Applied General Topology
© Universidad Politécnica de Valencia
Volume 4, No. 2, 2003
pp. 475–486

A note on separation in AP

R. Lowen and M. Sioen

Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

Abstract. It is our aim in this note to take a closer look at some separation axioms in the construct **AP** of approach spaces and contractions. Whereas lower separation axioms seem to be qualitative, the higher ones seem to have a quantitative nature. Also some characterizations for the corresponding epireflectors will be given.

2000 AMS Classification: 54A05, 54D10, 54D15, 54B30.
Keywords: approach space, separation property, reflection.

1. Introduction and preliminaries.

As one readily knows, metric structures behave badly with respect to the formation of initial structures or in particular, of products, since e.g. for an infinite family of metrics, their $p\mathbf{MET}^\infty$-product is not compatible with the topological product of the associated underlying topologies. As a remedy to this defect, the common supercategory **AP** (the objects of which are called approach spaces) of **TOP** and $pq\mathbf{MET}^\infty$ was introduced in [7], where e.g. for a considered (infinite) family of metric spaces, their **AP**-product carries precisely that part of the numerical information present, which can be retained if one demands compatibility with the topological product of the family of underlying metric topologies. The basic difference in nature between approach and metric spaces consists in the fact that in an approach space, one specifies all the point-set distances (subsequent to some axioms), where such a point-set distance does not have to equal the infimum over the considered set of all the point-point distances, like in the metric case. We now recall the definition of an approach space. In the sequel, X stands for an arbitrary set and 2^X stands for its powerset.

Definition 1.1. A map $\delta : X \times 2^X \longrightarrow [0, \infty]$ is called a distance on X if it satisfies the following conditions:

(D1) $\forall x \in X : \delta(x, \{x\}) = 0,$

(D2) $\forall x \in X : \delta(x, \varnothing) = \infty$,
(D3) $\forall x \in X, \forall A, B \in 2^X : \delta(x, A \cup B) = \delta(x, A) \wedge \delta(x, B)$,
(D4) $\forall x \in X, \forall A \in 2^X, \forall \varepsilon \geq 0$:

$$\delta(x, A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon,$$

with $A^{(\varepsilon)} := \{z \in X \mid \delta(z, A) \leq \varepsilon\}$.
The pair (X, δ) is called an approach space.

The morphisms to go along with approach spaces are the so-called contractions: if $(X, \delta), (X', \delta')$ are approach spaces, then a map $f : X \longrightarrow X'$ is called a contraction if

$$\forall x \in X, \forall A \in 2^X : \delta'(f(x), f(A)) \leq \delta(x, A).$$

It was shown in [7] that approach spaces and contractions constitute a topological construct, denoted by **AP**, into which **TOP** (resp. $p(q)\mathbf{MET}^\infty$ (via the usual definition of point-set distances in the metric case)) can be concretely embedded as a full concretely bireflective and concretely bicoreflective (resp. concretely bicoreflective) subconstruct. Given an approach space, the topological and $\infty p(q)-$metric coreflections have to be interpreted as the topology, resp. the $\infty p(q)$-metric "underlying" the given approach space, in the same sense as we think of the induced topology underlying a given metric. We will write $\mathcal{T}_\delta$ for the topological coreflection of a distance δ. For any background material of categorical nature, we refer to [1] and for detailed information on approach theory, we refer to [7] and [8]. Let us only mention some alternative axiomizations for approach spaces which are described in [7], [8] and which will be used throughout the paper: approach systems (parallelling neighbourhood systems in topology), regular function frames (parallelling closed subsets in topology), hulls (parallelling topological closures) and approach limits (parallelling the description of a topology via convergence of filters). For an approach space (X, δ), the corresponding approach system, resp. regular function frame, hull and approach limit will be denoted by $\mathcal{A} := (\mathcal{A}(x))_{x \in X}$, resp. $\mathcal{R}, h$ and λ and for explicit definitions and transition formulas between these equivalent characterizations, we again refer to [8]. If confusion might arise, we use a notation like $\mathcal{A}_\delta$ to denote the approach system corresponding to the distance δ. We will make no distinction between a distance and its associated approach system, regular function frame, hull and approach limit. To finish, we agree upon some notations. If $\mathbf{C}$ is a full subconstruct of **AP**, then $\mathcal{E}_{\mathbf{AP}}(\mathbf{C})$ stands for its epireflective hull in **AP**, being the full subconstruct of **AP** consisting of all subspaces of products of **C**-objects. The metric approach space corresponding to the real line equipped with the Euclidean metric will be denoted by $\mathbb{R}$. Also the following particular approach spaces will play a role in the sequel: the indiscrete 2-space $I_2 := (\{0,1\}, \delta_i)$, where $\delta_i(x, \varnothing) := \infty$ and $\delta_i(x, A) := 0$ if $A \neq \varnothing$, and $\mathbb{P} := ([0, \infty], \delta_{\mathbb{P}})$ with $\delta_{\mathbb{P}}(x, \varnothing) := \infty$ and $\delta_{\mathbb{P}}(x, A) := (x - \sup A) \vee 0$ for all x and $A \neq \varnothing$.

Separation axioms, measuring to which extent different points, or points not belonging to a closed set, resp. disjoint closed sets can be recognized as such by a given space play an important role in topology. In particular, many well-known extension theories only work (nicely) in the presence of certain separation axioms, e.g. the uniform completion for T_2-uniform spaces or metric spaces, the Wallman compactification for T_1-topological spaces or the Čech-Stone compactification for Tychonoff spaces. In this note we want to focus on some forms of separation axioms in the realm of approach theory and in case they determine categorically nice (i.e. epireflective) subconstructs of **AP**, to give a description for the corresponding epireflection arrows. We will use $\mathbf{TOP}_0$, resp. $\mathbf{TOP}_1$ and $\mathbf{TOP}_2$ for the full subconstructs of **TOP** consisting of all T_0, resp. T_1 and T_2-spaces.

2. Separation Axioms.

2.1. **The T_0-axiom.** In [10] the categorically correct definition of T_0-objects in the setting of topological constructs in the sense of H. Herrlich (see e.g. [1]) was given and it was shown there that the resulting subconstruct of all T_0-objects is the largest epireflective, not bireflective subconstruct of the considered topological construct. We will now identify the T_0-objects in **AP**.

Definition 2.1. We call an approach space (X, δ) a T_0-space if every contraction $f : I_2 \longrightarrow (X, \delta)$ is constant. We write $\mathbf{AP}_0$ for the full subconstruct of **AP** consisting of all T_0-objects.

Proposition 2.2. *For every $(X, \delta) \in |\mathbf{AP}|$, the following assertions are equivalent:*

(1) $(X, \mathcal{T}_\delta) \in |\mathbf{TOP}_0|$,
(2) $(X, \delta) \in \mathcal{E}_{\mathbf{AP}}(\mathbb{P})$,
(3) $\forall x, y \in X : x \neq y \Rightarrow$

$$((\exists \varphi \in \mathcal{A}(x) : \varphi(y) > 0) \vee (\exists \varphi \in \mathcal{A}(y) : \varphi(x) > 0))$$

(4) $\forall x, y \in X : x \neq y \Rightarrow (\exists \gamma \in \mathcal{R} : \gamma(x) \neq \gamma(y))$,
(5) $\forall f \in \mathbf{AP}(I_2, (X, \delta)) : f$ constant,
(6) $\forall x, y \in X : x \neq y \Rightarrow \mathcal{A}(x) \neq \mathcal{A}(y)$.

Proof. The implication (1)⇒(3) is obvious because it is proved in [8] that for every $x \in X$, $\{\{\psi < \varepsilon\} | \psi \in \mathcal{A}(x), \varepsilon > 0\}$ is a base for the $\mathcal{T}_\delta$-neighbourhoodsystem at x. To verify the implication (3) ⇒ (2), note that it was proved in [8] that the source

$$(\delta(\cdot, A) : (X, \delta) \longrightarrow \mathbb{P} : x \mapsto \delta(x, A))_{A \in 2^X}$$

is initial in **AP**. Therefore it suffices to show that it is point-separating in order to conclude that (X, δ) is a subspace of a power of $\mathbb{P}$, so pick $x, y \in X$ with $x \neq y$. Assume without loss of generality that $\varphi(y) > 0$ for some $\varphi \in \mathcal{A}(x)$. Then automatically $\delta(x, \{y\}) \geq \varphi(y) > 0$ and $\delta(y, \{y\}) = 0$. The implication (2)⇒(1) is obvious since the concrete bicoreflector from **AP** onto **TOP** preserves products and subspaces, because $([0, \infty], \mathcal{T}_{\delta_\mathbb{P}}) \in |\mathbf{TOP}_0|$ and because the latter is an epireflective subconstruct of **TOP**. The implication (1) ⇒(4) is proved using the implication (1) ⇒(3) because for all $x \in X$, $\delta(\cdot, \{x\}) \in$

$\mathcal{R}$. To prove the converse one, take $x, y \in X$, $x \neq y$ and assume without loss of generality that $\gamma(x) > \alpha > \gamma(y)$ for some $\gamma \in \mathcal{R}, \alpha \in \mathbb{R}^+$. Because $\gamma : (X, \mathcal{T}_\delta) \longrightarrow \mathbb{P}$ is lower semicontinuous, $\{\gamma > \alpha\}$ is a $\mathcal{T}_\delta$-neighbourhoood of x not containing y. Furthermore, it is clear that (1) implies (5), whereas the converse implication follows by contraposition, because if $x, y \in X$ were distinct points such that all neighbourhoods of x contain y and vice versa, $f : I_2 \longrightarrow (X, \delta)$ defined by $f(0) := x, f(1) := y$ would be a non-constant contraction. The implication (3) $\Rightarrow$(6) is obvious and we finish with the implication (6) $\Rightarrow$ (4). Take $x, y \in X$ with $x \neq y$. According to (6), we can assume without loss of generality that there exists $\varphi \in \mathcal{A}(x) \setminus \mathcal{A}(y)$. Therefore, it follows from the transition formula (distance $\longrightarrow$ approach system) that there exists $A \in 2^X$ with $\inf_{z \in A} \varphi(z) > \delta(y, A)$, whence automatically $\delta(x, A) > \delta(y, A)$ and because $\delta(\cdot, A) \in \mathcal{R}$, we are done.

□

This shows that the T_0-property in **AP** is in fact completely topological and it is proved in the next proposition that, again like in the topological case, the corresponding epireflection arrows are obtained as quotients.

Proposition 2.3. $\mathbf{AP}_0$ *is an epireflective subconstruct of* **AP**. *For any* $(X, \delta) \in |\mathbf{AP}|$, *we define an equivalence relation* $\sim$ *on* X *by*

$$x \sim y \Leftrightarrow (\forall \varphi \in \mathcal{R} : \varphi(x) = \varphi(y)) \Leftrightarrow \mathcal{A}(x) = \mathcal{A}(y).$$

Then the **AP**-*quotient of* (X, δ) *with respect to* $\sim$ *gives us is an* $\mathbf{AP}_0$-*epireflection arrow for* (X, δ).

Proof. For every $x \in X$, we write $\overline{x}$ for the corresponding equivalence class w.r.t. $\sim$ and we denote the corresponding projection by $\pi : X \longrightarrow X/\sim : x \mapsto \overline{x}$. By definition of $\sim$, it is obvious that for each $\gamma \in \mathcal{R}$, the map $\overline{\gamma} : X/\sim \longrightarrow [0, \infty] : \overline{x} \mapsto \gamma(x)$ is well-defined. From the descripiton of quotients in **AP**, it is now clear that the final regular function frame on $X/\sim$ with respect to $\pi : (X, \mathcal{R}) \longrightarrow X/\sim$ is exactly

$$\mathcal{R}/\sim := \{\varphi \in [0, \infty]^{X/\sim} | \varphi \circ \pi \in \mathcal{R}\} = \{\overline{\gamma} | \gamma \in \mathcal{R}\}.$$

Then clearly $(X/\sim, \mathcal{R}/\sim) \in |\mathbf{AP}_0|$ and for every $(X', \mathcal{R}') \in |\mathbf{AP}_0|$ and $f \in \mathbf{AP}((X, \mathcal{R}), (X', \mathcal{R}'))$, it is clear that

$$\overline{f} : (X/\sim, \mathcal{R}/\sim) \longrightarrow (X', \mathcal{R}') : \overline{x} \mapsto f(x)$$

is a well-defined contraction, being the unique one such that $f = \overline{f} \circ \pi$. □

The corresponding epireflector is denoted by $\mathrm{T}_0 : \mathbf{AP} \longrightarrow \mathbf{AP}_0$.

Remark 2.4. **AP** is universal, i.e. it is the bireflective hull of $\mathbf{AP}_0$ in **AP**. Moreover, every epireflector from **AP** onto one of its subconstructs is either a bireflector or the composition of a bireflector, followed by the $\mathbf{AP}_0-$epireflector.

Proof. The universality follows immediately from the result, proved in [8], that for each $(X, \delta) \in |\mathbf{AP}|$, the source

$$(\delta(\cdot, A) : (X, \delta) \longrightarrow \mathbb{P})_{A \in 2^X}$$

is initial in **AP**, because $\mathbb{P} \in |\mathbf{AP}_0|$. The second part is proved in [10]. □

2.2. The T_1-axiom.

Remark 2.5. For every $(X, \delta) \in |\mathbf{AP}|$, the following assertions are equivalent:

(1) $(X, \mathcal{T}_\delta) \in |\mathbf{TOP}_1|$,

(2) $\forall x, y \in X : x \neq y \Rightarrow (\exists \varphi, \psi \in \mathcal{R} : (\varphi(x) < \varphi(y)) \wedge (\psi(y) < \psi(x))$,

(3) $\forall x, y \in X : x \neq y \Rightarrow$

$$((\exists \varphi \in \mathcal{A}(x) : \varphi(y) > 0) \wedge (\exists \psi \in \mathcal{A}(y) : \psi(x) > 0)).$$

(4) $\forall x, y \in X : x \neq y \Rightarrow ((\mathcal{A}(x) \not\subset \mathcal{A}(y)) \wedge (\mathcal{A}(y) \not\subset \mathcal{A}(x)))$.

Proof. This is proved in the same way as 2.2 . □

Comparing this remark with the way T_1-objects are defined in **TOP** and the characterizations of T_0-objects in **AP**, yields that the following definition is plausible:

Definition 2.6. We call an approach space T_1 if it satisfies the equivalent statements from 2.5. We define $\mathbf{AP}_1$ to be the full subconstruct of **AP** defined by all T_1 approach spaces.

Corollary 2.7. $\mathbf{AP}_1$ *is an epireflective subconstruct of* **AP***.*

Next we want to give an internal description of the corresponding epireflector from **AP** onto $\mathbf{AP}_0$. It is well-known that a topological space $(X, \mathcal{T})$ is T_1 if and only if it is both T_0 and symmetric in the sense of [3], meaning that

$$\forall x, y \in X : x \in \mathrm{cl}(\{y\}) \Leftrightarrow y \in \mathrm{cl}(\{x\}).$$

This, together with remark 2.4 above motivates the following line of working:

Definition 2.8. An approach space (X, δ) is called R_0 if it satisfies the condition

$$\forall x \in X : \mathcal{A}(x) = \bigcap_{y : \delta(y, \{x\}) = 0} \mathcal{A}(y).$$

The full subconstruct of **AP** formed by all R_0-objects is denoted by $\mathbf{AP}_{R_0}$.

Constructing the T_1−epireflector will carry us outside of **AP**, into the superconstruct **PRAP** of pre-approach spaces and contractions, as introduced in [9]. Let us only recall that a pre-approach space is a pair (X, δ) with $\delta : X \times 2^X \longrightarrow [0, \infty]$ satisfying (D1), (D2) and (D3) (such δ is called a pre-distance) and contractions are defined in the same way as above. Just as in the approach case, a pre-approach distance δ can be equivalently characterized by resp. a pre-approach system $\mathcal{A}$, a pre-hull h and a pre-approach limit λ. For details we refer to [9], but we note that, just as for distances, stepping from **AP** to **PRAP** comes down to dropping the triangular axiom for $\mathcal{A}$ and λ and the idempotency for h. It was proved in [9] that **AP** is a concretely bireflective subconstruct of **PRAP**. Take $(X, h) \in |\mathbf{PRAP}|$. If $\gamma \in [0,\infty]^X$, define $h^0(\gamma) := \gamma$ and for each ordinal $\alpha \geq 1$,

$$h^\alpha(\gamma) := \begin{cases} h(h^{\alpha-1}(\gamma)) & \alpha \text{ not a limit ordinal} \\ \bigwedge_{\beta<\alpha} h^\beta(\gamma) & \alpha \text{ limit ordinal.} \end{cases}$$

Then there exists some ordinal κ such that $h^\kappa(\gamma) = h^{\kappa+1}(\gamma)$ for all $\gamma \in [0,\infty]^X$ and we put $h_*(\gamma) := h^\kappa(\gamma)$ for each $\gamma \in [0,\infty]^X$. Then it can be proved that $h_* : [0,\infty]^X \longrightarrow [0,\infty]^X$ is a hull on X and $\mathrm{id}_X : (X, h) \longrightarrow (X, h_*)$ is the reflection arrow. We will write $\mathrm{D} : \mathbf{PRAP} \longrightarrow \mathbf{AP}$ for the corncrete reflector and to simplify notations, we will write $(X, \mathrm{D}(\delta))$ instead of $\mathrm{D}((X,\delta))$ for $(X,\delta) \in |\mathbf{PRAP}|$, and analoguously for the associated pre-approach systems, pre-hulls and pre-approach limits. First note that obviously, $|\mathbf{AP}_1| = |\mathbf{AP}_{R_0}| \cap |\mathbf{AP}_0|$. We will first show that $\mathbf{AP}_{R_0}$ is a concretely bireflective subconstruct of **AP**, yielding at once a description of the concrete bireflector $\mathrm{R}_0 : \mathbf{AP} \longrightarrow \mathbf{AP}_{R_0}$.

If $(X, \mathcal{A}) \in |\mathbf{AP}|$, we define a relation $\sim_{R_0}$ on X as follows:

$$x \sim_{R_0} y \Leftrightarrow \delta(x, \{y\}) = 0.$$

If we put

$$\mathcal{A}_*(x) := \bigcap_{y \sim_{R_0} x} \mathcal{A}(y)$$

for all $x \in X$, and $\mathcal{A}_* := (\mathcal{A}_*(x))_{x\in X}$, then $(X, \mathcal{A}_*) \in |\mathbf{PRAP}|$. Fix $(X, \mathcal{A}) \in |\mathbf{AP}|$. Put $\mathcal{A}^0 := \mathcal{A}$ and for every ordinal $\alpha \geq 1$, define

$$\mathcal{B}^\alpha(x) := \begin{cases} (\mathcal{A}^{\alpha-1})_*(x) & \alpha \text{ not a limit ordinal,} \\ \bigcap_{\beta<\alpha} \mathcal{A}^\beta(x) & \alpha \text{ a limit ordinal} \end{cases}, \quad x \in X$$

(note that $(X, \mathcal{B}^\alpha := (\mathcal{B}^\alpha(x))_{x\in X}) \in |\mathbf{PRAP}|$) and define $\mathcal{A}^\alpha := \mathrm{D}(\mathcal{B}^\alpha)$, whence $(X, \mathcal{A}^\alpha) \in |\mathbf{AP}|$.

Proposition 2.9. *For every $(X, \mathcal{A}) \in |\mathbf{AP}|$, there exists an ordinal κ for which $\mathcal{A}^\kappa = \mathcal{A}^{\kappa+1}$. If we denote $\mathcal{A}_{R_0} := \mathcal{A}^\kappa$, $(X, \mathcal{A}_{R_0}) \in |\mathbf{AP}_{R_0}|$ and $\mathrm{id}_X : (X, \mathcal{A}) \longrightarrow (X, \mathcal{A}_{R_0})$ is the $\mathbf{AP}_{R_0}-$bireflection arrow.*

Proof. Pick $(X, \mathcal{A}) \in |\mathbf{AP}|$. Since for all ordinals $\beta < \alpha$, $\mathcal{A}^\beta \supset \mathcal{A}^\alpha$ and hence $0 \leq \delta_{\mathcal{A}^\alpha} \leq \delta_{\mathcal{A}^\beta}$, it is clear that $\mathcal{A}^{\kappa+1} = \mathcal{A}^\kappa$, if we take a fixed ordinal $\kappa > \mathrm{card}([0,\infty]^{X\times 2^X})$. Then define $\mathcal{A}_{R_0} := \mathcal{A}^\kappa$. By construction, it is obvious that $(X, \mathcal{A}_{R_0}) \in |\mathbf{AP}_{R_0}|$. Now fix $(X', \mathcal{A}') \in |\mathbf{AP}_{R_0}|$ and $f \in \mathbf{AP}((X, \mathcal{A}), (X', \mathcal{A}'))$. Then by definition, $f \in \mathbf{AP}((X, \mathcal{A}^0), (X', \mathcal{A}'))$. Now assume that

$$f \in \mathbf{AP}((X, \mathcal{A}^{\alpha-1}), (X', \mathcal{A}'))$$

for some non-limit ordinal α. In order to verify that

$$f \in \mathbf{PRAP}((X, \mathcal{B}^\alpha), (X', \mathcal{A}')),$$

pick $x \in X$ and $\varphi' \in \mathcal{A}'(f(x))$. We should prove that $\varphi' \circ f \in \mathcal{B}^\alpha(x)$, so let $y \in X$ such that $\delta_{\mathcal{A}^{\alpha-1}}(y, \{x\}) = 0$. Then automatically $\delta'(f(y), \{f(x)\}) = 0$, so because $(X', \mathcal{A}') \in |\mathbf{AP}_{R_0}|$,

$$\varphi' \in \bigcap_{z\in X' : \delta'(z,\{f(x)\})=0} \mathcal{A}'(z) \subset \mathcal{A}'(f(y)),$$

whence $\varphi' \circ f \in \mathcal{A}^{\alpha-1}(y)$. Since D is a concrete bireflector, it now immediately follows that $f \in \mathbf{AP}((X, \mathcal{A}^{\alpha}), (X', \mathcal{A}'))$. Next take a limit ordinal α and assume that $f \in \mathbf{AP}((X, \mathcal{A}^{\beta}), (X', \mathcal{A}'))$ for all $\beta < \alpha$. It then trivially follows that $f \in \mathbf{AP}((X, \mathcal{A}^{\alpha}), (X', \mathcal{A}'))$. By transfinite induction, it now follows in particular that $f \in \mathbf{AP}((X, \mathcal{A}_{R_0}), (X', \mathcal{A}'))$ so we are done. □

For $(X, \delta) \in |\mathbf{AP}|$, we will also use the notation $(X, \mathrm{R}_0(\delta))$ for $\mathrm{R}_0((X, \delta))$ and the same convention applies for the other equivalent axiomizations for approach spaces.

Proposition 2.10. *For every $(X, \mathcal{R}) \in |\mathbf{AP}|$, the $\mathbf{AP}_1$-epireflection is obtained by taking the $\mathbf{AP}_0$-epireflection of its $\mathbf{AP}_{R_0}-$bireflection, i.e. the corresponding $\mathbf{AP}_1$-epireflection arrow is given by*

$$\pi : (X, \mathcal{R}) \longrightarrow (X/\sim, \mathrm{R}_0(\mathcal{R})/\sim),$$

(where $\sim$ and π are determined by $\mathrm{R}_0(\mathcal{R})$.)

Proof. It suffices to verify that $(X/\sim, \mathrm{R}_0(\mathcal{R})/\sim) \in |\mathbf{AP}_1|$. First note that, with the notations as in 2.3, for all $x, y \in X$

$$\delta_{\mathrm{R}_0(\mathcal{R})/\sim}(\overline{x}, \{\overline{y}\}) = \sup_{\gamma \in \mathrm{R}_0(\mathcal{R}), \overline{\gamma}(\overline{y})=0} \overline{\gamma}(\overline{x}) = \sup_{\gamma \in \mathrm{R}_0(\mathcal{R}), \gamma(y)=0} \gamma(x) = \delta_{\mathrm{R}_0(\mathcal{R})}(x, \{y\}).$$

Because $(X, \mathrm{R}_0(\mathcal{R})) \in |\mathbf{AP}_{R_0}|$, this implies that

$$\forall x, y \in X : \delta_{\mathrm{R}_0(\mathcal{R})/\sim}(\overline{x}, \{\overline{y}\}) = 0 \Rightarrow \delta_{\mathrm{R}_0(\mathcal{R})/\sim}(\overline{y}, \{\overline{x}\}) = 0,$$

i.e. that $(X/\sim, \mathcal{T}_{\mathrm{R}_0(\mathcal{R})/\sim})$ is a symmetric space in the sense of [3] (called an R_0- space there) and since it also belongs to $|\mathbf{TOP}_0|$, it belongs to $|\mathbf{TOP}_1|$ and we are done. □

2.3. **The T_2-axiom.** If X is a set and $\mathfrak{F} \subset [0, \infty]^X$, we call

$$\langle \mathfrak{F} \rangle := \{\gamma \in [0, \infty]^X \mid \forall \varepsilon > 0, \forall M < \infty : \exists \gamma^{\varepsilon M} \in \mathfrak{F} : \gamma \wedge M \leq \gamma^{\varepsilon M} + \varepsilon\},$$

resp. $c(\mathfrak{F}) := \sup_{\gamma \in \mathfrak{F}} \inf_{x \in X} \gamma(x)$, the saturation, resp. the level of $\mathfrak{F}$. If $(X, \mathcal{A}) \in |\mathbf{AP}|$ and $x, y \in X$, then $\mathcal{A}(x) \vee \mathcal{A}(y) := \langle \mathcal{A}(x) \cup \mathcal{A}(y) \rangle$ is the supremum of $\mathcal{A}(x)$ and $\mathcal{A}(y)$ in the lattice of all saturated ideals in $[0, \infty]^X$.

Remark 2.11. If $(X, \delta) \in |\mathbf{AP}|$, the following assertions are equivalent:

(1) $\forall x, y \in X : x \neq y \Rightarrow c(\mathcal{A}(x) \vee \mathcal{A}(y)) > 0$,

(2) $\forall x, y \in X : x \neq y \Rightarrow (\exists \varphi \in \mathcal{A}(x), \exists \psi \in \mathcal{A}(y) : \inf_{s \in X}(\varphi \vee \psi)(s) > 0$,

(3) $(X, \mathcal{T}_\delta) \in |\mathbf{TOP}_2|$.

Proof. Obvious since $(\{\{\varphi < \varepsilon\} | \varepsilon > 0, \varphi \in \mathcal{A}(x)\})_{x \in X}$ is a base for the $\mathcal{T}_\delta$-neigbourhood system. □

Again, a comparison with the classical topological situation and taking into account that the role of neighbourhood filters in topology is played by the so-called approach systems in approach theory, makes it plausible to define Hausdorff objects in **AP** in the following, again topological, way:

Definition 2.12. We call an approach space T_2 if it satisfies the equivalent statements from 2.11. We define $\mathbf{AP}_2$ to be the full subconstruct of $\mathbf{AP}$ defined by all T_2 approach spaces.

Corollary 2.13. $\mathbf{AP}_2$ *is an epireflective subconstruct of* $\mathbf{AP}$.

As an answer to a question raised by H. Herrlich, an internal description of the epireflector from $\mathbf{TOP}$ onto $\mathbf{TOP}_2$ was given by V. Kannan in [5], making use of a transfinite construction. We will now derive an explicit description for the epireflector from $\mathbf{AP}$ onto $\mathbf{AP}_2$, along the same lines as was done for the T_1-case in the section above. First we define a property R for approach spaces, which is inspired by the notion of reciprocity for convergence spaces, as defined in [2].

If $(X, \mathcal{A}) \in |\mathbf{AP}|$, we define a relation $\sim_R$ on X by

$$\begin{aligned} & x \sim_R y \\ & \Leftrightarrow (\exists x_1 := x, \ldots, x_n := y : \forall i \in \{1, \ldots, n-1\} : c(\mathcal{A}(x_i) \vee \mathcal{A}(x_{i+1})) = 0). \end{aligned}$$

Definition 2.14. We call $(X, \delta) \in |\mathbf{AP}|$ an R- space if it fulfills the following condition

$$\forall x \in X : \mathcal{A}(x) = \bigcap_{y \sim_R x} \mathcal{A}(y)$$

and we denote the full subconstruct of $\mathbf{AP}$ formed by all R-spaces by $\mathbf{AP}_R$.

First note that $|\mathbf{AP}_2| = |\mathbf{AP}_0| \cap |\mathbf{AP}_R|$. To begin with, we will prove that $\mathbf{AP}_R$ is a bireflective subconstruct of $\mathbf{AP}$ by describing the bireflector $\mathrm{R} : \mathbf{AP} \longrightarrow \mathbf{AP}_R$. Let $(X, \mathcal{A}) \in |\mathbf{AP}|$. If we put

$$\mathcal{A}_\dagger(x) := \bigcap_{y \sim_R x} \mathcal{A}(y)$$

for all $x \in X$, and $\mathcal{A}_\dagger := (\mathcal{A}_\dagger(x))_{x \in X}$, then $(X, \mathcal{A}_\dagger) \in |\mathbf{PRAP}|$. Fix $(X, \mathcal{A}) \in |\mathbf{AP}|$. Put $\mathcal{A}^0 := \mathcal{A}$ and for every ordinal $\alpha \geq 1$, define

$$\mathcal{B}^\alpha(x) := \left\{ \begin{array}{ll} (\mathcal{A}^{\alpha-1})_\dagger(x) & \alpha \text{ not a limit ordinal,} \\ \bigcap_{\beta < \alpha} \mathcal{A}^\beta(x) & \alpha \text{ a limit ordinal} \end{array} \right. , \quad x \in X$$

(note that $(X, \mathcal{B}^\alpha := (\mathcal{B}^\alpha(x))_{x \in X}) \in |\mathbf{PRAP}|$ and define $\mathcal{A}^\alpha := \mathrm{D}(\mathcal{B}^\alpha)$, whence $(X, \mathcal{A}^\alpha) \in |\mathbf{AP}|$.

Proposition 2.15. *For all* $(X, \mathcal{A}) \in |\mathbf{AP}|$, *there exists an ordinal* κ *for which* $\mathcal{A}^{\kappa+1} = \mathcal{A}^\kappa$. *If we denote* $\mathcal{A}_R := \mathcal{A}^\kappa$, $(X, \mathcal{A}_R) \in |\mathbf{AP}_R|$ *and* $id_X : (X, \mathcal{A}) \longrightarrow (X, \mathcal{A}_R)$ *is the* $\mathbf{AP}_R$*−bireflection arrow.*

Proof. The proof is exactly the same as that of 2.9 except for verifying that for $(X', \mathcal{A}') \in |\mathbf{AP}_R|$ and $f \in \mathbf{AP}((X, \mathcal{A}), (X', \mathcal{A}'))$, if we assume that $f \in \mathbf{AP}((X, \mathcal{A}^{\alpha-1}), (X', \mathcal{A}'))$ for some non-limit ordinal α (*), it follows that $f \in \mathbf{PRAP}((X, \mathcal{B}^\alpha), (X', \mathcal{A}'))$. Therefore, pick $x \in X$ and $\varphi' \in \mathcal{A}'(f(x))$. Assume that $y \in X$ and $x_1 := y, \ldots, x_n := x$ such that $c(\mathcal{A}^{\alpha-1}(x_i) \vee \mathcal{A}^{\alpha-1}(x_{i+1})) = 0$

for all $i \in \{1, \ldots, n-1\}$. Then it follows from our assumption (*) that for all $i \in \{1, \ldots, n-1\}$

$$c(\mathcal{A}'(f(x_i)) \vee \mathcal{A}'(f(x_{i+1}))) = \sup_{\gamma \in \mathcal{A}'(f(x_i))} \sup_{\mu \in \mathcal{A}'(f(x_{i+1}))} \inf_{z' \in X'} (\gamma \vee \mu)(z')$$

$$\leq \sup_{\gamma \in \mathcal{A}'(f(x_i))} \sup_{\mu \in \mathcal{A}'(f(x_{i+1}))} \inf_{z \in X} (\gamma \vee \mu)(f(z)) \leq c(\mathcal{A}^{\alpha-1}(x_i) \vee \mathcal{A}^{\alpha-1}(x_{i+1})) = 0,$$

showing that $f(y) \sim_R f(x)$. Because $(X', \mathcal{A}') \in |\mathbf{AP}_R|$, this implies that $\varphi' \in \mathcal{A}'(f(y))$ whence $\varphi' \circ f \in \mathcal{A}^{\alpha-1}(y)$ and this completes the proof. □

For $(X, \delta) \in |\mathbf{AP}|$, we will also use the notation $(X, \mathrm{R}(\delta))$ for $\mathrm{R}((X, \delta))$ and the same convention applies for the other equivalent axiomizations for approach spaces.

Proposition 2.16. *For every $(X, \mathcal{R}) \in |\mathbf{AP}|$, the $\mathbf{AP}_2$-epireflection is obtained by taking the $\mathbf{AP}_0$-epireflection of its $\mathbf{AP}_R-$bireflection, i.e. the corresponding $\mathbf{AP}_2$-epireflection arrow is given by*

$$\pi : (X, \mathcal{R}) \longrightarrow (X/\sim, \mathrm{R}(\mathcal{R})/\sim),$$

(where $\sim$ and π are detremined by $\mathrm{R}(\mathcal{R})$.)

Proof. We only need to check that $(X/\sim, \mathrm{R}(\mathcal{R})/\sim) \in |\mathbf{AP}_2|$. Now assume that $x, y \in X$ for which every $\mathcal{T}_{\mathrm{R}(\mathcal{R})/\sim}$-neighbourhood of $\overline{x}$ meets every $\mathcal{T}_{\mathrm{R}(\mathcal{R})/\sim}$-neighbourhood of $\overline{y}$. Note that $\mathcal{T}_{\mathrm{R}(\mathcal{R})} = \{\{\mu > 0\} \mid \mu \in \mathrm{R}(\mathcal{R})\}$ and that with the notation introduced in 2.3 $\mathrm{R}(\mathcal{R})/\sim = \{\overline{\gamma} | \gamma \in \mathrm{R}(\mathcal{R})\}$. This yields that $x \sim_R y$ where the $\sim_R$-relation is taken with respect to $\mathrm{R}(\mathcal{R})$, and because $(X, \mathrm{R}(\mathcal{R})) \in |\mathbf{AP}_R|$, it follows that $\mathcal{A}_{\mathrm{R}(\mathcal{R})}(x) = \mathcal{A}_{\mathrm{R}(\mathcal{R})}(y)$, whence $\overline{x} = \overline{y}$.

□

2.4. **Regularity.** In [11], three different suggestions for regularity were proposed, and in [5], it was motivated that the strongest one of them is the correct notion of regularity in the construct **AP**. We simply recall this definition for the sake of completeness. For any set X, let $\mathbf{F}(X)$ stand for the set of all filters on X and for each $\mathcal{F} \in \mathbf{F}(X)$ and $\varepsilon \geq 0$, let $\mathcal{F}^{(\varepsilon)}$ denote the filter generated by $\{F^{(\varepsilon)} \mid F \in \mathcal{F}\}$.

Definition 2.17. [11], [5] An approach space (X, δ) is called regular if

$$\forall \varepsilon \geq 0, \forall \mathcal{F} \in \mathbf{F}(X) : \lambda(\mathcal{F}^{(\varepsilon)}) \leq \lambda(\mathcal{F}) + \varepsilon.$$

We write **RAP** for the full subconstruct of **AP** formed by all regular spaces, and it was proved in [11] that it moreover is concretely bireflective. For some other equivalent characterizations of regularity in terms of distances and approach systems, we refer to [11].

Note that, where the lower separation axioms we discussed in **AP** all turn out to be topological in the sense that an approach space (X, δ) is T_i if and only if its topological coreflection in T_i in the classical sense (with $i \in \{0, 1, 2\}$), the regularity condition stated above is of a purely quantitative nature. This was already noted in [11], where it is shown that

$$\delta(x,A) := \begin{cases} \infty & A = \varnothing \\ 0 & x \in A \\ 1 & x \notin A, A \text{ infinite} \\ 2 & \text{other cases} \end{cases} \qquad x \in \mathbb{R}, A \subset \mathbb{R}$$

defines an approach distance on $\mathbb{R}$ such that $(\mathbb{R},\delta) \notin |\mathbf{RAP}|$ but $(\mathbb{R},\mathcal{T}_\delta)$ is a regular topological space. It was also already stated in [11] that for topological spaces, this notion of regularity is equivalent to the classical one.

2.5. **Complete Regularity.** We recall the definition of uniform approach spaces from [8].

Definition 2.18. An approach space (X,δ) is called uniform if and only if there exists a collection $\mathcal{D}$ of ∞p-metrics on X which is closed with respect to taking finite suprema and such that

$$\forall x \in X, \forall A \in 2^X : \delta(x,A) = \sup_{d \in \mathcal{D}} \delta_d(x,A).$$

The full subconstruct of **AP** formed by all uniform approach spaces is denoted by **UAP** and it can be shown (see e.g. [8]) that $\mathbf{UAP} = \mathcal{E}_{\mathbf{AP}}(p\mathbf{MET}^\infty)$. It was also proved in [8] that for every $(X,\delta) \in |\mathbf{AP}|$, the corresponding **UAP**−epireflection arrow, which in fact is a concrete bireflection arrow, is given by

$$\mathrm{id}_X : (X,\delta) \longrightarrow (X, \delta^u := \sup_{d \in \mathcal{G}^s(\delta)} \delta_d),$$

with

$$\mathcal{G}^s(\delta) := \{d | d \ \infty p - \text{metric on } X, \delta_d \leq \delta\}.$$

The following proposition shows that 'being uniform' is precisely the correct quantified generalization of complete regularity to the approach setting and it was proved in [8] that for topological objects, these two notions are equivalent.

Proposition 2.19. *For every* $(X,\delta) \in |\mathbf{AP}|$*, the following assertions are equivalent:*

(1) $(X,\delta) \in |\mathbf{UAP}|$,

(2) $\forall x \in X, \forall A \in 2^X, \forall \varepsilon > 0, \forall \omega < \infty : \exists f_\varepsilon^\omega \in \mathbf{AP}((X,\delta),\mathbb{R}) :$

$$f_\varepsilon^\omega(x) = 0 \text{ and } \forall z \in A : f_\varepsilon^\omega(z) + \varepsilon \geq \delta(x,A) \wedge \omega.$$

(3) $\forall x \in X, \forall A \in 2^X, \forall \varepsilon > 0, \forall \omega < \infty : \exists f_\varepsilon^\omega \in \mathbf{AP}((X,\delta),\mathbb{R})$ *bounded* :

$$f_\varepsilon^\omega(x) = 0 \text{ and } \forall z \in A : f_\varepsilon^\omega(z) + \varepsilon \geq \delta(x,A) \wedge \omega.$$

Proof. We first show that (1) implies (2). Therefore let $\mathcal{D}$ be a collection of ∞p-metrics on X which is closed w.r.t. the formation of finite suprema and such that $\delta = \sup_{d \in \mathcal{D}} \delta_d$ and fix $x \in X, A \in 2^X, \varepsilon > 0$ and $\omega < \infty$. Then we can find $d_\varepsilon^\omega \in \mathcal{D}$ with

$$\delta_{d_\varepsilon^\omega}(x,A) + \varepsilon > \delta(x,A) \wedge \omega$$

and because $f_\varepsilon^\omega := d_\varepsilon^\omega(x,\cdot) \in \mathbf{AP}((X,\delta),\mathbb{R})$, we are done. That (2) implies (3) is clear since for every $f \in \mathbf{AP}((X,\delta),\mathbb{R})$ and $\omega < \infty$, obviously $|f| \wedge \omega \in \mathbf{AP}((X,\delta),\mathbb{R})$. Finally we show that (3) implies (1). In order to verify that $(X,\delta) \in |\mathbf{UAP}|$, it sufices to show that $\delta \leq \delta^u$, so fix $x \in X, A \in 2^X, \varepsilon > 0$ and $\omega < \infty$. According to (3), we can find $f_\varepsilon^\omega \in \mathbf{AP}((X,\delta),\mathbb{R})$ bounded such that $f_\varepsilon^\omega(x) = 0$ and such that $f_\varepsilon^\omega(a) + \varepsilon \geq \delta(x,A) \wedge \omega$ for each $a \in A$. If we denote the Euclidean metric on $\mathbb{R}$ by d_E, it is clear that $d_E \circ (f_\varepsilon^\omega \times f_\varepsilon^\omega)$ is an ∞p-metric on X, which belongs to $\mathcal{G}^s(\delta)$ because f_ε^ω is a contraction. It then is clear that

$$\delta^u(x,A) + \varepsilon \geq \inf_{a\in A} |f_\varepsilon^\omega(x) - f_\varepsilon^\omega(a)| + \varepsilon \geq \delta(x,A) \wedge \omega,$$

completing the proof.

□

To see that, like regularity, this is a numerical, non-topological separation axiom, note that

$$d(x,y) := \begin{cases} |x-y|/2 & x \leq y \\ |x-y| & x > y \end{cases} \qquad x,y \in \mathbb{R}$$

defines a pseudo-quasi-metric on $\mathbb{R}$, such that $(\mathbb{R}, \mathcal{T}_d)$ is a completely regular topological space but $(X,\delta_d) \notin |\mathbf{UAP}|$.

References

[1] Adámek, J. Herrlich, H. and Strecker, G. *Abstract and Concrete Categories*, John Wiley, New York (1990)

[2] Bentley, H. L., Herrlich H. and Lowen-Colebunders E. *Convergence* , J. Pure Appl. Alg. **68** (1990), pp. 27-45

[3] Herrlich, H. *Topological Structures*, Math. Centre Tracts **52** (1974), pp. 59-122

[4] Herrlich, H. *Categorical Topology 1971-1981*, Proc. 5th Prague Top. Symp. 1981, Heldermann Verlag Berlin (1983), pp. 279-383

[5] Brock, P and Kent, D. *On Convergence Approach Spaces*, Appl. Cat. Struct. **6** (1998), pp. 117-125

[6] Kannan, V. *On a Problem of Herrlich*, J. Madurai. Univ. **6** (1976), pp. 101-104

[7] Lowen, R. *Approach Spaces: a Common Supercategory of* **TOP** *and* **MET**, Math. Nachr. **141** (1989), pp. 183-226

[8] Lowen, R. *Approach Spaces: the Missing Link in the Topology-Uniformity-Metric Triad*, Oxford Mathematical Monographs, Oxford University Press (1997)

[9] Lowen, E and Lowen, R. *A Quasitopos Containing* **CONV** *and* **MET** *as full subcategories* , Int. J. Math. and Math. Sci., **11**(3) (1988), pp. 417-438

[10] Marny, T. *On Epireflective Subcategories of Topological Categories*, Gen. Top. Appl. **10** (1979), pp. 175-181

[11] Robeys, K., *Extensions of Products of Metric Spaces*, PhD Thesis Universiteit Antwerpen, RUCA (1992)

Received November 2001
Revised August 2002

R. Lowen
University of Antwerp, RUCA, Department of Mathematics and Computer Science, Middelheimlaan 1, B2020 Antwerp, Belgium

E-mail address: rlow@ruca.ua.ac.be

M. Sioen
Free University of Brussels, Department of Mathematics, Pleinlaan 2, B1000 Brussels, Belgium

E-mail address: msioen@vub.ac.be