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Partial metrizability in value quantales

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Abstract. Partial metrics are metrics except that the distance from a point to itself need not be 0. These are useful in modelling partially defined information, which often appears in computer science. We generalize this notion to study "partial metrics" whose values lie in a value quantale which may be other than the reals. Then each topology arises from such a generalized metric, and for each continuous poset, there is such a generalized metric whose topology is the Scott topology, and whose dual topology is the lower topology. These are both corollaries to our result that a bitopological space is pairwise completely regular if and only if there is such a generalized metric whose topology is the first topology, and whose dual topology is the second.

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1. Introduction

A *partial metric* [7] is a generalised metric for modelling partially defined information. For example, if a person is told to visit *London* in the UK they would instantly ask where in London. A more precise instruction might be to visit London's *Hyde Park*, yet again they would ask, where in Hyde Park. A sufficiently precise instruction might be to visit the *Prince Albert Memorial*, or the *Serpentine Gallery* in Hyde Park. Let the relation $L_1 \leq L_2$ on such

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locations be defined by L_2 is a place in L_1. Then this is a partial ordering. For example,

$$London \leq \begin{cases} Hyde\ Park & \leq \begin{cases} Serpentine\ Gallery \\ Prince\ Albert\ Memorial \end{cases} \\ Trafalgar\ Square & \leq \ Nelson's\ Column \end{cases}$$

A partial metric is a means of extending the notion of metric, such as the Euclidean distance between the Serpentine Gallery and Nelson's Column, to posets. To see how and, most importantly, why this is useful a brief discussion of the programming problem in computer science which led to partial metrics is now given.

A *concurrent program* is a computer program consisting of two or more processes to be executed in parallel, and also to communicate with each other in some way. A problem arises when two processes are each waiting upon the other for a communication before each itself can communicate. This is a deadly embrace situation known as *deadlock* where two processes remain alive yet doing nothing as each waits for a communication that can never arrive. In general it is not decidable whether or not an arbitrary concurrent program may deadlock at some time in its execution. However, in a concurrent program for a safety critical application, such as for use in a hospital's intensive care unit, it is essential that there be some means of proving that deadlock can never occur. The best that can be done is to consider certain concurrent programs where it can be proven that deadlock can never arise. Deadlock is a problem between two processes which are directly or indirectly connected by some path of communication, and so a consideration of such paths provides an analysis of possible deadlock. A *cycle* is a set of processes $P_1,\ P_2,\ \ldots\ ,\ P_n$ where each P_i communicates by sending messages to $P_{(i+1)\ mod\ n}$, and so receiving messages from $P_{(i-1)\ mod\ n}$. For each P_i let c_i be the largest possible difference at any time between the number of messages sent minus the number received. For example, consider the case for $n = 2$. Suppose $c_1 = 5$, then at any time P_1 has sent at least five more messages than it has received. Suppose also that $c_2 = -3$, then P_2 has sent at least -3 more (i.e. at most 3 fewer) messages than it has received. This cycle of two processes cannot deadlock as at any time there is a net surplus of at least $c_1 + c_2 = 5 + (-3) = 2$ messages being produced by the cycle than being received. If the so-called *cycle sum* $\sum_{i=1}^{n}\ c_i\ >\ 0$ then this cycle of processes can never deadlock. To prove that a concurrent program will not deadlock it is thus sufficient to prove that each and every cycle sum is positive.

This is the *cycle sum theorem* [10], later extended to a more sophisticated model of concurrent programming as the *cycle contraction mapping theorem* [6]. The virtue of these theorems is that they are an *extensional* treatment of deadlock, one which does not require a detailed understanding of exactly how programs are executed. They prove that a deadlock free computation is necessarily the only possible behaviour for a concurrent program. This is in contrast to the more usual (but more difficult) procedure of constructing the sequence of all intermediate states of an execution, and demonstrating that the limit is a deadlock free computation. The behaviour of a cycle can be studied as the fixed point of a function. The purpose of the cycle sum and cycle contraction mapping theorems is to firstly prove that the fixed point is unique, and secondly that this point is, in a desired sense, totally defined. Being an extensional treatment of deadlock we require not the details of how programs are executed, only the distinction between the so-called *total* (the word *total* is used here in place of *complete* as in [7]) computations (i.e. the desirable deadlock free executions) and the *partial*, that is, those initial parts of a total computation. To appreciate the distinction between total and partial objects we return to the analogy of a visitor to London. A reference to *London* is only partially informative, as it does not refer to a more specific place of interest such as *Hyde Park* or *Trafalgar Square*. London is thus a *partialization* of Hyde Park, Trafalgar Square, etc. where the (extent of) totalness of a place can be measured by the area of ground upon which it stands. London is a partial approximation to Hyde Park as the latter stands upon a smaller space and contained within that of London.

The cycle sum and cycle contraction mapping theorems are in essence inductive results to prove that eventually a *total* result must follow from the inductive hypothesis that for each step in a computation the next step will steadily increase the (extent of) totalness. To formulate such an hypothesis requires a function to measure the (extent of) totalness of a computation. For example, a tourist might 'visit' the partial places *London*, *Trafalgar Square*, *Nelson's column*. The (extent of) totalness of each partial place, as measured by its area, becomes increasingly precise at each step. In the theorems under discussion the property of *deadlock free* for programs is *partialized* precisely to the extent to which it applies to each initial part of a deadlock free computation. This is a *downward* approach in which a *partial object* is viewed as a partialized total one. This is in contrast to the established Scott-Strachey least fixed point semantics [9] where the behaviour of a program is viewed as the limit of a chain of partial approximations, an *upward* approach where a total object (if the notion exists) is a *completion* of partial objects. While the upward view is necessary to define the semantics for an arbitrary program, the downward view is sufficient to reason about well behaved programs such as those which are deadlock free.

The general problem arising from these deadlock studies [10, 6] is how to *partialize* theories. Given a theory of (now to be known as) *total* objects how can additional (to be known as) *partial* objects be introduced and the theory extended yet weakened to apply to them? In each of the above deadlock studies the total objects form a metric space of infinite sequences. For a set X let, $d : X^\omega \times X^\omega \to \Re^+$ be the metric such that $d(x,y) = 2^{-sup\{n | \forall m < n, x_m = y_m\}}$. The set of partial objects used in each of the two studies is very different, yet each can be understood as an instance of the same problem of how to partialize the theory of metric spaces. Firstly this involves generalising the notion of a *metric*. A *partial metric* (or *pmetric*) [7] is a function $p : X\times X \to \Re^+$ satisfying the following conditions.

1) For every $x, y \in X, p(x, y) \geq p(x, x)$
2) For every $x, y \in X, p(x, y) = p(y, x)$
3) For every $x, y, z \in X, p(x, z) \ \leq \ p(x, y) + p(y, z) - p(y, y)$
4) For every $x, y \in X, x = y$ iff $p(x, y) = p(x, x) = p(y, y)$

A partial metric is a generalisation of the notion of a metric such that self distance is not necessarily zero. From a partial metric can be defined a metric $d : X\times X \to \Re^+$, a partial ordering $\leq\subseteq X\times X$, a *weight* function $|\cdot| : X \to \Re^+$, and notions of *total* & *partial* objects.

$$\begin{array}{rcl} d(x,y) & = & 2\times p(x,y) - p(x,x) - p(y,y) \\ x \leq y & \Leftrightarrow & p(x,x) = p(x,y) \\ |x| & = & p(x,x) \\ x \ total & \Leftrightarrow & p(x,x) = 0 \\ x \ partial & \Leftrightarrow & p(x,x) > 0 \end{array}$$

Note that a pmetric restricted to the total objects is a metric. For all $x, y \in X$, $x < y \Rightarrow |x| > |y|$, and so $|\cdot|$ can be used to measure the (extent of) totalness of each member of X. The Banach contraction mapping theorem can be extended to partial metrics. A *contraction* is a function $f : X \to X$ for which there exists a $0 < c < 1$ such that $\forall\ x, y \ \in \ X,\ p(f(x),\ f(y)) \ \leq \ c \times p(x, y)$. A *Cauchy sequence* is an $x \in X^\omega$ such that there exists $a > 0$ such that for each $\epsilon > 0$ there exists $k \in \omega$ such that for all $n, m > k$, $|p(x_n, x_m) - a| < \epsilon$. A sequence $x \in X^\omega$ *converges* if there exists $a \in X$ such that for each $\epsilon > 0$ there exists $k > 0$ such that for all $n > k$, $p(x_n, a) - p(a, a) < \epsilon$. p is *complete* if every Cauchy sequence converges. The *partial metric contraction mapping theorem* [6] is that each contraction for a complete partial metric has a unique fixed point, and this point is total.

Although originally developed as a partialized theory for extensional reasoning about properties of programs such as deadlock, partial metrics have since been developed in computer science as a *theory of partiality* for studying continuous lattices, using the induced ordering $x \leq y$ iff $p(x, x) = p(x, y)$. A partial metric $p : X \times X \to \Re^+$ generalises the theory of metric spaces by dropping the requirement that self distance always be zero, and in so doing opens up the study of T_0 spaces to a symmetric (in contrast to quasimetrics) metric style treatment, which in addition incorporates a weight function $|\cdot| : X \to \Re^+$. The present paper takes the process of generalisation further by replacing the range $\Re^+$ of a pmetric by a value quantale.

2. P-metrics and qmetrics in value lattices

Definition 2.1. *A* value lattice *is a poset* $(\mathcal{V}, \leq)$*, whose least element is denoted* 0 *and largest is* ∞*, such that* $(\mathcal{V}, \geq)$ *is a continuous lattice; together with an associative, commutative operation* $+ : \mathcal{V}\times\mathcal{V} \to \mathcal{V}$ *such that* 0 *is an identity and for each* $R, S \supseteq \mathcal{V}$, $(\bigwedge R) + (\bigwedge S) = \bigwedge\{r + s | r \in R, s \in S\}$*, where*

$\bigwedge$ *denotes inf. Here are some simple but useful consequences of this infinite distributive law:*

- *(s1) For all $p \in V$, $p + \infty = \infty$.*
- *(s2) For all $p, q, r, s \in V$, $p \geq q$ and $r \geq s$ implies $p + r \geq q + s$.*

A value lattice $\mathcal{V}$, is Boolean *if for each $a \in \mathcal{V}$, $a + a = a$.*

Like any map preserving arbitrary infima, for each p, the map $\_+p$, $\_+p(q) = p + q$ has a right adjoint (see [4], chap. 0.3), $\_ \dot{-} p$ defined by $\_ \dot{-} p(q) = \bigwedge\{r \in V | p + r \geq q\} (= q \dot{-} p)$. The properties of $\dot{-}$ (see [2]) include for any $p, q, r \in V$, $S \subseteq V$:

- (d1) $p + r \geq q$ iff $r \geq q \dot{-} p$.
- (d2) $q \dot{-} p = 0$ iff $p \geq q$;
- (d3) $p + (q \dot{-} p) \geq q$;

The reversal of order above – that is, the requirement that $(\mathcal{V}, \geq)$, rather than the expected $(\mathcal{V}, \leq)$ be a continuous lattice – is due our need to maintain the traditional way of writing axioms for metrics, in order to allow easy comparison between metric spaces and our structures. The reader must be careful when looking at references (except [2]), which use the traditional order. Notice that products of value lattices are again value lattices, and that with the way-above relation denoted $\gg$, that in a product $\prod_I \mathcal{V}_i$, $a \gg b$ if and only if each $a_i \gg b_i$ and $\{i | a_i \neq \infty\}$ is finite. When we write $\ll$, we mean the inverse of $\gg$.

Key examples of value lattices include the extended nonnegative reals, $\mathbb{E} = [0, \infty]$, the unit interval $\mathbb{I} = [0, 1]$, and the two-point set, $\mathbb{B} = \{0, \infty\}$, together with the usual $\leq, +$, except that in $\mathbb{I}$ we truncate addition, via $a + b = \min\{a +_u b, 1\}$, where $+_u$ denotes the usual sum. By a qmetric from a set X into a value quantale $\mathcal{V}$, we mean a map $q : X \times X \to \mathcal{V}$ such that for each $x, y, z \in X$, $q(x, z) \leq q(x, y) + q(y, z)$, and $q(x, x) = 0$.

Definition 2.2. *A* $\mathcal{V}$-pseudopmetric space *is a pair (X, p), consisting of a set X and a function $p : X \times X \to \mathcal{V}$ satisfying the following conditions:*

P1) For every $x, y \in X$, $p(x, y) \geq p(x, x)$,

P2) For every $x, y \in X$, $p(x, y) = p(y, x)$,

P3) For every $x, y, z \in X$, $p(x, z) \leq p(x, y) + (p(y, z) \dot{-} p(y, y))$.

Its associated qmetric *is $q_p : X \times X \to \mathcal{V}$, defined by $q_p(x, y) = p(x, y) \dot{-} p(y, y)$. The* dual *of any qmetric is the qmetric defined by $q^*(x, y) = q(y, x)$ (so $q_p^*(x, y) = p(x, y) \dot{-} p(x, x)$).*

A $\mathcal{V}$-pmetric is a $\mathcal{V}$-pseudopmetric which also satisfies:

P4) For every $x, y \in X$, $x = y$ iff $p(x, y) = p(x, x) = p(y, y)$.

Definition 2.3. *Given a $\mathcal{V}$-qmetric $q : X \times X \to \mathcal{V}$, the* ball *(or* closed ball*) about $x \in X$ of radius $r \in \mathcal{V}$ for is the set $N_r(x) = \{y \in X | q(x, y) \leq r\}$; also, $N_r^*(x) = \{y \in X | q^*(x, y) \leq r\}$, and the* open ball *about x of radius r is $B_r(x) = \{y | q(x, y) \ll r\}$.*

A subset U of X is open in X if for each $x \in U$ there is an $r \gg 0$ such that $N_r(x) \subseteq U$. We write τ_q, for the collection of all open subsets of a $\mathcal{V}$-qmetric space (X,q), $\tau_p = \tau_{q_p}$ and $\tau_{p^*} = \tau_{(q_p)^*}$ for a $\mathcal{V}$-pseudopmetric space.

Theorem 2.4. *Let $(\mathcal{V}, \geq, +)$ be a value lattice.*

(a) If $p : X \times X \to \mathcal{V}$ is a $\mathcal{V}$-pseudopmetric, then q_p is a $\mathcal{V}$-qmetric.

(b) For each $\mathcal{V}$-qmetric, τ_q is a topology. For each $x \in X$, $\{B_r(x)|r \gg 0\}$ and $\{N_r(x)|r \gg 0\}$, are neighborhood bases for τ_q at x. For each $x \in X, r \in \mathcal{V}$, $N^*_r(x)$ is a closed set in τ_q. Also, if $r \gg 0$ then $B_r(x)$ is an open set in τ_q, so in particular, the set of all open balls is a base for the topology τ_q.

Proof. (a) Certainly, if $p : X \times X \to \mathcal{V}$ is a $\mathcal{V}$-pseudopmetric and $q(x,y) = p(x,y) \dot{-} p(y,y)$, then $q(x,x) = 0$. Next, we show that if $\mathcal{V}$ is a value lattice and $a,b,c \in \mathcal{V}$ then $(a+b) \dot{-} c \leq (a \dot{-} c)+b$. By definition of $\dot{-}$, $a \leq (a \dot{-} c)+c$. Thus for every $b \in \mathcal{V}$, $a+b \leq (a \dot{-} c)+c+b$. Therefore $(a+b) \dot{-} c \leq (a \dot{-} c)+b$. Now by the above for every $x,y,z \in X$, $q(x,z) = p(x,z) \dot{-} p(z,z) \leq (p(z,y) + (p(x,y) \dot{-} p(y,y))) \dot{-} p(z,z) \leq (p(x,y) \dot{-} p(y,y)) + (p(y,z) \dot{-} p(z,z)) = q(x,y)+q(y,z)$.

(b) These proofs are left to the reader. That τ_q is a topology, is straightforward (or see [5]). The others use facts about the continuous lattice $(\mathcal{V}, \geq)$, which generalize those about $\mathbb{I}$:

$$a \ll r,s \text{ iff } r \wedge s \gg a \text{ and } r,s \geq a \text{ iff } r \wedge s \geq a.$$

$$\gg \text{ is interpolative, so if } r \gg 0 \text{ then for some } s,\ r \gg s \gg 0.$$

By Scott continuity of $+$, if $q(x,y) \ll r$ then $\bigwedge\{q(x,y)+s|s \gg 0\} = q(x,y)+\bigwedge\{s \gg 0\} = q(x,y) \ll r$ so for some $s \gg 0$, $q(x,y)+s \ll r$, thus if $z \in N_s(y)$ then $z \in B_r(x)$, so $B_r(x)$ is open. □

Theorem 2.5. *In any $\mathcal{V}$-qmetric space, $x \in \mathsf{cl}(y)$ if and only if $q(x,y) = 0$. In any $\mathcal{V}$-pseudopmetric space, $x \in \mathsf{cl}(y)$ if and only if $p(x,y) = p(y,y)$.*

Proof. By Theorem 2.4, $N^*_0(y)$ is closed, so $\mathsf{cl}(y) \subseteq N^*_0(y)$. But if $x \notin \mathsf{cl}(y)$ then for some open T, $x \in T$ and $y \notin T$. By Theorem 2.4, $N_\epsilon(x) \subseteq T$ for some $\epsilon \gg 0$, thus in particular, $q(x,y) \not\leq \epsilon$, so $q(x,y) \neq 0$. This shows the reverse inclusion, $N^*_0(y) \subseteq \mathsf{cl}(y)$. The assertion about $\mathcal{V}$-pseudopmetric spaces is seen by noting that by the above, $x \in \mathsf{cl}(y)$ if and only if $q_p(x,y) = 0$, and certainly this happens if and only if $p(y,y) \geq p(x,y)$, so they are equal by P1). □

Corollary 2.6. *For a $\mathcal{V}$-qmetric, τ_q is T_0 if and only if for each $x,y \in X$, $q(x,y) = q(y,x) = 0 \Rightarrow x = y$. A $\mathcal{V}$-pseudopmetric p is a $\mathcal{V}$-pmetric if and only if τ_p is T_0.*

Proof. It is known that a topology is T_0 if and only if, for each $x,y \in X$, $x \in \mathsf{cl}(y) \& y \in \mathsf{cl}(x) \Rightarrow x = y$. Thus by Theorem 2.5, τ_q is T_0 if and only if $q(x,y) = q(y,x) = 0 \Rightarrow x = y$, and so τ_p is T_0 if and only if $p(x,y) = p(y,y) = p(x,x) \Rightarrow x = y$. □

Lemma 2.7. *$p(x,y) = \max(x,y)$ is an $\mathbb{I}$-pmetric on $\mathbb{I}$ and is also a $\mathbb{B}$-pmetric on $\mathbb{B}$. Also, τ_p is the Scott (or upper) topology, $\sigma = \{(x,1]|x \in (0,1)\} \cup \{\varnothing, \mathbb{I}\}$, and τ_{p^*} is the lower topology, $\omega = \{[0,x)|x \in (0,1)\} \cup \{\varnothing, \mathbb{I}\}$.*

Proof. Since for every $x, y \in \mathbb{I}$, $\max(y,x) = \max(x,y) \geq \max(x,x)$, p satisfies P1 and P2; also P4 is clear. For P3, if $y = \max(x,y,z)$ then $p(x,z) \leq y + p(y,z) - p(y,y) = p(x,y) + p(y,z) - p(y,y)$; if $z = \max(x,y,z)$ then $p(x,z) \leq p(x,y) + z - p(y,y) = p(x,y) + p(y,z) - p(y,y)$ and the case $x = \max(x,y,z)$ is similar. Also if $p(x,x) = p(y,y)$, then $x = y$. Thus p is a $\mathbb{I}$-pmetric on $\mathbb{I}$.

Now we show that $\tau_p = \sigma$ and $\tau_{p^*} = \omega$: Let $A \in \tau_p$. Then for each $x \in A$ there exists $r > 0$ (notice that here $r \gg 0$ if and only if $r > 0$), such that $N_r(x) \subseteq A$. Thus $\{y | x - r \leq y\} = N_r(x) \subseteq A$, hence $\uparrow (x - r) \subseteq A$. Also, if $\bigvee D \in A$ and D is directed, then there is some $r > 0$ such that $N_r(\bigvee D) \subseteq A$, and by properties of $\bigvee$, for some $d \in D$ we have $\bigvee D \dot{-} r \leq d$, showing $A \in \sigma$. If $x \in A \in \sigma$, then $\uparrow x \subseteq A$. But $x = \bigvee(x - 1/n) \in A$, $n \in \mathbb{N}$ and $\{x - 1/n | n \in \mathbb{N}\}$ is a directed set, thus there is $m \in \mathbb{N}$ such that $x - 1/m \in A$. So $\uparrow (x - 1/m) \subseteq A$. Therefore $N_{1/m}(x) \subseteq A$ and hence $A \in \tau_p$. Now let $A \in \tau_{p^*}$; then for every $x \in A$ there is $r > 0$ such that $N^*_r(x) \subseteq A$. Thus $\{y | y < x + r/2\} \subseteq \{y | y - x \leq r\} = N^*_r(x) \subseteq A$. So $(X - \uparrow (x + r/2)) \subseteq A$. Therefore $A \in \omega$. For the reverse inclusion, assume $A \in \omega$. Then for every $x \in A$ there is $a \in X$ such that $x \in (X - \uparrow a) \subseteq A$. Thus $x < a$ and hence $r = (a - x)/2 > 0$ and $x \in N^*_r(x) \subseteq A$.

For the assertions about $\mathbb{B}$, since $\mathbb{B} \subseteq \mathbb{I}$, p is a $\mathbb{B}$-pmetric on $\mathbb{B}$. Note that $N_0(\infty) = \{\infty\}$ and $N_0(0) = N_\infty(x) = \mathbb{B}$ for $x \in \mathbb{B}$, so $\tau_p = \sigma$, and a similar proof shows $\tau_{p^*} = \omega$. □

Lemma 2.8. *If $f : X \to Y$ and p is a $\mathcal{V}$-pseudopmetric on Y, then $p_f(x,y) = p(f(x), f(y))$ defines a $\mathcal{V}$-pseudopmetric on X. Also, f is continuous from (X, τ) to (Y, τ_p), if and only if $\tau_{p_f} \subseteq \tau$.*

Proof. Since p is a $\mathcal{V}$-pseudopmetric on Y, p_f is on X. We distinguish open q_p-balls in Y from open q_{p_f}-balls in X, denoting the former by $B^Y_r(y)$, the latter by $B^X_r(x)$. By definition of p_f, $B^X_r(x) = \{y | p(f(x), f(y)) \dot{-} p(f(y), f(y)) \ll r\} = f^{-1}[B^Y_r(f(x))]$. Of course, f is continuous if and only if the inverse image of each set in the base of open q_p-balls is open, that is, if and only if each open q_{p_f}-ball is open; the latter occurs if and only if $\tau_{p_f} \subseteq \tau$. □

Recall that a bitopological space (X, τ, τ^*) is *completely regular* if whenever $x \in T \in \tau$ then there is a pairwise continuous $f : (X, \tau, \tau^*) \to (\mathbb{I}, \sigma, \omega)$, such that $f(x) = 1$ and f is 0 off T; it is *zero-dimensional* if whenever $x \in T \in \tau$ then there is a pairwise continuous $f : (X, \tau, \tau^*) \to (\mathbb{B}, \sigma, \omega)$ such that $f(x) = \infty$ and f is 0 off T. A bitopological space (X, τ, τ^*) is said to have a property *pairwise*, if both (X, τ, τ^*) and its *dual,* (X, τ^*, τ) have the property.

Theorem 2.9. *If (X, τ, τ^*) is completely regular then there is a value lattice $\mathcal{V}$ and a $\mathcal{V}$-pseudopmetric such that $\tau = \tau_p$ and $\tau^* \supseteq \tau_{p^*}$. Further, if (X, τ, τ^*) is pairwise completely regular then there is a value lattice $\mathcal{V}$ and a $\mathcal{V}$-pseudopmetric such that $\tau = \tau_p$ and $\tau^* = \tau_{p^*}$. The analogous result holds for zero-dimensionality in place of complete regularity, with* Boolean *value lattices.*

Conversely, if there is a value lattice and a $\mathcal{V}$-pseudopmetric such that $\tau = \tau_p$ and $\tau^* \supseteq \tau_{p^*}$, then (X, τ, τ^*) is completely regular, and converses also hold in the other three cases.

Throughout the above, p is a $\mathcal{V}$-pmetric if and only if τ is T_0.

Proof. Let $PC(X, \mathbb{I})$ be the collection of all pairwise continuous functions from (X, τ, τ^*) to $(\mathbb{I}, \sigma, \omega)$, and define $A = \mathbb{I}^{PC(X,\mathbb{I})}$. With the pointwise order, A is a value lattice and for $\phi, \psi \in A$, $\phi \gg \psi$ if and only if $\phi(f) > \psi(f)$, for every $f \in PC(X, \mathbb{I})$ and $\{f | \phi(f) \neq 1\}$ is finite. Now define $p : X \times X \to A$ such that $p(x, y)(f) = max\{f(x), f(y)\}$ for every $f \in PC(X, \mathbb{I})$. A coordinatewise proof then shows that p is an A-pseudopmetric on X. Now we show that $\tau = \tau_p$. Let $T \in \tau$. If $x \in T$ then there is a pairwise continuous function $g : (X, \tau, \tau^*) \to (\mathbb{I}, \sigma, \omega)$ such that $g(x) = 1$ and g is 0 off T. Now take $r \in A$ such that $r(g) = 1/2$ and $r(f) = 1$ for $f \neq g$. Then $N_r(x) = \{y | g(y) \geq 1/2\} \subseteq T$. Thus $\tau \subseteq \tau_p$. Now assume that $T \in \tau_p$. For each $x \in T$, there exists $r \gg 0$ such that $N_r(x) \subseteq T$. Let $I = \{f \in PC(X, \mathbb{I}) | r(f) \neq 1\}$. Then $N_r(x) = \bigcap\{y | f_i(x) - f_i(y) \leq r(f_i)\}$, where $i \in I$. Hence $x \in \bigcap\{y | f_i(x) - f_i(y) < r(f_i)/2\} \subseteq N_r(x) \subseteq T$, where $i \in I$. Therefore $x \in \bigcap\{y | y \in f_i^{-1}((f_i(x) - r(f_i)/2, 1])\}$. Since every f_i is continuous, $\{y | y \in f_i^{-1}((f_i(x) - r(f_i)/2, 1])\} \in \tau$ and since I is finite, $\bigcap\{y | y \in f_i^{-1}((f_i(x) - r(f_i)/2, 1])\} \in \tau$. Thus $T \in \tau$ and by the above we now have $\tau = \tau_p$. Similarly $\tau_{p^*} \subseteq \tau^*$. For zero-dimensionality replace A by $\mathbb{B}^{PC(X,\mathbb{B})}$ and proceed as in the completely regular case.

In the "pairwise" cases, the above p satisfies $\tau = \tau_q$ and $\tau_{p^*} = \tau^*$.

For the converses, consider the relation $\lhd_p$ on subsets of X, defined by $S \lhd_p T \Leftrightarrow (\exists r \gg 0)(N_r(S) \subseteq T)$, where $N_r(S)$ is defined to be $\bigcup_{x \in S} N_r(x)$. Then $\lhd_p$ can easily be seen to satisfy the properties (a1)-(a3) of an auxiliary relation given below (where (a3) results from the interpolation property of the way-below relation $\gg$ in the continuous lattice $(\mathcal{V}, \geq)$). Further, it is clear that $R, S \lhd_p T \Rightarrow R \cup S \lhd_p T$, $R \lhd_p S, T \Rightarrow R \lhd_p S \cap T$, $\varnothing \lhd_p \varnothing$ and $X \lhd_p X$. These are the defining properties of a quasiproximity (see [3]). Each quasiproximity $\lhd$ has a *dual,* $\lhd^*$, defined by $S \lhd^* T \Leftrightarrow X \setminus T \lhd X \setminus S$, and gives rise to a topology, $\tau_\lhd = \{T | x \in T \Rightarrow \{x\} \lhd T\}$. Certainly, $\tau_p = \tau_{\lhd_p}$ and $\tau_{p*} = \tau_{\lhd_p^*}$. In the reference just mentioned, Urysohn's lemma is shown for quasiproximities; thus, if $S \lhd T$ then there is a pairwise continuous function $f : (X, \tau_\lhd, \tau_{\lhd^*}) \to ([0, 1], \sigma, \omega)$ such that $f[S] = \{1\}$, $f[X \setminus T] = \{0\}$, and as a result, if $x \in T \in \tau_p$ then, letting $S = \{x\}$, there is a pairwise continuous function $f : (X, \tau_p, \tau_{p^*}) \to ([0, 1], \sigma, \omega)$ such that $f(x) = 1$ and $f[X \setminus T] = \{0\}$; the same then holds for (X, τ_{p^*}, τ_p) using $\lhd^*$. This yields the results for complete regularity and pairwise complete regularity, and those involving 0-dimensionality are simpler, since in this case, for each $x \in X$, $r \gg 0$, the function defined by $f(y) = \begin{cases} 1 & y \in N_r(x) \\ 0 & y \notin N_r(x) \end{cases}$, is pairwise continuous from (X, τ_p, τ_{p^*}) to $(\{0, 1\}, \sigma, \omega)$.

The last statement is immediate from Corollary 2.6. □

If $\leq$ is a transitive, reflexive relation on X (= a pre-order) then the *Alexandroff topology* is $\alpha(\leq) = \{T \subseteq X | x \in T \;\&\; x \leq y \Rightarrow y \in T\}$.

Theorem 2.10. *(a) For any topology τ, $(X, \tau, \alpha(\geq_\tau))$ is pairwise 0-dimensional. Thus there is a Boolean value lattice $\mathcal{V}$ and a $\mathcal{V}$-pseudopmetric such that $\tau = \tau_p$.*

(b) For each continuous bounded dcpo, there is a value lattice $\mathcal{V}$ and a $\mathcal{V}$-pmetric such that its Scott topology, σ, is τ_p and its lower topology, ω, is τ_{p^*}. If the dcpo is algebraic as well, then $\mathcal{V}$ can be assumed Boolean.

Proof. (a) Notice that $U \in \alpha(\geq_\tau)$ if and only if $x \in U$ and $y \in \mathsf{cl}(\{x\})$ imply that $y \in U$. Now consider $x \in T \in \tau$ then define $f : (X, \tau, \alpha(\geq_\tau)) \to (\mathbb{B}, \sigma, \omega)$ such that $f = \infty$ on T and $f = 0$ on $X \setminus T$. Since $T \in \tau$ implies $X \setminus T \in \alpha(\geq_\tau)$, f is pairwise continuous. Now let $x \in U \in \alpha(\geq_\tau)$. Since $(X \setminus \mathsf{cl}(\{x\})) \in \tau$, $\mathsf{cl}(\{x\}) \in \alpha(\geq_\tau)$. Define $f : (X, \alpha(\geq_\tau), \tau) \to (\mathbb{B}, \sigma, \omega)$ by $f = \infty$ on $\mathsf{cl}(\{x\})$ and $f = 0$ on $X \setminus \mathsf{cl}(\{x\})$. By the construction, f is pairwise continuous.

(b) For each continuous bounded dcpo, (P, σ, ω) is pairwise completely regular and σ is T_0 (see [3]); additionally if P is algebraic, then (P, σ, ω) is pairwise 0-dimensional, using the fact that for compact x, $\uparrow x = \Uparrow x$, so a base for the open sets in σ is a subbase for the closed sets in ω. □

3. Value quantales

First, we recall the definition and a few basic properties of value quantales from [2]. Assume V is a complete lattice. Then V is *completely distributive* if for any family $\{x_{i,j} \mid j \in J, k \in K_j\}$ of elements of V,

$$\bigwedge_{j\in J} \bigvee_{k\in K_j} x_{j,k} = \bigvee_{f\in M} \bigwedge_{j\in J} x_{j,f(j)},$$

where $M = \prod_{j\in J} K_j$.

Assume V is a complete lattice and $p, q \in V$. Then q is *well above* p, denoted by $q \succ p$, iff for any subset $S \subseteq V$, if $p \geq \bigwedge S$, then for some $r \in S$, $q \geq r$.

Raney ([8]) has shown that a complete lattice is completely distributive if and only if each $p = \bigwedge\{q \mid q \succ p\}$. This is the criterion we shall use below.

The "way-above" relation $\gg$ for a continuous lattice $(L, \geq)$ and $\succ$ on a completely distributive lattice $(V, \geq)$ differ in that for the former (unlike the latter), the set S must be directed (by $\geq$). Like $\gg$, $\succ$ satisfies most of the axioms for an *auxiliary relation*, $\triangleright$ on a poset $(P, \geq)$: if $p, q, r, s \in P$ and $S \subseteq P$,

- (a1) $q \triangleright p$ implies $q \geq p$;
- (a2) $s \geq q$, $q \triangleright p$ and $p \geq r$ implies $s \triangleright r$; and
- (a3) (*Interpolation Property*) If $q \triangleright p$, then for some r, $q \triangleright r$ and $r \triangleright p$.

Property (a3) is more special, and holds for $\gg$ in any continuous lattice $(L, \geq)$ ([4]), and for $\succ$ in any completely distributive lattice $(V, \geq)$ ([8]). The definitions of completely distributive lattice (for $\succ$) and continuous lattice (for $\gg$) amount to the statement that the relation is *approximating:*

- (a4) Each p is the inf, $\bigwedge\{r | r \triangleright p\}$.

For an arbitrary set in a completely distributive lattice, $q \succ \bigwedge S$ iff for some $r \in S$, $q \geq r$, and for directed set D in a continuous lattice, that $q \gg \bigwedge D$ iff for some $r \in D$, $q \geq r$.

However, many auxiliary relations, like $\gg$, are *subdirecting* : for each $x \in X$, $\{y|y \rhd x\}$ is directed by $\geq$. This need not hold for $\succ$; for example, in $\mathbb{B}^F$ $r \succ 0$ if and only if, there is at most one f such that $r(f) < \infty$, and this collection is clearly not directed. But we need this property for 0:

A *value distributive lattice* is a completely distributive lattice V satisfying the following two conditions:

- (v1) $\infty \neq 0$.
- (v2) If $p \succ 0$ and $q \succ 0$, then $p \wedge q \succ 0$.

A *value quantale* $\mathcal{V} =< V, \leq, + >$ consists of a value distributive lattice $< V, \leq >$ and an operation $+$ on V satisfying definition 1.

We now describe a special case of the construction of value quantale from [2]. Assume $(L, \geq)$ is a continuous lattice in which $\perp \neq \top$, and let $A = A(L) = \{x \in L \mid x \gg \perp\}$, where $\gg$ denotes the way-above relation on L. Then $\top \in A$ and if $a, b \in A$, then $a \wedge b \in A$. By a *round upper set in* L, we mean a nonempty $I \subseteq A$ for which:

(r1) $j \geq k \in I \Rightarrow j \in I$, and

(r2) $(\forall j \in I) \Downarrow j \cap I \neq \varnothing$.

(Note in particular, that we do not require that I be directed by $\geq$.) Let $\mathcal{R} = (R[L], \supseteq)$ denote the poset of round upper sets in L, with reverse set inclusion. Since $\mathcal{R}$ is an inf-closed subset of $(2^A, \supseteq)(\cong (\mathbb{B}^A, \geq))$, $\mathcal{R}$ is a complete lattice with $\bigwedge S = \bigcup S \cup \{\top\}$.

For $a \in L$, define $\theta(a) = \{x \in A | x \gg a\}$. Then $\theta(a) \in \mathcal{R}$ and for $a \in A$, and $I \in \mathcal{R}$, notice that $a \in I \Rightarrow \theta(a) \succ I$, since if $a \in I \geq \bigwedge S$ then $a \in I \subseteq \bigcup S$ so for some $J \in S$, $a \in J$, showing $\theta(a) \subseteq J$, thus $\theta(a) \geq J$. Thus $I = \bigcup\{\theta(a)|a \in I\} \subseteq \bigcup\{J \succ I\} \subseteq \bigcup\{J \subseteq I\} = I$, so in particular, each $I = \bigwedge\{J \succ I\}$, thus $\mathcal{R}$ is completely distributive by Raney's result.

Also note that if $J \succ I$ then for some $a \in I$, $J \geq \theta(a)$, that is, $J \subseteq \theta(a)$; this with the previous paragraph shows $J \succ I \Leftrightarrow (\exists a \in I) J \subseteq \theta(a)$. Here are some other properties of θ that we need later:

(θ_1) θ preserves direct inf: For let D be directed by $\geq$. Then for $x \in A$, $x \in \theta(\bigwedge D) \Leftrightarrow x \gg \bigwedge D \Leftrightarrow$ for some $y \in L$, $x \gg y \gg \bigwedge D \Leftrightarrow$ for some $z \in L$ there is a $d \in D$ such that $x \gg z \geq d \Leftrightarrow$ there is a $d \in D$ such that $x \gg d \Leftrightarrow x \in \bigcup_{d \in D} \theta(d) = \bigwedge \theta[D]$. Hence $\bigwedge \theta[D] = \theta(\bigwedge D)$ as required.

(θ_2) For each $a, b \in L$, $a \geq b \Leftrightarrow \theta(a) \geq \theta(b)$: If $a \geq b$ then $b = \bigwedge\{a, b\}$ a directed set, so by θ_1, $\theta(b) = \bigwedge\{\theta(a), \theta(b)\} \leq \theta(a)$. Conversely, if $\theta(a) \geq \theta(b)$, then $\{x|x \gg a\} \subseteq \{x|x \gg b\}$, so $\bigwedge\{x|x \gg a\} \geq \bigwedge\{x|x \gg b\}$. But since L is a continuous lattice, $a = \bigwedge\{x|x \gg a\}$ and $b = \bigwedge\{x|x \gg b\}$. Hence $a \geq b$.

In $\mathcal{R}$, clearly the smallest element, called 0, is A and the largest, ∞, is $\{\top\}$. The two differ, since $\perp \neq \top$, so by (a4), for some $a \neq \top$, $a \gg \perp$; thus $a \in A$ so $0 \neq \{\top\} = \infty$.

Note also that if $I, J \succ 0$ then for some $a, b \in A$ we have $I \subseteq \theta(a)$, $J \subseteq \theta(b)$ so $I, J \subseteq \theta(a \wedge b)$, and $a \wedge b \in A$, showing that $I, J \geq \theta(a \wedge b) \succ 0$. Thus $\{I \succ 0\}$ is $\geq$-directed, so that $\mathcal{R}$ is a value distributive lattice in the terminology of [2].

Now suppose that $\star : L \times L \to L$ is a binary operation on L such that $(L, \star, \bot)$ is a commutative monoid and for any $a \in L$, the function $a\star\_ : L \to L$ preserves infs and the way above relation; that is, any indexed family $\{b_i\}_{i\in I}$ in L, $a \star \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \star b_i)$, and whenever $b \gg b'$, then $a \star b \gg a \star b'$. Then $\star : A \times A \to A$.

For $I, J \in \mathcal{R}$, define $I + J = \{a|(\exists x \in I, y \in J)a \geq x \star y\}$. Clearly, $+$ is associative, commutative and monotone. Also 0 is a unit for $+$, because of the following: If $a \in I + A$ then for some $x \in I$ and $y \in A$, $a \geq x \star y$. Thus $a \geq x \star y \geq x\star \bot = x$, and since I is an upper set, $a \in I$. For the reverse set inclusion, if $a \in I$ then by (r2) there is $m \in I$ such that $a \gg m$. Thus $a \gg m = m\star \bot = m \star \bigwedge_{k\gg\bot} k = \bigwedge_{k\gg\bot} m \star k$. Thus there is $k_0 \gg\bot$ such that $a \geq m \star k_0$. Hence $a \in I + A$. Thus $I + A = I$ for every I.

Note also that for $S \subseteq \mathcal{R}$ and $I \in \mathcal{R}$, $I + \bigwedge S = I + \bigcup(S \cup \{\top\}) = \{a|(\exists x \in I, y \in J \in S \cup \{\top\})(a \geq x \star y)\} = \bigcup_{J\in S\cup\{\top\}}\{a|(\exists x \in I, y \in J)(a \geq x \star y)\} = \bigcup_{J\in S\cup\{\top\}}(I + J) = \bigwedge_{J\in S}(I + J)$, so $(\mathcal{R}, +)$ is a value quantale. Further:

(θ_3) Each $\theta(a \star b) = \theta(a) + \theta(b)$. For if $t \in \theta(a) + \theta(b)$ then $(\exists x \in \theta(a), y \in \theta(b))t \geq x\star y$ thus $t \geq x\star y \gg a\star b$, hence $t \in \theta(a\star b)$. But if $t \in \theta(a\star b)$. Then $t \gg a \star b = \bigwedge\{x \star y|x \gg a, y \gg b\}$. Thus by definition of $\gg$ there are $w \gg a$ and $v \gg b$ such that $t \geq w \star v$. Therefore $t \in \theta(a) + \theta(b)$. As a result of this and θ_2, we also have:

(θ_4) For each $a, b, c \in L$, $a \stackrel{\star}{-} b \leq c \Leftrightarrow \theta(a) \dot{-} \theta(b) \leq \theta(c)$, since $a \stackrel{\star}{-} b \leq c \Leftrightarrow a \leq b \star c \Leftrightarrow \theta(a) \leq \theta(b \star c) = \theta(b) + \theta(c) \Leftrightarrow \theta(a) \dot{-} \theta(b) \leq \theta(c)$

We have two special cases in mind: Assume K is a nonempty set and let $\mathbb{F}$ be $(\mathbb{I}, \leq, +)$ (with truncated addition $+$ as introduced preceding Definition 1) or $(\mathbb{B}, \leq, +)$. Let K be any nonempty set. Then as a product of continuous lattices, $L = \mathbb{F}^K$ is also a continuous lattice with the pointwise order, and in it, $a \gg\bot$ if and only if for each $i \in K$, $a(i) \gg\bot_i$ and for all but a finite number of i, $a(i) = \top_i$. Thus in particular, for $\mathbb{F} = \mathbb{B}$, $A = \{r \in \mathbb{B}^K|r^{-1}[\{\infty\}]$ is cofinite$\}$, and for $\mathbb{F} = \mathbb{I}$, $A = \{r \in [0, 1)^K|r^{-1}[\{1\}]$ is cofinite$\}$.

Let $\star$ be pointwise addition; certainly $\star$ is in both cases an associative, commutative operation preserving $\leq$ and $\gg$, for which 0 is the unit, since these all hold coordinatewise. Thus $\star$ obeys the assumptions made of it, so $(\mathcal{R}[\mathbb{F}^K], +)$ is a value quantale. Following [2], we call the $\mathcal{R}[\mathbb{B}^K]$ the *value quantales of subsets*, and denote them by $\Gamma(K)$, and we call the $\mathcal{R}[\mathbb{I}^K]$ the *value quantales of fuzzy subsets*, and denote them by $\Lambda(K)$.

One special property of $\Gamma(K)$ worth noting is that (r2) is trivial, since for $r \in A$, it is easy to see that $r \gg r$; for $\Lambda(K)$ it is worth noticing for (r2) that for $r, s \in A$, $r \gg s$ if and only if $r(i) > s(i)$ whenever $r(i) \neq 1$.

Theorem 3.1. *In theorem 2.9 and its corollaries, "value lattice" can be improved to "value quantale".*

Proof. By Theorem 2.9, for $K = PC(X, \mathbb{F})$ there is a K-pseudopmetric p such that $\tau = \tau_p$ and $\tau^* \supseteq \tau_{p^*}$. Let $\theta : K \to \mathcal{R}[\mathbb{F}^K]$ be as defined above. In addition to the properties already established, notice that $\theta(0) = 0$ and $\theta(\infty) = \infty$.

We finish the proof by showing that for the $\mathcal{R}[\mathbb{F}^K]$-pseudopmetric $d = \theta \circ p$, $\tau_d = \tau_p$ and $\tau_{d^*} = \tau_{p^*}$.

For suppose $T \in \tau_p$; if $x \in T$ then for some $r \gg 0$, $N_r(x) \subseteq T$. But $r \in A(\mathbb{F}^K)$ so $\theta(r) \succ A(\mathbb{F}^K)$, the 0 of $\mathcal{R}[\mathbb{F}^K]$. Further, if $y \in N_{\theta(r)}(x)$, we have $q_d(x,y) \leq \theta(r)$ so $\theta(p(x,y)) \dot{-} \theta(p(y,y)) \leq \theta(r)$, thus by (θ_4), $p(x,y) \dot{-} p(y,y) = q_p(x,y) \leq r$, showing $y \in T$. This shows $N_{\theta(r)}(x) \subseteq T$, so $T \in \tau_d$.

Conversely, suppose $T \in \tau_d$; if $x \in T$ then for some $s \succ 0$, $s \in \mathcal{R}[\mathbb{F}^K]$, $N_s(x) \subseteq T$. By the beginning of the discussion of θ there is an $r \in K$, $r \gg 0$, so that $\theta(r) \leq s$. If $q_p(x,y) \leq r$ then $p(x,y) \dot{-} p(y,y) \leq r$, thus $\theta(p(x,y)) \dot{-} \theta(p(y,y)) \leq \theta(r) \leq s$, so $y \in T$. This shows $N_r(x) \subseteq T$, so $T \in \tau_p$.

The above show that $\tau_p = \tau_d$, and a similar proof shows that $\tau_{p^*} = \tau_{d^*}$. This completes the proof that throughout Theorem 2.9, the value lattice $\mathbb{F}^K$ and pseudopmetric p can be replaced by the value quantale $\mathcal{R}[\mathbb{F}^K]$ and pseudopmetric d. □

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