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Star-Hurewicz and related properties

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Abstract. We continue the investigation of star selection principles first considered in [9]. We are concentrated onto star versions of the Hurewicz covering property and star selection principles related to the classes of open covers which have been recently introduced.

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1. Introduction

A number of the results in the literature show that many topological properties can be described and characterized in terms of star covering properties (see [3], [13], [2], [12]). The method of stars has been used to study the problem of metrization of topological spaces, and for definitions of several important classical topological notions. We use here such a method in investigation of selection principles for topological spaces.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a topological space X.

The symbol $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\} \in \mathcal{B}$ [18].

The symbol $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$ [18].

In [9], Kočinac introduced star selection principles in the following way.

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Definition 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X. Then:*

(a) *The symbol $\mathsf{S}_1^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{St(U_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$;*

(b) *The symbol $\mathsf{S}_{fin}^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}}\{St(V,\mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$;*

(c) *By $\mathsf{U}_{fin}^*(\mathcal{A},\mathcal{B})$ we denote the selection hypothesis: for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of members of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{St(\cup\mathcal{V}_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$ or there is some $n \in \mathbb{N}$ such that $St(\cup\mathcal{V}_n,\mathcal{U}_n) = X$.*

Definition 1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X and let $\mathcal{K}$ be a family of subsets of X. Then we say that X belongs to the class $\mathsf{SS}_{\mathcal{K}}^*(\mathcal{A},\mathcal{B})$ if X satisfies the following selection hypothesis: for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(K_n : n \in \mathbb{N})$ of elements of $\mathcal{K}$ such that $\{St(K_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.*

When $\mathcal{K}$ is the collection of all one-point [resp., finite, compact] subspaces of X we write $\mathsf{SS}_1^*(\mathcal{A},\mathcal{B})$ [resp., $\mathsf{SS}_{fin}^*(\mathcal{A},\mathcal{B})$, $\mathsf{SS}_{comp}^*(\mathcal{A},\mathcal{B})$] instead of $\mathsf{SS}_{\mathcal{K}}^*(\mathcal{A},\mathcal{B})$.

Here, as usual, for a subset A of a space X and a collection $\mathcal{P}$ of subsets of X, $\mathrm{St}(A,\mathcal{P})$ denotes the star of A with respect to $\mathcal{P}$, that is the set $\cup\{P \in \mathcal{P} : A \cap P \neq \varnothing\}$; for $A = \{x\}$, $x \in X$, we write $\mathrm{St}(x,\mathcal{P})$ instead of $St(\{x\},\mathcal{P})$.

In [10] it was explained that selection principles in uniform spaces are actually a kind of star selection principles.

Let X be a space. If $\mathcal{U}$ and $\mathcal{V}$ are families of subsets of X we denote by $\mathcal{U} \wedge \mathcal{V}$ the set $\{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. The symbols $[X]^{<\omega}$ and $[X]^{\leq n}$ denote the collection of all finite subsets of X and all subsets of X having $\leq n$ elements, respectively.

In this paper all spaces will be Hausdorff. $\mathcal{A}$ and $\mathcal{B}$ will be collections of the following open covers of a space X:

$\mathcal{O}$: the collection of all open covers of X;

Ω: the collection of ω-covers of X. An open cover $\mathcal{U}$ of X is an *ω-cover* [5] if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in an element of $\mathcal{U}$;

Γ: the collection of γ-covers of X. An open cover $\mathcal{U}$ of X is a *γ-cover* [5] if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

$\mathcal{O}^{gp}$: the collection of groupable open covers. An open cover $\mathcal{U}$ of X is *groupable* [11] if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that each $x \in X$ belongs to $\cup\mathcal{U}_n$ for all but finitely many n;

$\mathcal{O}^{wgp}$: the collection of weakly groupable open covers. A cover $\mathcal{U}$ of X is a *weakly groupable* [1] if it is a countable union of finite, pairwise disjoint sets $\mathcal{U}_n$, $n \in \mathbb{N}$, such that for each finite set $F \subset X$ we have $F \subset \cup\mathcal{U}_n$ for some n.

We consider only spaces X whose each ω-cover contains a countable ω-subcover (or equivalently, for each $n \in \mathbb{N}$, every open cover of X^n has a countable subcover). Thus all considered covers are assumed to be *countable*.

Recall that a space X is said to have the *Menger property* [15], [6], [16], [8] (resp. the *Rothberger property* [17], [16], [18] if the selection hypothesis $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ (resp. $\mathsf{S}_1(\mathcal{O},\mathcal{O})$) is true for X.

The following terminology was introduced in [9]. A space X is said to have:

1. the *star-Rothberger property* SR,
2. the *star-Menger property* SM,
3. the *strongly star-Rothberger property* SSR,
4. the *strongly star-Menger property* SSM,

if it satisfies the selection hypothesis:

1. $\mathsf{S}_1^*(\mathcal{O},\mathcal{O})$,
2. $\mathsf{S}_{fin}^*(\mathcal{O},\mathcal{O})$ (or, equivalently, $\mathsf{U}_{fin}^*(\mathcal{O},\mathcal{O})$),
3. $\mathsf{SS}_1^*(\mathcal{O},\mathcal{O})$,
4. $\mathsf{SS}_{fin}^*(\mathcal{O},\mathcal{O})$,

respectively.

In 1925 in [6] (see also [7]), W. Hurewicz introduced the *Hurewicz covering property* for a space X in the following way:

H: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each n $\mathcal{V}_n \subset \mathcal{U}_n$, and for each $x \in X$, for all but finitely many n, $x \in \cup\mathcal{V}_n$.

Two star versions of this property are:

SH: A space X satisfies the *star-Hurewicz property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$ $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\mathrm{St}(\cup\mathcal{V}_n,\mathcal{U}_n)$ for all but finitely many n.

SSH: A space X satisfies the *strongly star-Hurewicz property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to $\mathrm{St}(A_n,\mathcal{U}_n)$ for all but finitely many n (i.e. if X satisfies $\mathsf{SS}_{fin}^*(\mathcal{O},\Gamma)$).

In this paper we study some properties of these spaces. We also consider SM and SSM spaces, in particular in connection with new classes of covers that appeared recently in the literature - groupable and weakly groupable covers.

2. Spaces related to SM spaces

Theorem 2.1. *If each finite power of a space X is SM, then X satisfies $\mathsf{U}_{fin}^*(\mathcal{O},\Omega)$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ be a partition of $\mathbb{N}$ into infinitely many infinite subsets. For each k and each $m \in N_k$ let $\mathcal{W}_m = \{U_1 \times \cdots \times U_k : U_1, \cdots, U_k \in \mathcal{U}_m\}$. Then

$(\mathcal{W}_m : m \in N_k)$ is a sequence of open covers of X^k, and since X^k is a star-Menger space, one can choose a sequence $(\mathcal{H}_m : m \in N_k)$ such that for each m, $\mathcal{H}_m \in [\mathcal{W}_m]^{<\omega}$ and $\bigcup_{m \in N_k}\{\mathrm{St}(H, \mathcal{W}_m) : H \in \mathcal{H}_m\}$ is an open cover of X^k. For every $m \in N_k$ and every $H \in \mathcal{H}_m$ we have $H = U_1(H) \times \cdots \times U_k(H)$, where $U_i(H) \in \mathcal{U}_m$ for every $i \leq k$. Put $\mathcal{V}_m = \{U_i(H) : i \leq k, H \in \mathcal{H}_m\}$. Then for each $m \in N_k$ $\mathcal{V}_m$ is a finite subset of $\mathcal{U}_m$. We claim that $\{\mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an ω-cover of X. Let $F = \{x_1, \cdots, x_s\}$ be a finite subset of X. Then $x = (x_1, \cdots, x_s) \in X^s$ so that there is an $n \in N_s$ such that $x \in \mathrm{St}(H, \mathcal{W}_n)$ for some $H \in \mathcal{H}_n$. But $H = U_1(H) \times \cdots \times U_s(H)$, where $U_1(H), \cdots, U_s(H) \in \mathcal{V}_n$. The point x belongs to some $W \in \mathcal{W}_n$ of the form $V_1 \times \cdots \times V_s$, $V_i \in \mathcal{U}_n$ for each $i \leq s$, which meets $U_i(H) \times \cdots \times U_s(H)$. This means that for each $i \leq s$ we have $x_i \in \mathrm{St}(U_i(H), \mathcal{U}_n) \subset \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)$, i.e. $F \subset \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)$. So, X satisfies $\mathsf{U}^*_{fin}(\mathcal{O}, \Omega)$. □

Now we shall see that the previous theorem can be given in another form.

Theorem 2.2. *For a space X the following are equivalent:*

(1) *X satisfies $\mathsf{U}^*_{fin}(\mathcal{O}, \Omega)$;*

(2) *X satisfies $\mathsf{U}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$.*

Proof. Because each countable ω-cover is weakly groupable, (1) implies (2) is trivial, so that we have to prove only (2) $\Rightarrow$ (1). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Let for each n, $\mathcal{H}_n := \bigwedge_{i \leq n} \mathcal{U}_i$. Apply (2) to the sequence $(\mathcal{H}_n; n \in \mathbb{N})$. There is a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n $\mathcal{W}_n \in [\mathcal{H}_n]^{<\omega}$ and $\{\mathrm{St}(\cup\mathcal{W}_n, \mathcal{H}_n) : n \in \mathbb{N}\}$ is a weakly groupable cover of X. There is, therefore, a sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that for each finite set F in X one has $F \subset \cup\{\mathrm{St}(\cup\mathcal{W}_i, \mathcal{H}_i) : n_k \leq i < n_{k+1}\}$ for some k. Consider the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ defined in the following way:

$\mathcal{V}_n = \bigcup_{i < n_1} \mathcal{W}_i$, for $n < n_1$;

$\mathcal{V}_n = \bigcup_{n_k \leq i < n_{k+1}} \mathcal{W}_i$, for $n_k \leq n < n_{k+1}$.

Then for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ satisfying: for each finite set $F \subset X$ there is an n such that $F \subset \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)$. This means that X satisfies $\mathsf{U}^*_{fin}(\mathcal{O}, \Omega)$. □

Problem 2.3. *Is it true that $\mathsf{S}^*_{fin}(\mathcal{O}, \Omega) = \mathsf{S}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$?*

3. Spaces related to SSM spaces

Theorem 3.1. *If all finite powers of a space X are strongly star-Menger, then X satisfies $\mathsf{SS}^*_{fin}(\mathcal{O}, \Omega)$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ be a partition of $\mathbb{N}$ into infinite (pairwise disjoint) sets. For every $k \in \mathbb{N}$ and every $m \in N_k$ let $\mathcal{W}_m = \mathcal{U}^k_m$. Then $(\mathcal{W}_m : m \in N_k)$ is a sequence of open covers of X^k. Applying to this sequence the fact that X^k is SSM we find a sequence $(A_m : m \in N_k)$ of finite subsets of X^k such that $\{\mathrm{St}(A_m, \mathcal{W}_m) : m \in N_k\}$ is an open cover of X^k. For each m take a finite set

$S_m \subset X$ such that $S_m^k \supset A_m$. We claim that $\{\mathrm{St}(S_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an ω-cover of X. Let $F = \{x_1, \cdots, x_p\}$ be a finite subset of X. Then $(x_1, \cdots, x_p) \in X^p$ so that there is $n \in N_p$ such that $(x_1, \cdots, x_p) \in \mathrm{St}(A_n, \mathcal{W}_n) \subset \mathrm{St}(S_n^p, \mathcal{W}_n)$, and consequently $F \subset \mathrm{St}(S_n, \mathcal{U}_n)$. □

The following theorem shows that spaces whose all finite powers are SSM can be also related to spaces defined in terms of weakly groupable covers.

Theorem 3.2. *For a space X the following are equivalent:*

(1) *X satisfies $\mathsf{SS}^*_{fin}(\mathcal{O}, \Omega)$;*
(2) *X satisfies $\mathsf{SS}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$.*

Proof. (1) $\Rightarrow$ (2): It is evident because countable ω-covers are weakly groupable and SS^*_{fin} is monotone in the second variable.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let for each $n \in \mathbb{N}$, $\mathcal{V}_n = \bigwedge_{i \leq n} \mathcal{U}_i$. For each n $\mathcal{V}_n$ is an open cover of X that refines $\mathcal{U}_i$ for all $i \leq n$. Apply (2) to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$. There is a sequence $(S_n : n \in \mathbb{N})$ of finite subsets of X such that $\{\mathrm{St}(S_n, \mathcal{V}_n) : n \in \mathbb{N}\}$ is a weakly groupable open cover of X. Therefore, there is a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of natural numbers such that for every finite subset F of X we have $F \subset \bigcup_{n_k \leq i < n_{k+1}} \mathrm{St}(S_i, \mathcal{V}_i)$ for some $k \in \mathbb{N}$. For each n let

$A_n = \bigcup_{i < n_1} S_i$ for $n < n_1$, and
$A_n = \bigcup_{n_k \leq i < n_{k+1}} S_i$ for $n_k \leq n < n_{k+1}$.

Then $(A_n : n \in \mathbb{N})$ is a sequence of finite subsets of X which witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X satisfies $\mathsf{SS}^*_{fin}(\mathcal{O}, \Omega)$. □

4. Star-Hurewicz spaces

It is clear that each Hurewicz space is strongly star-Hurewicz and each strongly star-Hurewicz space is star-Hurewicz.

The following proposition shows that in the class of paracompact spaces, the Hurewicz, strongly star-Hurewicz and star-Hurewicz properties are equivalent.

Proposition 4.1. *A paracompact Hausdorff space X has the star-Hurewicz property if and only if it has the Hurewicz property.*

Proof. We already noted that if a space X has the Hurewicz property then X has also the star-Hurewicz property. So let X be a paracompact space having star-Hurewicz property and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. By the Stone characterization of paracompactness [4] for every $n \in \mathbb{N}$, $\mathcal{U}_n$ has an open star-refinement, say $\mathcal{V}_n$. Since X has the star-Hurewicz property there exists a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{W}_n \in [\mathcal{V}_n]^{<\omega}$ and each $x \in X$ belongs to all but finitely many of the sets $\mathrm{St}(\cup\mathcal{W}_n, \mathcal{V}_n)$. For each $W \in \mathcal{W}_n$, let U_W be a member of $\mathcal{U}_n$ such that $\mathrm{St}(W, \mathcal{V}_n) \subset U_W$. For every $n \in \mathbb{N}$, we have that $\mathcal{H}_n := \{U_W : W \in \mathcal{W}_n\} \in [\mathcal{U}_n]^{<\omega}$. It is easily seen that the sequence $(\mathcal{H}_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X is a Hurewicz space. □

Proposition 4.2. *The product $X \times Y$ of a star-Hurewicz space X and a compact space Y is star-Hurewicz.*

Proof. Let $(\mathcal{W}_n : n \in \mathbb{N})$ be a sequence of open covers of $X \times Y$. Without loss of generality, we can assume that for every $n \in \mathbb{N}$, $\mathcal{W}_n = \mathcal{U}_n \times \mathcal{V}_n$, where $\mathcal{U}_n$ is an open cover of X and $\mathcal{V}_n$ is an open cover of Y. Fix $x \in X$. Since $\{x\} \times Y$ is a compact space and for each $n \in \mathbb{N}$ $\mathcal{W}_n$ is an open cover of $\{x\} \times Y$, there exists $(\mathcal{U}_{n,x} \times \mathcal{V}_{n,x}) \in [\mathcal{W}_n]^{<\omega}$ such that $\cup(\mathcal{U}_{n,x} \times \mathcal{V}_{n,x}) \supset \{x\} \times Y$. For every $n \in \mathbb{N}$, the set $U_{n,x} = \cap \mathcal{U}_{n,x}$ is open in X, so that the family $\mathcal{G}_n = \{U_{n,x} : x \in X\}$ is an open cover of X. Since X has the star-Hurewicz property, there exists a sequence $(\mathcal{H}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$ $\mathcal{H}_n \in [\mathcal{G}_n]^{<\omega}$ and each $x \in X$ belongs to all but finitely many sets $\{\mathrm{St}(\cup\mathcal{H}_n, \mathcal{G}_n) : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, let $\mathcal{H}_n = \{U_{n,x_1}, \cdots, U_{n,x_{k(n)}}\}$. We have that for every n, $\mathcal{K}_n := (\mathcal{U}_{n,x_1} \times \mathcal{V}_{n,x_1}) \cup \cdots \cup (\mathcal{U}_{n,x_{k(n)}} \times \mathcal{V}_{n,x_{k(n)}}) \in [\mathcal{W}_n]^{<\omega}$. The sequence $(\mathcal{K}_n : n \in \mathbb{N})$ is such that $\{\mathrm{St}(\cup\mathcal{K}_n, \mathcal{W}_n) : n \in \mathbb{N}\}$ is a γ-cover of $X \times Y$. So the space $X \times Y$ has the star-Hurewicz property. □

Let us mention that there is a SSH space X and a Lindelöf space Y such that the product $X \times Y$ is not SSH. Take $X = [0, \omega_1)$ with the usual order topology and Y to be the one-point Lindelöfication of X.

Theorem 4.3. *For a space X the following are equivalent:*

(1) *X is a star-Hurewicz space;*
(2) *X satisfies $\mathsf{U}^*_{fin}(\mathcal{O}, \mathcal{O}^{gp})$.*

Proof. Each countable γ-cover is groupable and thus (1) implies (2). Let us show (2) $\Rightarrow$ (1). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let for each n $\mathcal{G}_n$ be the open cover $\bigwedge_{i \leq n} \mathcal{U}_i$. Apply (2) to the sequence of these $\mathcal{G}_n$'s and for each $n \in \mathbb{N}$ select a finite set $\mathcal{V}_n \subset \mathcal{G}_n$ such that the set $\{\mathrm{St}(\cup\mathcal{V}_n, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a groupable cover of X. Let $n_1 < n_2 < \cdots$ be a sequence of natural numbers which witnesses the last fact: for each $x \in X$ x belongs to $\cup\{\mathrm{St}(\cup\mathcal{V}_i, \mathcal{G}_i) : n_k \leq i < n_{k+1}\}$ for all but finitely many k. As in the proof of Theorem 2.2 put

$\mathcal{W}_n = \bigcup_{i<n_1} \mathcal{V}_i$, for $n < n_1$;
$\mathcal{W}_n = \bigcup_{n_k \leq i < n_{k+1}} \mathcal{V}_i$, for $n_k \leq n < n_{k+1}$.

Then the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ shows that X satisfies (1), because, evidently, each $x \in X$ belongs to all but finitely many sets $\mathrm{St}(\cup\mathcal{W}_n, \mathcal{U}_n)$. □

Problem 4.4. *Is it true that $\mathsf{S}^*_{fin}(\mathcal{O}, \Gamma) = \mathsf{S}^*_{fin}(\mathcal{O}, \mathcal{O}^{gp})$?*

Consider now what happens if a space X satisfies the following condition closely related to the star-Hurewicz property:

$\mathrm{SH}_{\leq n}$: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n $\mathcal{V}_n \in [\mathcal{U}_n]^{\leq n}$ and the set $\{\mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a γ-cover of X.

The answer to this question is given by the next theorem.

Theorem 4.5. *Let a space X satisfies $SH_{\leq n}$. Then X satisfies $\mathsf{S}_1^*(\mathcal{O},\mathcal{O}^{gp})$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. We define new covers $\mathcal{V}_n$, $n \in \mathbb{N}$, of X putting for each n

$$\mathcal{V}_n = \wedge\{\mathcal{U}_i : (n-1)n/2 < i \leq n(n+1)/2\}.$$

Apply $\mathrm{SH}_{\leq n}$ to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$. We find a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n, $|\mathcal{W}_n| \leq n$, $\mathcal{W}_n \subset \mathcal{V}_n$ and $\{\mathrm{St}(\cup\mathcal{W}_n,\mathcal{V}_n) : n \in \mathbb{N}\}$ is a γ-cover of X. Write $\mathcal{W}_n = \{W_i : (n-1)n/2 < i \leq n(n+1)/2\}$. For each W_i take also the set $U_i \in \mathcal{U}_i$ which is a term in the representation of W_i given above. We shall prove now that the set $\{\mathrm{St}(U_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is an open groupable cover of X. Consider the sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of natural numbers defined by $n_k = k(k-1)/2$. Then, by construction, for each $x \in X$ we have $x \in \bigcup_{n_k < i \leq n_{k+1}} \mathrm{St}(W_i,\mathcal{U}_i)$ for all but finitely many k which implies that the cover $\{\mathrm{St}(U_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is groupable. □

5. Strongly star-Hurewicz spaces

Recall that a space X is *strongly starcompact* [3], [13] if for each open cover $\mathcal{U}$ of X there is a finite set $F \subset X$ such that $\mathrm{St}(F,\mathcal{U}) = X$. Clearly, every strongly starcompact space is SSH. But we have a bit more. Call a space X *σ-strongly starcompact* if it is a union of countably many strongly starcompact spaces.

Theorem 5.1. *Every σ-strongly starcompact space is SSH.*

Proof. Let $X = \bigcup_{n\in\mathbb{N}} X_n$, where each X_n is strongly starcompact, and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. One may suppose that $X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$, because the union of finitely many strongly starcompact spaces is also strongly starcompact. For each n let A_n be a finite subset of X such that $\mathrm{St}(A_n,\mathcal{U}_n) \supset X_n$. It follows that each point of X belongs to all but finitely many sets $\mathrm{St}(A_n,\mathcal{U}_n)$, i.e. the sequence $(A_n : n \in \mathbb{N})$ shows that X is a strongly star-Hurewicz space. □

The following theorem gives a characterization of SSH spaces in terms of groupable covers.

Theorem 5.2. *For a space X the following are equivalent:*

(1) *X has the strongly star-Hurewicz property;*
(2) *X satisfies the selection principle $\mathsf{SS}^*_{fin}(\mathcal{O},\mathcal{O}^{gp})$.*

Proof. (1) $\Rightarrow$ (2): It follows from the fact that countable γ-covers are groupable and monotonicity of SS^*_{fin} in the second variable.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let for each $n \in \mathbb{N}$, $\mathcal{V}_n = \wedge_{i\leq n}\mathcal{U}_n$. Applying (2) to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of open covers of X find a sequence $(T_n : n \in \mathbb{N})$ of finite subsets of X such that $\{\mathrm{St}(T_n,\mathcal{V}_n) : n \in \mathbb{N}\}$ is a groupable open cover of X. Let $n_1 < n_2 < \cdots < n_k < \cdots$ be a sequence of natural numbers such that for each $x \in X$,

$x \in \bigcup_{n_k \leq i < n_{k+1}} \mathrm{St}(T_i, \mathcal{V}_i)$ for all but finitely many k. For each $n \in \mathbb{N}$ put $S_n = \bigcup_{i<n_1} T_i$ for $n < n_1$, and $S_n = \bigcup_{n_k \leq i < n_{k+1}} T_i$ for $n_k \leq n < n_{k+1}$. Each S_n is a finite subset of X. We prove that the family $\{\mathrm{St}(S_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a γ-cover of X. Let $x \in X$. There exists $k_0 \in \mathbb{N}$ such that $x \in \bigcup_{n_k \leq i < n_{k+1}} T_i$ for all $k > k_0$. Since $\mathrm{St}(T_i, \mathcal{V}_i) \subset \mathrm{St}(S_i, \mathcal{U}_i)$ for all i with $n_k \leq i < n_{k+1}$, we have that for each $k > k_0$, $x \in \mathrm{St}(S_k, \mathcal{U}_k)$, i.e. $\{\mathrm{St}(S_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a γ-cover of X. $\square$

The previous theorem suggests to consider also the selection principle $\mathsf{SS}_1^*(\mathcal{O}, \mathcal{O}^{gp})$ that is naturally related to the SSH property. We have the following result.

Theorem 5.3. *Let a space X satisfies the following condition: for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(A_n : n \in \mathbb{N})$ of subsets of X such that for each n $|A_n| \leq n$ and $\{St(A_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a γ-cover of X. Then X satisfies $\mathsf{SS}_1^*(\mathcal{O}, \mathcal{O}^{gp})$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. For each n let $\mathcal{V}_n = \bigwedge_{(n-1)n/2 < i \leq n(n+1)/2} \mathcal{U}_i$. We apply the assumption of the theorem to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ to find a sequence $(A_n : n \in \mathbb{N})$ of subsets of X such that for each n, $|A_n| \leq n$ and $\{\mathrm{St}(A_n, \mathcal{V}_n) : n \in \mathbb{N}\}$ is a γ-cover of X: for each $x \in X$ there exists n_0 such that $x \in \mathrm{St}(A_n, \mathcal{V}_n)$ for all $n > n_0$. For each n write A_n as $A_n = \{x_i : (n-1)n/2 < i \leq n(n+1)/2\}$. Then $\{\mathrm{St}(x_i, \mathcal{U}_i) : i \in \mathbb{N}\}$ is an open groupable cover of X. Indeed, consider the sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of natural numbers defined by $n_k = k(k-1)/2$. Then for each point $x \in X$ we have $x \in \bigcup_{n_k < i \leq n_{k+1}} \mathrm{St}(x_i, \mathcal{U}_i)$ for all but finitely many k. $\square$

To each selection principle mentioned above one can associate a game [18], [8], [11], [9]. A number of selection principles of the S_{fin} and S_1 form have been characterized by the corresponding games.

The following game G is naturally associated to the SSH property. We shall call this game the *strongly star-Hurewicz game*.

Let X be a space. Two players, ONE and TWO, play a round per each natural number n. In the n–th round ONE chooses an open cover $\mathcal{U}_n$ of X and TWO responds by choosing a finite set $A_n \subset X$. A play $\mathcal{U}_1, A_1; \cdots; \mathcal{U}_n, A_n; \cdots$ is won by TWO if $\{\mathrm{St}(A_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a γ-cover of X; otherwise, ONE wins.

Evidently, if ONE has no winning strategy in the strongly star-Hurewicz game, then X is an SSH space.

Conjecture 5.4. *The strongly star-Hurewicz property of a space X need not imply ONE does not have a winning strategy in the strongly star-Hurewicz game played on X.*

Notice that similar situations might be expected also for the other star selection principles and corresponding games.

6. Relativization

In this section we define a relative version of the SSH property and show that there is a subspace Y of a SSM space X which is relatively SSH in X but Y with the subspace topology is not SSH.

Let Y be a subspace of a space X. We say that Y *is strongly star-Hurewicz in* X (or Y *is relatively strongly star-Hurewicz in* X) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that each point $y \in Y$ belongs to all but finitely many sets $\mathrm{St}(A_n, \mathcal{U}_n)$.

Remark 6.1. The following two definitions also give relative versions of the SSH property. Below Y is a subspace of X.

(R1) For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of Y by sets open in X there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that each point $y \in Y$ belongs to all but finitely many sets $\mathrm{St}(A_n, \mathcal{U}_n)$.

(R2) For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X (or of Y by sets open in X) there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of Y such that each point $y \in Y$ belongs to all but finitely many sets $\mathrm{St}(A_n, \mathcal{U}_n)$.

We do not consider these two properties, but they can be also investigated.

Denote by ${}^{\mathbb{N}}\mathbb{N}$ the set of all functions from $\mathbb{N}$ into $\mathbb{N}$. For $f, g \in {}^{\mathbb{N}}\mathbb{N}$ we write $f \prec^* g$ if and only if there is $n_0 \in \mathbb{N}$ such that $f(n) < g(n)$ for each $n \geq n_0$. A family $\mathcal{F} \subset {}^{\mathbb{N}}\mathbb{N}$ is called *bounded* if there exists a function g in ${}^{\mathbb{N}}\mathbb{N}$ such that $f \prec^* g$ for each $f \in \mathcal{F}$. The symbol $\mathfrak{b}$ denotes the minimal cardinality of a unbounded subset of ${}^{\mathbb{N}}\mathbb{N}$.

A family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ is called *almost disjoint* if the intersection of any two distinct sets in $\mathcal{A}$ is finite.

Let $\mathcal{A}$ be an almost disjoint family. The symbol $\Psi(\mathcal{A})$ denotes the space $\mathbb{N} \cup \mathcal{A}$ with the following topology: all points of $\mathbb{N}$ are isolated; a basic neighborhood of a point a in $\mathcal{A}$ is of the form $\{a\} \cup (\mathbb{N} \setminus F)$, where F is a finite subset of $\mathbb{N}$.

Example 6.2. Let $\mathcal{A}$ be an almost disjoint family of cardinality $< \mathfrak{b}$. Then $\mathcal{A}$ is relatively SSH in $\Psi(\mathcal{A})$, but $\mathcal{A}$ with the subspace topology is not SSH.

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $\Psi(\mathcal{A})$. One can suppose that for each n elements of $\mathcal{U}_n$ are basic sets:

$$\mathcal{U}_n = \{U_n(a) : a \in \mathcal{A}\} \cup \{\{n\} : n \in \mathbb{N} \setminus \cup_{a \in \mathcal{A}} U_n(a)\}.$$

Also, we can suppose that to each $a \in \mathcal{A}$ only one neighborhood $U_n(a) \in \mathcal{U}_n$ is assigned. For each $a \in \mathcal{A}$ define a function $f : \mathbb{N} \to \mathbb{N}$ by $f_a(n) = \min\{k \in \mathbb{N} : k \in U_n(a)\}$. By assumption the family $\{f_a : a \in \mathcal{A}\}$ is bounded, so that there exists a function $f \in {}^{\mathbb{N}}\mathbb{N}$ such that $f_a \prec^* f$ for each $a \in \mathcal{A}$. This means that for each $a \in \mathcal{A}$ there is $n_a \in \mathbb{N}$ with $f(n) > f_a(n)$ for every $n \geq n_a$. Further, consider for each $n \in \mathbb{N}$ the finite set $A_n := \{1, 2, \cdots, f(n)\}$ subset of $\mathbb{N}$. We claim that the sequence $(A_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that $\mathcal{A}$ is relatively SSH in $\Psi(A)$. Indeed, for each $a \in \mathcal{A}$ the intersection

$U_n(a) \cap A_n \neq \varnothing$ (because $f_a(n) \in A_n \cap U_n(a)$) for each $n \geq n_a$, i.e. each point $a \in \mathcal{A}$ belongs to all but finitely many sets $\mathrm{St}(A_n, \mathcal{U}_n)$.

On the other hand, the subspace $\mathcal{A}$ of $\Psi(\mathcal{A})$ is the discrete space of cardinality $\mathfrak{b}$ and thus it can not be SSH.

Let us remark that according to a result from [14] this space $\Psi(\mathcal{A})$ is SSM.

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