{(i_1,i_2,\ldots i_n)} :\ i_k = 1\}, \quad \{A_{(i_1,i_2,\ldots i_n)} :\ i_k = 0\},$$

are partitions of $A_k$ and $\Omega \setminus A_k$, respectively.

**Lemma 3.3.** *If $\mu$ is non-atomic, then there is a sequence $(A_n)_{n=1}^\infty$ in $\Sigma(\Omega)$ such that, for each $n$,*

$$\mu(A_{(1,1,\ldots,1)}) = \mu(A_{(0,0,\ldots,0)}) = (\frac{1}{4} + \frac{1}{2^{n+1}})\mu(\Omega),$$

$$\mu(A_{(i_1,i_2,\ldots,i_n)}) = \frac{1}{2^{n+1}}\mu(\Omega), (i_1, i_2, \ldots i_n) \notin \{(0, 0, ..., 0), (1, 1, ..., 1)\}.$$

*Proof.* We give an inductive sketch of proof. The first set $A_1$ appears courtesy of the non-atomicity of the measure $\mu$ and Proposition 3.1. Once already obtained $A_1, A_2, ..., A_n$, again Proposition 3.1 guarantees the existence of measurable sets

$$B_{(i_1,i_2,...,i_n)} \subseteq A_{(i_1,i_2,...,i_n)}, \quad (i_1, i_2, ...i_n) \in \{0,1\}^n,$$

such that

$$\mu(B_{(i_1,i_2,...,i_n)}) = \frac{1}{2^{n+2}}\mu(\Omega), \quad (i_1, i_2, ..., i_n) \neq (1, 1, ..., 1),$$

$$\mu(B_{(1,1,...,1)}) = (\frac{1}{4} + \frac{1}{2^{n+2}})\mu(\Omega).$$

Defining $A_{n+1} := \cup\{B_{(i_1,i_2,...,i_n)} : \ (i_1, i_2, ...i_n) \in \{0,1\}^n\}$, the induction process is done. □

**Proposition 3.4.** *Nikodym-Saks' space* $(\Sigma(\Omega), q^*)$ *is compact if and only if* $\mu$ *is purely atomic.*

*Proof.* Assuming $(\Sigma(\Omega), q^*)$ is compact, it all reduces to show that every set $A \in \Sigma(\Omega)$, with $\mu(A) > 0$, contains atoms. If this were not so, then the restricted measure $\mu_{|A}$ would be non-atomic in the restricted Nikodym-Saks' space $\Sigma(A)$. Applying the former lemma, we would find a sequence $(A_n)_{n=1}^{\infty} \subseteq \Sigma(A)$ satisfying the conditions there stated. It may be easily seen that the distance $q^*(A_n, A_m) = \frac{1}{4}\mu(A)$, $n \neq m$. Hence, such a sequence cannot admit any Cauchy subsequence, which contradicts the fact that $\Sigma(\Omega)$ is compact.

Conversely, let $(E_n)_{n=1}^{\infty}$ be an atomic partition of $\Sigma(\Omega)$. We show that $(\Sigma(\Omega), q^*)$ is homeomorphic to Cantor's space $2^{\mathbb{N}}$. Consider the one-to-one and onto mapping $T : \ 2^{\mathbb{N}} \longrightarrow \Sigma(\Omega)$ such that

$$T(x) \ := \ \cup\{E_n : \ x_n = 1\}, \quad x \in 2^{\mathbb{N}}.$$

Given $x \in 2^{\mathbb{N}}$, $\varepsilon > 0$, let $A = T(x)$. Since

$$\mu(A) = \sum\{\mu(E_n) : \ x_n = 1\}, \quad \mu(\Omega \setminus A) = \sum\{\mu(E_n) : \ x_n = 0\},$$

there are two finite subsets $F, F'$ of $\mathbb{N}$ such that $x(F) = 1$, $x(F') = 0$, and

$$\sum\{\mu(E_n) : \ n \in F\} > \mu(A) - \frac{\varepsilon}{2}, \quad \sum\{\mu(E_n) : \ n \in F'\} > \mu(\Omega \setminus A) - \frac{\varepsilon}{2}.$$

It follows that the set $V = \{y \in 2^{\mathbb{N}} : \ y(F) = 1, y(F') = 0\}$ is a neighborhood of $x$ such that $\mu(A \bigtriangledown T(y)) < \varepsilon$, $y \in V$. Finally, to see that $T^{-1}$ is also continuous, it suffices to notice that, for each $p \in \mathbb{N}$, the intervals $\Sigma_+(E_p)$ and $\Sigma(\Omega \setminus E_p)$ are closed subsets of $\Sigma(\Omega)$. □

## 4. Nikodym-Saks' space as a topological group.

We know that $\Sigma(\Omega)$ is an abelian nilpotent group and that the symmetric difference $\bigtriangledown$ is continuous, thus $\Sigma(\Omega)$ may be regarded also as a topological group. Besides, for each $A \in \Sigma(\Omega)$, $\Sigma(\Omega)$ can be expressed as the topological direct sum of the closed subgroups $\Sigma(A)$ and $\Sigma(\Omega \setminus A)$, i.e., $\Sigma(\Omega) = \Sigma(A) \oplus \Sigma(\Omega \setminus A)$. Our aim in this section is to show that Nikodym-Saks' space can be decomposed, in a unique way, as the topological direct sum of a connected subgroup (the component of the zero element $\varnothing$) plus a compact totally disconnected subgroup.

**Proposition 4.1.** *There exists a unique measurable set* $M$ *such that* $\Sigma(\Omega) = \Sigma(M) \oplus \Sigma(\Omega \setminus M)$, *with* $\Sigma(M)$ *connected and* $\Sigma(\Omega \setminus M)$ *compact.*

*Proof.* Let $\mathbf{A}$ denote the countable collection, possibly empty, of atoms of $\Sigma(\Omega)$. Since the set $M := \Omega \setminus (\cup\{E : E \in \mathbf{A}\})$ contains no atoms, it follows after Proposition 3.1 that the topological group $\Sigma(M)$ is connected. Now, assuming $M \neq \Omega$ (otherwise, $\Sigma(\Omega \setminus M) = \{\varnothing\}$, clearly compact), we have that the restricted measure $\mu_{|\Omega\setminus M}$ is purely atomic, Proposition 3.4 then shows that $\Sigma(\Omega \setminus M)$ is compact. Notice also that in this case, since $\Sigma(\Omega \setminus M)$ is a copy of a Cantor space, it is totally disconnected. □

**Proposition 4.2.** *The connected subgroup* $\Sigma(M)$ *before obtained coincides with the connected component of* $\varnothing$.

*Proof.* Let us denote by $\mathbf{C}$ the connected component of $\Sigma(\Omega)$ containing $\varnothing$. It is well known that $\mathbf{C}$ is a closed topological subgroup of $\Sigma(\Omega)$. Hence, since $\varnothing \in \Sigma(M)$ and $\Sigma(M)$ is connected, it follows that $\Sigma(M) \subseteq \mathbf{C}$. We show the reverse inclusion.

Let $A \in \mathbf{C}$. Assume $A \setminus M \neq \varnothing$. Hence, after what we have seen before, $\Sigma(A \setminus M)$ is a totally disconnected subgroup. But, the mapping $\tau : \mathbf{C} \longrightarrow \Sigma(A \setminus M)$ such that $\tau(X) = X \cap (A \setminus M)$ is continuous, and so $\tau(\mathbf{C})$ is connected in $\Sigma(A \setminus M)$. Thus, $\tau(\mathbf{C})$ must be a singleton, but this is not so since $\varnothing, A \setminus M \in \tau(\mathbf{C})$. □

We finish by studying the properties of compactness and connectedness related with their local properties.

**Corollary 4.3.** $\Sigma(\Omega)$ *is compact if and only if it is locally compact.*

*Proof.* Since $\Sigma(\Omega)$ is Hausdorff, we need only show the sufficiency part. So, if $\Sigma(\Omega)$ is locally compact, we may find $\delta > 0$ such that the closed ball $\overline{V}^*_\delta(\varnothing) = \{X \in \Sigma(\Omega) : \ q^*(\varnothing, X) \leq \delta\}$ is compact. Let, as before, $M \in \Sigma(\Omega)$ be such that $\Sigma(M)$ is the connected component of $\varnothing$. It all reduces to see that $M = \varnothing$. If not, since $\Sigma(M)$ is connected, we can find, after Proposition 3.1, $A \in \Sigma(M)$ such that $0 < \mu(A) \leq \delta$, and so $\Sigma(A) \subseteq \overline{V}^*_\delta(\varnothing)$. Hence $\Sigma(A)$ is compact and, after Proposition 3.4, $A$ must contain atoms. This would imply that, $M$ containing $A$ would also contain atoms, thus contradicting the definition of $M$. □

**Corollary 4.4.** *If* $\Sigma(\Omega)$ *is connected, then it is locally connected.*

*Proof.* Let us assume that $\Sigma(\Omega)$ is connected. We first show that every ball centered at $\varnothing$ is connected. This is a simple consequence of the facts

$$V^*_\delta(\varnothing) \ = \ \cup\{\Sigma(X) : \ X \in V^*_\delta(\varnothing)\}, \quad \varnothing \in \cap\{\Sigma(X) : \ X \in V^*_\delta(\varnothing)\},$$

and that, for any $X$, $\Sigma(X)$ is connected. Now, being in a topological group, since every ball centered at $A$ is a translation of the form

$$V_\delta^*(A) \;=\; A \;\triangledown\, V_\delta^*(\varnothing),$$

it follows that $V_\delta^*(A)$ is also connected. □

As it was pointed out by Professor Paolo De Lucia one can easily find examples of locally connected disconnected (but not totally disconnected) Nikodym-Saks spaces.

**Example 4.5.** Consider the measure space $(\Omega, \Sigma, \mu)$ given by ($\lambda$ represents the Lebesgue measure in $[0,1]$ and we denote by $\mathbf{L}$ the class of Lebesgue-measurable subsets of $[0,1]$):

$$\Omega \;=\; [0,1] \;\cup \{2\}, \quad \Sigma \;=\; \mathbf{L} \;\cup \{A \cup \{2\} :\; A \in \mathbf{L}\},$$
$$\mu_{|\mathbf{L}} = \lambda,\;\; \mu(A \cup \{2\}) = \lambda(A) + 1,\;\; A \in \mathbf{L}.$$

Clearly, since $\{2\}$ is a $\mu$-atom, the corresponding Nikodym-Saks' space $\Sigma(\Omega)$ is not connected. Notice that it is neither totally disconnected, since $\Sigma([0,1])$ is the connected component of $\varnothing$. And, after recalling that $\Sigma(\Omega) = \Sigma([0,1]) \oplus \Sigma(\{2\})$, it can be easily shown that $\Sigma(\Omega)$ is locally connected.

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Jesús Ferrer
*Departamento de Análisis Matemático, Universidad de Valencia, Dr. Moliner 50, 46100 Burjasot (Valencia), Spain*

*E-mail address*: Jesus.Ferrer@uv.es