

## Relations that preserve compact filters

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ABSTRACT. Many classes of maps are characterized as (possibly multi-valued) maps preserving particular types of compact filters.

### 1. INTRODUCTION

A filter  $\mathcal{F}$  on  $X$  is *compact at*  $A \subset X$  if every finer ultrafilter has a limit point in  $A$ . As a common generalization of compactness (in the case of a principal filter) and of convergence, it is not surprising that the notion turned out to be very useful in a variety of context (see for instance [5], [6], [2] under the name of compactoid filter, [16], [17], [18] under the name of total filter). The purpose of this paper is to build on the results of [5] and [6] to show that a large number of classes of single and multi-valued maps classically used in topology, analysis and optimization are instances of *compact relation*, that is, relation that preserves compactness of filters. It is well known (see for instance [5], [2]) that upper semi-continuous multivalued maps and compact valued upper semi-continuous maps are such instances. S. Dolecki showed [6] that closed, countably perfect, inversely Lindelöf and perfect maps are other examples of compact relations. In this paper, it is shown that continuous maps as well as various types of quotient maps (hereditarily quotient, countably biquotient, biquotient) are also compact relations. Moreover, I show that maps among these variants of quotient and of perfect maps with ranges satisfying certain local topological properties (such as Fréchetness, strong Fréchetness and bisequentiality) can be directly characterized in similar terms. This requires to work in the category of convergence spaces rather than in the category of topological spaces. Therefore, I recall basic facts on convergence spaces in the next section.

The companion paper [15], which should be seen as a sequel to the present paper, uses these characterizations to present applications of product theorems for compact filters to theorems of stability under product of variants of compactness, of local topological properties (Fréchetness and its variants, among others) and of the classes of maps discussed above.

## 2. TERMINOLOGY AND BASIC FACTS

**2.1. Convergence spaces.** By a *convergence space*  $(X, \xi)$  I mean a set  $X$  endowed with a relation  $\xi$  between points of  $X$  and filters on  $X$ , denoted  $x \in \lim_{\xi} \mathcal{F}$  or  $\mathcal{F} \rightarrow x$ , whenever  $x$  and  $\mathcal{F}$  are in relation, and satisfying  $\lim \mathcal{F} \subset \lim \mathcal{G}$  whenever  $\mathcal{F} \leq \mathcal{G}$ ;  $\{x\}^{\uparrow} \rightarrow x$  <sup>(1)</sup> for every  $x \in X$  and  $\lim(\mathcal{F} \wedge \mathcal{G}) = \lim \mathcal{F} \cap \lim \mathcal{G}$  for every filters  $\mathcal{F}$  and  $\mathcal{G}$  <sup>(2)</sup>. A map  $f : (X, \xi) \rightarrow (Y, \tau)$  is *continuous* if  $f(\lim_{\xi} \mathcal{F}) \subset \lim_{\tau} f(\mathcal{F})$ . If  $\xi$  and  $\tau$  are two convergences on  $X$ , we say that  $\xi$  is *finer than*  $\tau$ , in symbols  $\xi \geq \tau$ , if  $Id_X : (X, \xi) \rightarrow (X, \tau)$  is continuous. The category **Conv** of convergence spaces and continuous maps is topological <sup>(3)</sup> and cartesian-closed <sup>(4)</sup>.

Two families  $\mathcal{A}$  and  $\mathcal{B}$  of subsets of  $X$  *mesh*, in symbols  $\mathcal{A} \# \mathcal{B}$ , if  $A \cap B \neq \emptyset$  whenever  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ . A subset  $A$  of  $X$  is  $\xi$ -*closed* if  $\lim_{\xi} \mathcal{F} \subset A$  whenever  $\mathcal{F} \# A$ . The family of  $\xi$ -closed sets defines a topology  $T\xi$  on  $X$  called *topological modification* of  $\xi$ . The neighborhood filter of  $x \in X$  for this topology is denoted  $\mathcal{N}_{\xi}(x)$  and the closure operator for this topology is denoted  $cl_{\xi}$ . A convergence is a topology if  $x \in \lim_{\xi} \mathcal{N}_{\xi}(x)$ . By definition, the adherence of a filter (in a convergence space) is:

$$(2.1) \quad \text{adh}_{\xi} \mathcal{F} = \bigcup_{\mathcal{G} \# \mathcal{F}} \lim_{\xi} \mathcal{G}.$$

In particular, the adherence of a subset  $A$  of  $X$  is the adherence of its *principal filter*  $\{A\}^{\uparrow}$ . The *vicinity filter*  $\mathcal{V}_{\xi}(x)$  of  $x$  for  $\xi$  is the infimum of the filters converging to  $x$  for  $\xi$ . A convergence  $\xi$  is a *pretopology* if  $x \in \lim_{\xi} \mathcal{V}_{\xi}(x)$ . A convergence  $\xi$  is respectively a topology, a pretopology, a *paratopology*, a *pseudotopology* if  $x \in \lim_{\xi} \mathcal{F}$  whenever  $x \in \text{adh}_{\xi} \mathcal{D}$ , for every  $\mathbb{D}$ -filter  $\mathcal{D} \# \mathcal{F}$  where  $\mathbb{D}$  is respectively, the class  $cl_{\xi}^{\sharp}(\mathbb{F}_1)$  of principal filters of  $\xi$ -closed sets <sup>(5)</sup>, the class  $\mathbb{F}_1$  of principal filters, the class  $\mathbb{F}_{\omega}$  of countably based filters, the class  $\mathbb{F}$  of all filters. In other words, the map  $\text{Adh}_{\mathbb{D}} [4]$  defined by

$$(2.2) \quad \lim_{\text{Adh}_{\mathbb{D}} \xi} \mathcal{F} = \bigcap_{\mathbb{D} \# \mathcal{D} \# \mathcal{F}} \text{adh}_{\xi} \mathcal{D}$$

<sup>1</sup>If  $\mathcal{A} \subset 2^X$ ,  $\mathcal{A}^{\uparrow} = \{B \subset X : \exists A \in \mathcal{A}, A \subset B\}$ .

<sup>2</sup>Several different variants of these axioms have been used by various authors under the name *convergence space*.

<sup>3</sup>In other words, for every sink  $(f_i : (X_i, \xi_i) \rightarrow X)_{i \in I}$ , there exists a *final convergence* structure on  $X$ : the finest convergence on  $X$  making each  $f_i$  continuous. Equivalently, for every source  $(f_i : X \rightarrow (Y_i, \tau_i))_{i \in I}$  there exists an *initial convergence*: the coarsest convergence on  $X$  making each  $f_i$  continuous.

<sup>4</sup>In other words, for any pair  $(X, \xi), (Y, \tau)$  of convergence spaces, there exists the coarsest convergence  $[\xi, \tau]$  -called *continuous convergence*- on the set  $C(\xi, \tau)$  of continuous functions from  $X$  to  $Y$  making the evaluation map

$$ev : (X, \xi) \times (C(\xi, \tau), [\xi, \tau]) \rightarrow (Y, \tau)$$

(jointly) continuous.

<sup>5</sup>More generally, if  $o : 2^X \rightarrow 2^X$  and  $\mathcal{F} \subset 2^X$  then  $o^{\sharp} \mathcal{F}$  denotes  $\{o(F) : F \in \mathcal{F}\}$  and if  $\mathbb{D}$  is a class of filters (or of family of subsets) then  $o^{\sharp}(\mathbb{D})$  denotes  $\{\mathcal{F} : \exists \mathcal{D} \in \mathbb{D}, \mathcal{F} = o^{\sharp} \mathcal{D}\}$ .

is the (restriction to objects of the) reflector from **Conv** onto the full subcategory of respectively topological, pretopological, paratopological and pseudotopological spaces when  $\mathbb{D}$  is respectively, the class  $\text{cl}_\xi^{\text{h}}(\mathbb{F}_1)$ ,  $\mathbb{F}_1$ ,  $\mathbb{F}_\omega$  and  $\mathbb{F}$ .

A convergence space is *first-countable* if whenever  $x \in \lim \mathcal{F}$ , there exists a countably based filter  $\mathcal{H} \leq \mathcal{F}$  such that  $x \in \lim \mathcal{H}$ . Of course, a topological space is first-countable in the usual sense if and only if it is first-countable as a convergence space. Analogously, a convergence space is called *sequentially based* if whenever  $x \in \lim \mathcal{F}$ , there exists a sequence  $(x_n)_{n \in \omega} \leq \mathcal{F}$  <sup>(6)</sup> such that  $x \in \lim(x_n)_{n \in \omega}$ .

A class of filters  $\mathbb{D}$  (under mild conditions on  $\mathbb{D}$ ) defines a reflective subcategory of **Conv** (and the associated reflector) via (2.2). Dually, it also defines (under mild conditions on  $\mathbb{D}$ ) the coreflective subcategory of **Conv** of  $\mathbb{D}$ -based convergence spaces [4], and the associated (restriction to objects of the) coreflector  $\text{Base}_{\mathbb{D}}$  is

$$(2.3) \quad \lim_{\text{Base}_{\mathbb{D}}} \xi \mathcal{F} = \bigcup_{\mathbb{D} \triangleright \mathcal{D} \leq \mathcal{F}} \lim_{\xi} \mathcal{D}.$$

For instance, if  $\mathbb{D} = \mathbb{F}_\omega$  is the class of countably based filter, then  $\text{Base}_{\mathbb{D}}$  is the coreflector on first-countable convergence spaces. If  $\mathbb{D}$  is the class  $\mathbb{E}$  of filters generated by sequences, then  $\text{Base}_{\mathbb{D}}$  is the coreflector on sequentially based convergences.

**2.2. Local properties and special classes of filters.** Recall that a topological space is *Fréchet* (respectively, *strongly Fréchet*) if whenever  $x$  is in the closure of a subset  $A$  (respectively,  $x$  is in the intersection of closures of elements of a decreasing sequence  $(A_n)_n$  of subsets of  $X$ ) there exists a sequence  $(x_n)_{n \in \omega}$  of elements of  $A$  (respectively, such that  $x_n \in A_n$ ) such that  $x \in \lim(x_n)_{n \in \omega}$ . In other words, if  $x$  is in the adherence of a principal (resp. countably based) filter, then there exists a sequence meshing with that filter that converges to  $x$ . These are special cases of the following general notion, defined for convergence spaces.

Let  $\mathbb{D}$  and  $\mathbb{J}$  be two classes of filters. A convergence space  $(X, \xi)$  is called  $(\mathbb{J}/\mathbb{D})$ -*accessible* if

$$\text{adh}_{\xi} \mathcal{J} \subset \text{adh}_{\text{Base}_{\mathbb{D}}} \xi \mathcal{J},$$

for every  $\mathcal{J} \in \mathbb{J}$ . When  $\mathbb{D} = \mathbb{F}_\omega$  and  $\mathbb{J}$  is respectively the class  $\mathbb{F}$ ,  $\mathbb{F}_\omega$  and  $\mathbb{F}_1$ , then  $(\mathbb{J}/\mathbb{D})$ -accessible topological spaces are respectively bisequential, strongly Fréchet and Fréchet spaces. Analogously, if  $\mathbb{D}$  is the class of filters generated by long sequences (of arbitrary length) and  $\mathbb{J} = \mathbb{F}_1$  then  $(\mathbb{J}/\mathbb{D})$ -accessible topological spaces are radial spaces. We use the same names for these instances of  $(\mathbb{J}/\mathbb{D})$ -accessible convergence spaces (see [4] for details).

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<sup>6</sup>From the viewpoint of convergence, there is no reason to distinguish between a sequence and the filter generated by the family of its tails. Therefore, in this paper, sequences are identified to their associated filter and I will freely treat sequences as filters. Hence the notation  $(x_n)_{n \in \omega} \leq \mathcal{F}$ .

A filter  $\mathcal{F}$  is called  $\mathbb{J}$  to  $\mathbb{D}$  *meshable-refinable*, in symbol  $\mathcal{F} \in (\mathbb{J}/\mathbb{D})_{\# \geq}$ , if

$$\mathcal{J} \in \mathbb{J}, \mathcal{J} \# \mathcal{F} \implies \exists \mathcal{D} \in \mathbb{D}, \mathcal{D} \# \mathcal{J} \text{ and } \mathcal{D} \geq \mathcal{F}.$$

It follows immediately from the definitions that a *topological* space is  $(\mathbb{J}/\mathbb{D})$ -accessible if and only if every neighborhood filter is  $\mathbb{J}$  to  $\mathbb{D}$  meshable-refinable, and more generally that:

**Theorem 2.1.** *Let  $\mathbb{D}$  and  $\mathbb{J}$  be two classes of filters.*

- (1) *A convergence space  $(X, \xi)$  is  $(\mathbb{J}/\mathbb{D})$ -accessible if and only if  $\xi \geq \text{Adh}_{\mathbb{J}} \text{Base}_{\mathbb{D}} \xi$ ;*
- (2) *If  $\xi = \text{Base}_{(\mathbb{J}/\mathbb{D})_{\# \geq}} \xi$ , then  $\xi$  is  $(\mathbb{J}/\mathbb{D})$ -accessible. If moreover  $\xi$  is pre-topological (in particular topological) then the converse is true.*

The following gathers the most common cases of  $(\mathbb{J}/\mathbb{D})$ -accessible (topological) spaces and  $(\mathbb{J}/\mathbb{D})_{\# \geq}$ -filters when  $\mathbb{D} = \mathbb{F}_{\omega}$ . Denote by  $\mathbb{F}_{\wedge \omega}$  the class of *countably deep* <sup>(7)</sup> filters. The names for  $(\mathbb{J}/\mathbb{F}_{\omega})_{\# \geq}$ -filters come from the fact that a topological space is  $(\mathbb{J}/\mathbb{F}_{\omega})$ -accessible if and only if every neighborhood filter is a  $(\mathbb{J}/\mathbb{F}_{\omega})_{\# \geq}$ -filter.

class $\mathbb{J}$	$(\mathbb{J}/\mathbb{F}_{\omega})$ -accessible space	$(\mathbb{J}/\mathbb{F}_{\omega})_{\# \geq}$ -filter
$\mathbb{F}$	bisequential [14]	bisequential
$\mathbb{F}_{\omega}$	strongly Fréchet or countably bisequential [14]	strongly Fréchet
$(\mathbb{F}_{\omega}/\mathbb{F}_{\omega})_{\# \geq}$	productively Fréchet [12]	productively Fréchet
$\mathbb{F}_{\wedge \omega}$	weakly bisequential [1]	weakly bisequential
$\mathbb{F}_1$	Fréchet [14]	Fréchet

Table 1

**2.3. Compactness.** Let  $\mathbb{D}$  be a class of filters on a convergence space  $(X, \xi)$  and let  $\mathcal{A}$  be a family of subsets of  $X$ . A filter  $\mathcal{F}$  is  $\mathbb{D}$ -compact at  $\mathcal{A}$  (for  $\xi$ ) [7] if <sup>(8)</sup>

$$(2.4) \quad \mathcal{D} \in \mathbb{D}, \mathcal{D} \# \mathcal{F} \implies \text{adh}_{\xi} \mathcal{D} \# \mathcal{A}.$$

Notice that a subset  $K$  of a convergence space  $X$  (in particular of a topological space) is respectively compact, countably compact, Lindelöf if  $\{K\}^{\uparrow}$  is  $\mathbb{D}$ -compact at  $\{K\}$  if  $\mathbb{D}$  is respectively, the class  $\mathbb{F}$  of all,  $\mathbb{F}_{\omega}$  of countably based,  $\mathbb{F}_{\wedge \omega}$  of countably deep filters. On the other hand

**Theorem 2.2.** *Let  $\mathbb{D}$  be a class of filters. A filter  $\mathcal{F}$  is  $\mathbb{D}$ -compact at  $\{x\}$  for  $\xi$  if and only if*

$$x \in \lim_{\text{Adh}_{\mathbb{D}}} \xi \mathcal{F}.$$

In particular, if  $\xi$  is a topology, then  $x \in \lim \mathcal{F}$  if and only if  $\mathcal{F}$  is compact at  $\{x\}$  if and only if  $\mathcal{F}$  is  $\mathbb{F}_1$ -compact at  $\{x\}$ .

For a topological space  $X$ , a subset  $K$  is compact if and only if every open cover of  $K$  has a finite subcover of  $K$ , if and only if every filter on  $K$  has

<sup>7</sup>A filter  $\mathcal{F}$  is *countably deep* if  $\bigcap \mathcal{A} \in \mathcal{F}$  whenever  $\mathcal{A}$  is a countable subfamily of  $\mathcal{F}$ .

<sup>8</sup>Notice that (2.4) makes sense not only for a filter but for a general family  $\mathcal{F}$  of subsets of  $X$ . Such general compact families play an important role for instance in [8].

adherent points in  $K$ . In contrast, for general convergence spaces, the definition of compactness in terms of covers (cover-compactness) and in terms of filters (compactness) are different. If  $(X, \xi)$  is a convergence space, a family  $\mathcal{S} \subset 2^X$  is a *cover of  $K \subset X$*  if every filter converging to a point of  $K$  contains an element of  $\mathcal{S}$ . Hence a subset  $K$  of a convergence space is called *cover-(countably) compact* if every (countable) cover of  $K$  has a finite subcover. It is easy to see that a cover-compact convergence is compact, but in general not conversely. For instance, in a pseudotopological but not pretopological convergence, points are compact, but not cover-compact.

Notice that in this definition, we can assume the original cover  $\mathcal{S}$  to be stable under finite union, in which case we call  $\mathcal{S}$  an *additive cover*. The family  $\mathcal{S}_c$  of complements of elements of an additive cover  $\mathcal{S}$  is a filter-base on  $X$  with empty adherence. Hence  $K$  is cover-(countably) compact if every (countable) additive cover of  $K$  has an element that is a cover of  $K$ , or equivalently, if every (countable) filter-base with no adherence point in  $K$  has an element with no adherence point in  $K$ . In other words,  $K$  is cover-(countably) compact if every (countably based) filter whose every member has adherent points in  $K$ , has adherent points in  $K$ .

More generally, we will need the following characterization of cover-compactness in terms of filters [6]. Let  $\mathbb{D}$  and  $\mathbb{J}$  be two classes of filters. A filter  $\mathcal{F}$  is  $(\mathbb{D}/\mathbb{J})$ -compact at  $\mathcal{B}$  if

$$\mathcal{D} \in \mathbb{D}, \forall \mathcal{J} \in \mathbb{J}, \mathcal{J} \leq \mathcal{D}, \text{adh } \mathcal{J} \# \mathcal{F} \implies \text{adh } \mathcal{D} \# \mathcal{B}.$$

It is clear than if  $\mathcal{F}$  is  $(\mathbb{D}/\mathbb{F}_1)$ -compact (at  $\mathcal{B}$ ), then it is  $\mathbb{D}$ -compact (at  $\mathcal{B}$ ). More precisely, we have the following relationship between  $(\mathbb{D}/\mathbb{F}_1)$ -compactness and  $\mathbb{D}$ -compactness (which could be deduced from the results of [6, section 8])

**Proposition 2.3.** *Let  $\mathbb{D}$  be a class of filters on a convergence space  $(X, \xi)$ . A filter  $\mathcal{F}$  is  $(\mathbb{D}/\mathbb{F}_1)$ -compact at  $\mathcal{B}$  if and only if  $\mathcal{V}_\xi(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} \bigcap_{x \in F} \mathcal{V}_\xi(x)$  is  $\mathbb{D}$ -compact at  $\mathcal{B}$ .*

*Proof.* By definition,  $\mathcal{F}$  is  $(\mathbb{D}/\mathbb{F}_1)$ -compact at  $\mathcal{B}$  if and only if

$$\mathcal{D} \in \mathbb{D}, \left( \text{adh}_\xi^{\mathbb{D}} \mathcal{D} \right) \# \mathcal{F} \implies \text{adh } \mathcal{D} \# \mathcal{B}.$$

It is easy to verify that  $\left( \text{adh}_\xi^{\mathbb{D}} \mathcal{D} \right) \# \mathcal{F}$  if and only if  $\mathcal{D} \# \mathcal{V}_\xi(\mathcal{F})$ , which concludes the proof.  $\square$

Calling a convergence  $\xi$  *pretopologically diagonal*, or *P-diagonal*, if  $\lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{V}_\xi(\mathcal{F})$  for every filter  $\mathcal{F}$ , we obtain the following result, which is a particular case of a combination of Propositions 8.1 and 8.3 and of Theorem 8.2 in [6], even though the assumption that  $\text{adh}_\xi^{\mathbb{D}} \mathbb{D} \subset \mathbb{D}$  seems to be erroneously missing in [6].

**Corollary 2.4.** *If  $\xi$  is P-diagonal (in particular if  $\xi$  is a topology) and if  $\text{adh}_\xi^{\mathbb{D}} \mathbb{D} \subset \mathbb{D}$ , then  $(\mathbb{D}/\mathbb{F}_1)$ -compactness amounts to  $\mathbb{D}$ -compactness for  $\xi$ .*

*Proof.* Assume that  $\mathcal{F}$  is  $\mathbb{D}$ -compact (at  $\mathcal{B}$ ). To show that it is  $(\mathbb{D}/\mathbb{F}_1)$ -compact (at  $\mathcal{B}$ ), we only need to show that  $\mathcal{V}_\xi(\mathcal{F})$  is  $\mathbb{D}$ -compact (at  $\mathcal{B}$ ). But  $\mathcal{D}\#\mathcal{V}_\xi(\mathcal{F})$  if and only if  $(\text{adh}_\xi^{\natural}\mathcal{D})\#\mathcal{F}$ . Therefore,  $\text{adh}_\xi(\text{adh}_\xi^{\natural}\mathcal{D})\#\mathcal{B}$  because  $\text{adh}_\xi^{\natural}\mathcal{D} \in \mathbb{D}$ . Now,  $x \in \text{adh}_\xi(\text{adh}_\xi^{\natural}\mathcal{D})$  if there exists a filter  $\mathcal{G}\#\text{adh}_\xi^{\natural}\mathcal{D}$  with  $x \in \lim_\xi \mathcal{G}$ . Note that  $\mathcal{V}_\xi(\mathcal{G})\#\mathcal{D}$  and that, by  $P$ -diagonality,  $x \in \lim_\xi \mathcal{V}_\xi(\mathcal{G})$ . Hence  $\text{adh}_\xi(\text{adh}_\xi^{\natural}\mathcal{D}) \subset \text{adh}_\xi \mathcal{D}$  and  $\text{adh}_\xi \mathcal{D}\#\mathcal{B}$ .  $\square$

In some sense, the converse is true:

**Proposition 2.5.** *If  $\xi = \text{Adh}_{\mathbb{D}} \xi$  and  $\mathbb{D}$ -compactness implies  $(\mathbb{D}/\mathbb{F}_1)$ -compactness in  $\xi$ , then  $\xi$  is  $P$ -diagonal.*

*Proof.* If  $x \in \lim_\xi \mathcal{F}$  then  $\mathcal{F}$  is  $\mathbb{D}$ -compact at  $\{x\}$ , hence  $(\mathbb{D}/\mathbb{F}_1)$ -compact at  $\{x\}$ . Since  $\xi = \text{Adh}_{\mathbb{D}} \xi$ , we only need to show that  $x \in \text{adh}_\xi \mathcal{D}$  whenever  $\mathcal{D}$  is a  $\mathbb{D}$ -filter meshing with  $\mathcal{V}_\xi(\mathcal{F})$ . For any such  $\mathcal{D}$ , we have  $\text{adh}_\xi^{\natural}\mathcal{D}\#\mathcal{F}$  so that  $x \in \text{adh}_\xi \mathcal{D}$  because  $\mathcal{F}$  is  $(\mathbb{D}/\mathbb{F}_1)$ -compact at  $\{x\}$ .  $\square$

**2.4. Contour filters.** If  $\mathcal{F}$  is a filter on  $X$  and  $\mathcal{G} : X \rightarrow \mathbb{F}X$  then the *contour of  $\mathcal{G}$  along  $\mathcal{F}$*  is the filter on  $X$  defined by

$$\int_{\mathcal{F}} \mathcal{G} = \bigvee_{F \in \mathcal{F}} \bigwedge_{x \in F} \mathcal{G}(x).$$

This type of filters have been used in many situations, among others by Frolík under the name of sum of filters for a ZFC proof of the non-homogeneity of the remainder of  $\beta\mathbb{N}$  [11], by C. H. Cook and H. R. Fisher [3] under the name of compression operator of  $\mathcal{F}$  relative to  $\mathcal{G}$ , by H. J. Kowalsky [13] under the name of diagonal filter, and after them by many other authors to characterize topologicity and regularity of convergence spaces. To generalize this construction, I need to reproduce basic facts on cascades and multifilters. Detailed information on this topic can be found in [9].

If  $(W, \sqsubseteq)$  is an ordered set, then we write

$$W(w) = \{x \in W : w \sqsubseteq x\}.$$

An ordered set  $(W, \sqsubseteq)$  is *well-capped* if its every non empty subset has a maximal point <sup>(9)</sup>. Each well-capped set admits the (upper) *rank* to the effect that  $r(w) = 0$  if  $w \in \max W$ , and for  $r(w) > 0$ ,

$$r(w) = r_W(w) = \sup_{v \sqsupset w} (r(v) + 1).$$

A well-capped tree with least element is called a *cascade*; the least element of a cascade  $V$  is denoted by  $\emptyset = \emptyset_V$  and is called the *estuary* of  $V$ . The *rank* of a cascade is by definition the rank of its estuary. A cascade is a *filter cascade* if its every (non maximal) element is a filter on the set of its immediate successors.

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<sup>9</sup>In other words, a well-capped ordered set is a well-founded ordered set for the inverse order.

A map  $\Phi : V \setminus \{\emptyset_V\} \rightarrow X$ , where  $V$  is a cascade, is called a *multifilter on  $X$* . We talk about a multifilter  $\Phi : V \rightarrow X$  under the understanding that  $\Phi$  is not defined at  $\emptyset_V$ .

A couple  $(V, \Phi_0)$  where  $V$  is a cascade and  $\Phi_0 : \max V \rightarrow A$  is called a *perifilter on  $A$* . In the sequel we will consider  $V$  implicitly talking about a perifilter  $\Phi_0$ . If  $\Phi|_{\max V} = \Phi_0$ , then we say that the multifilter  $\Phi$  is an *extension* of the perifilter  $\Phi_0$ . The rank of a multifilter (perifilter) is, by definition, the rank of the corresponding cascade. If  $\mathbb{D}$  is a class of filters, we call  *$\mathbb{D}$ -multifilter* a multifilter with a cascade of  $\mathbb{D}$ -filters as domain.

The contour of a multifilter  $\Phi : V \rightarrow X$  depends entirely on the underlying cascade  $V$  and on the restriction of  $\Phi$  to  $\max V$ , hence on the corresponding perifilter  $(V, \Phi|_{\max V})$ . Therefore we shall not distinguish between the contours of multifilters and of the corresponding perifilters. The *contour* of  $\Phi : W \rightarrow X$  is defined by induction to the effect that  $\int \Phi = \Phi_{\natural}(\emptyset_W)$  if  $r(\Phi) = 1$ , and <sup>(10)</sup>

$$\int \Phi = \int_{\emptyset_W} \left( \int \Phi|_{W(\cdot)} \right)$$

otherwise. With each class  $\mathbb{D}$  of filters we associate the class  $\int \mathbb{D}$  of all  $\mathbb{D}$ -contour filters, i. e., the contours of  $\mathbb{D}$ -multifilter.

If  $\mathbb{D}$  and  $\mathbb{J}$  are two classes of filters, we say that  $\mathbb{J}$  is  *$\mathbb{D}$ -composable* if for every  $X$  and  $Y$ , the (possibly degenerate) filter  $\mathcal{H}\mathcal{F} = \{HF : H \in \mathcal{H}, F \in \mathcal{F}\}^\uparrow$  <sup>(11)</sup> belongs to  $\mathbb{J}(Y)$  whenever  $\mathcal{F} \in \mathbb{J}(X)$  and  $\mathcal{H} \in \mathbb{D}(X \times Y)$ , with the convention that every class of filters contains the degenerate filter. If a class  $\mathbb{D}$  is  $\mathbb{D}$ -composable, we simply say that  $\mathbb{D}$  is *composable*. Notice that

$$(2.5) \quad \mathcal{H}\#(\mathcal{F} \times \mathcal{G}) \iff \mathcal{H}\mathcal{F}\#\mathcal{G} \iff \mathcal{H}^- \mathcal{G}\#\mathcal{F},$$

where  $\mathcal{H}^- \mathcal{G} = \{H^-G = \{x \in X : (x, y) \in H \text{ and } y \in G\} : H \in \mathcal{H}, G \in \mathcal{G}\}^\uparrow$ .

**Lemma 2.6.** *Let  $\mathbb{D}$  and  $\mathbb{J}$  be two classes of filters. If  $\mathbb{D}$  is a  $\mathbb{J}$ -composable class of filters, then  $\int \mathbb{D}$  is also  $\mathbb{J}$ -composable.*

*Proof.* We proceed by induction on the rank of a  $\mathbb{D}$ -multifilter. The case of rank 1 is simply  $\mathbb{J}$ -composability of  $\mathbb{D}$ . Assume that for each  $\mathbb{D}$ -multifilter  $\Phi$  on  $X$  of rank  $\beta$  smaller than  $\alpha$  and each  $\mathbb{J}$ -filter  $\mathcal{J}$  on  $X \times Y$ , the filter  $\mathcal{J}(\int \Phi)$  is the contour of some  $\mathbb{D}$ -multifilter on  $Y$ . Consider now a  $\mathbb{D}$ -multifilter  $(\Phi, V)$  on  $X$  of rank  $\alpha$  and a  $\mathbb{J}$ -filter  $\mathcal{J}$  on  $X \times Y$ . Then

$$\int \Phi = \int_{\emptyset_V} \left( \int \Phi|_{V(\cdot)} \right) = \bigvee_{F \in \emptyset_V} \bigwedge_{v \in F} \int \Phi|_{V(v)},$$

and

$$\mathcal{J} \left( \int \Phi \right) = \bigvee_{F \in \emptyset_V} \bigwedge_{v \in F} \mathcal{J} \left( \int \Phi|_{V(v)} \right).$$

<sup>10</sup> $\Phi(v)$  is the image by  $\Phi$  of  $v$  treated as a point of  $V$ , while  $\Phi_{\natural}(v)$  is the filter generated by  $\{\Phi(F) : F \in v\}$ .

<sup>11</sup> $HF = \{y \in Y : (x, y) \in H \text{ and } x \in F\}$ .

As each  $\Phi|_{V(v)}$  is a multifilter of rank smaller than  $\alpha$ , each  $\mathcal{J}(\int \Phi|_{V(v)})$  is a  $(\int \mathbb{D})$ -filter. Moreover  $\emptyset_V$  is a  $\mathbb{D}$ -filter, so that  $\mathcal{J}(\int \Phi)$  is a contour of  $(\int \mathbb{D})$ -filters along a  $\mathbb{D}$ -filter, hence a  $(\int \mathbb{D})$ -filter.  $\square$

### 3. COMPACT RELATIONS

A relation  $R : (X, \xi) \rightrightarrows (Y, \tau)$  is  $\mathbb{D}$ -compact if for every subset  $A$  of  $X$  and every filter  $\mathcal{F}$  that is  $\mathbb{D}$ -compact at  $A$ , the filter  $R\mathcal{F}$  is  $\mathbb{D}$ -compact at  $RA$ .

**Proposition 3.1.** *If  $\mathbb{D}$  is  $\mathbb{F}_1$ -composable, then  $R : (X, \xi) \rightrightarrows (Y, \tau)$  is  $\mathbb{D}$ -compact if and only if  $R\mathcal{F}$  is  $\mathbb{D}$ -compact at  $Rx$  whenever  $x \in \lim_\xi \mathcal{F}$ .*

*Proof.* Only the "if" part needs a proof, so assume that  $R\mathcal{F}$  is  $\mathbb{D}$ -compact at  $Rx$  whenever  $x \in \lim_\xi \mathcal{F}$ , and consider a filter  $\mathcal{G}$  on  $X$  which is  $\mathbb{D}$ -compact at  $A$ . Let  $\mathcal{D}\#R\mathcal{G}$  be a  $\mathbb{D}$ -filter on  $Y$ . Then  $R^-\mathcal{D}\#\mathcal{G}$  so that there exists  $x \in A \cap \text{adh}_\xi R^-\mathcal{D}$ . Therefore, there exists  $\mathcal{U}\#R^-\mathcal{D}$  such that  $x \in \lim_\xi \mathcal{U}$ . By assumption,  $R\mathcal{U}$  is  $\mathbb{D}$ -compact at  $Rx \subset RA$ . Since  $\mathcal{D}\#R\mathcal{U}$ , the filter  $\mathcal{D}$  has adherent points in  $Rx$  hence in  $RA$ .  $\square$

**Corollary 3.2.** *Let  $\mathbb{D}$  be an  $\mathbb{F}_1$ -composable class of filters and let  $f : (X, \xi) \rightarrow (Y, \tau)$  with  $\tau = \text{Adh}_\mathbb{D} \tau$ . The following are equivalent:*

- (1)  *$f$  is continuous;*
- (2)  *$f$  is a compact relation;*
- (3)  *$f$  is a  $\mathbb{D}$ -compact relation.*

*Proof.* (1  $\implies$  2). If  $x \in \lim_\xi \mathcal{F}$ , then  $f(x) \in \lim_\tau f(\mathcal{F})$  so that  $f(\mathcal{F})$  is compact at  $f(x)$  and  $f$  is a compact relation by Proposition 3.1. (2  $\implies$  3) is obvious and (3  $\implies$  1) follows from Proposition 2.2.  $\square$

In particular,  $\mathbb{F}_1$ -compact (equivalently compact) maps between pretopological spaces (in particular between topological spaces) are exactly the continuous ones.

Notice that when  $\mathbb{D}$  contains the class of principal filters, then a  $\mathbb{D}$ -compact relation  $R$  is  $\mathbb{F}_1$ -compact and  $Rx$  is  $\mathbb{D}$ -compact for each  $x$  in the domain of  $R$ , because  $\{x\}^\uparrow$  is  $\mathbb{D}$ -compact at  $\{x\}$ . When the cover and filter versions of compactness coincide (in particular, in a topological space), the converse is true:

**Proposition 3.3.** *Let  $\mathbb{D}$  be an  $\mathbb{F}_1$ -composable class of filters. If  $R : (X, \xi) \rightrightarrows (Y, \tau)$  is an  $\mathbb{F}_1$ -compact relation and if  $Rx$  is  $(\mathbb{D}/\mathbb{F}_1)$ -compact in  $\tau$  for every  $x \in X$ , then  $R$  is  $\mathbb{D}$ -compact.*

*Proof.* Using Proposition 3.1, we need to show that  $R\mathcal{F}$  is  $\mathbb{D}$ -compact at  $Rx$  whenever  $x \in \lim_\xi \mathcal{F}$ . Consider a  $\mathbb{D}$ -filter  $\mathcal{D}\#R\mathcal{F}$ . Then,  $\text{adh}_\tau \mathcal{D}\#R\mathcal{F}$  for every  $D \in \mathcal{D}$  so that  $\text{adh}_\tau \mathcal{D}\#Rx$ , because  $Rx$  is  $\frac{\mathbb{D}}{\mathbb{F}_1}$ -compact.  $\square$

In view of Corollary 2.4, we obtain:



**Corollary 3.4.** *Let  $\mathbb{D}$  be an  $\mathbb{F}_1$ -composable class of filters such that  $\text{adh}_\tau^{\mathbb{D}} \mathbb{D}(\tau) \subset \mathbb{D}(\tau)$  and let  $\tau$  be a  $P$ -diagonal convergence (for instance a topology). Then  $R : (X, \xi) \rightrightarrows (Y, \tau)$  is  $\mathbb{D}$ -compact if and only if it is  $\mathbb{F}_1$ -compact and  $Rx$  is  $\mathbb{D}$ -compact in  $\tau$  for every  $x \in X$ .*

An immediate corollary of [9, Theorem 8.1] is that for a topology,  $\mathbb{D}$ -compactness amounts to  $(\int \mathbb{D})$ -compactness, provided that  $\mathbb{D}$  is a composable class of filters. However, the proof of [9, Theorem 8.1] only uses  $\mathbb{F}_1$ -composability of  $\mathbb{D}$ . Consequently,

**Corollary 3.5.** *Let  $\mathbb{D}$  be an  $\mathbb{F}_1$ -composable class of filters and let  $\tau$  be a topology such that  $\text{adh}_\tau^{\mathbb{D}} \mathbb{D}(\tau) \subset \mathbb{D}(\tau)$ . Let  $R : (X, \xi) \rightrightarrows (Y, \tau)$  be a relation. The following are equivalent:*

- (1)  $R$  is  $\mathbb{D}$ -compact;
- (2)  $R$  is  $\mathbb{F}_1$ -compact and  $Rx$  is  $\mathbb{D}$ -compact in  $\tau$  for every  $x \in X$ ;
- (3)  $R$  is  $\mathbb{F}_1$ -compact and  $Rx$  is  $(\int \mathbb{D})$ -compact in  $\tau$  for every  $x \in X$ ;
- (4)  $R$  is  $(\int \mathbb{D})$ -compact.

*Proof.* (1  $\iff$  2) and (3  $\iff$  4) follow from Corollary 3.4 and (1  $\iff$  4) follows from [9, Theorem 8.1]. □

The observation that perfect, countably perfect and closed maps can be characterized as  $\mathbb{D}$ -compact relations is due to S. Dolecki [6, section 10]. Recall that a surjection  $f : X \rightarrow Y$  between two topological spaces is *closed* if the image of a closed set is closed and *perfect* (resp. *countably perfect*, resp. *inversely Lindelöf*) if it is closed with compact (resp. countably compact, resp. Lindelöf) fibers. Once the concept of closed maps is extended to convergence spaces, all the other notions extend as well in the obvious way. As observed in [6, section 10], preservation of closed sets by a map  $f : (X, \xi) \rightarrow (Y, \tau)$  is equivalent to  $\mathbb{F}_1$ -compactness of the inverse map  $f^-$  when  $(X, \xi)$  is topological, but not if  $\xi$  is a general convergence. More precisely, calling a map  $f : (X, \xi) \rightarrow (Y, \tau)$  *adherent* [6] if

$$y \in \text{adh}_\tau f(H) \implies \text{adh}_\xi H \cap f^-y \neq \emptyset,$$

we have:

**Lemma 3.6.** (1) *A map  $f : (X, \xi) \rightarrow (Y, \tau)$  is adherent if and only if  $f^- : (Y, \tau) \rightrightarrows (X, \xi)$  is an  $\mathbb{F}_1$ -compact relation;*  
 (2) *If  $f : (X, \xi) \rightarrow (Y, \tau)$  is adherent, then it is closed;*  
 (3) *If  $f : (X, \xi) \rightarrow (Y, \tau)$  is closed and if adherence of sets are closed in  $\xi$  (in particular if  $\xi$  is a topology), then  $f$  is adherent.*

*Proof.* (1) follows from the definition and is observed in [6, section 10].

(2) If  $f(H)$  is not  $\tau$ -closed, then there exists  $y \in \text{adh}_\tau f(H) \setminus f(H)$ . Since  $f$  is adherent, there exists  $x \in \text{adh}_\xi H \cap f^-y$ . But  $x \notin H$  because  $f(x) = y \notin f(H)$ . Therefore  $H$  is not  $\xi$ -closed.

(3) is proved in [6, Proposition 10.2] even if this proposition is stated with a stronger assumption. □

Hence, a map  $f : (X, \xi) \rightarrow (Y, \tau)$  with a domain in which adherence of subsets are closed (in particular, a map with a topological domain) is adherent if and only if it is closed if and only if  $f^- : (Y, \tau) \rightrightarrows (X, \xi)$  is an  $\mathbb{F}_1$ -compact relation. If the domain and range of a map are topological spaces, it is well known that closedness of the map amounts to upper semicontinuity of the inverse relation. It was observed (for instance in [5]) that a (multivalued) map is upper semicontinuous (usc) if and only if it is an  $\mathbb{F}_1$ -compact relation.

A surjection  $f : X \rightarrow Y$  is  $\mathbb{D}$ -perfect if it is adherent with  $\mathbb{D}$ -compact fibers. In view of Corollary 3.5, compact valued usc maps between topological spaces, known as *usco maps*, are compact relations. Another direct consequence of Lemma 1 and of Corollary 3.5 is:

**Theorem 3.7.** *Let  $f : (X, \xi) \rightarrow (Y, \tau)$  be a surjection, let  $\mathbb{D}$  be an  $\mathbb{F}_1$ -composable class of filters, and let  $\xi$  be a topology such that  $\text{adh}_\xi^{\mathbb{D}} \mathbb{D} \subset \mathbb{D}$ . The following are equivalent:*

- (1)  $f$  is  $\mathbb{D}$ -perfect;
- (2)  $f^- : Y \rightrightarrows X$  is  $\mathbb{D}$ -compact;
- (3)  $f^- : Y \rightrightarrows X$  is  $(f \mathbb{D})$ -compact;
- (4)  $f$  is  $(f \mathbb{D})$ -perfect.

The equivalence between the first two points was first observed in [6, Proposition 10.2] but erroneously stated for general convergences as domain and range. Indeed, if  $f : (X, \xi) \rightarrow (Y, \tau)$  is a surjective map between two convergence spaces and if  $f^- : (Y, \tau) \rightrightarrows (X, \xi)$  is  $\mathbb{D}$ -compact, then  $f$  is adherent and has  $\mathbb{D}$ -compact fibers; if on the other hand  $f$  is adherent and has  $(\mathbb{D}/\mathbb{F}_1)$ -compact fibers then  $f^- : (Y, \tau) \rightrightarrows (X, \xi)$  is  $\mathbb{D}$ -compact. Hence, the two concepts are equivalent only when  $\mathbb{D}$ -compact sets are  $(\mathbb{D}/\mathbb{F}_1)$ -compact in  $\xi$ , for instance if  $\text{adh}_\xi^{\mathbb{D}}(\mathbb{D}) \subset \mathbb{D}$  and  $\xi$  is a  $P$ -diagonal convergence (in particular if  $\xi$  is a topology).

S. Dolecki offered in [4] a unified treatment of various classes of quotient maps and preservation theorems under such maps in the general context of convergences. He extended the usual notions of quotient maps to convergence spaces in the following way. A surjection  $f : (X, \xi) \rightarrow (Y, \tau)$  is  $\mathbb{D}$ -quotient if

$$(3.1) \quad y \in \text{adh}_\tau \mathcal{H} \implies f^-(y) \cap \text{adh}_\xi f^- \mathcal{H} \neq \emptyset,$$

for every  $\mathcal{H} \in \mathbb{D}(Y)$ . When  $\mathbb{D}$  is the class of all (resp. countably based, principal, principal of closed sets) filters, then continuous  $\mathbb{D}$ -quotient maps between topological spaces are exactly biquotient (resp. countably biquotient, hereditarily quotient, quotient) maps. Now, I present a new characterization of  $\mathbb{D}$ -quotient maps as  $\mathbb{D}$ -compact relations, in this general context of convergence spaces. As mentioned before, the category of convergence spaces and continuous maps is topological, hence if  $f : (X, \xi) \rightarrow Y$ , there exists the finest convergence — called *final convergence* and denoted  $f\xi$  — on  $Y$  making  $f$  continuous. Analogously, if  $f : X \rightarrow (Y, \tau)$ , there exists the coarsest convergence — called *initial convergence* and denoted  $f^- \tau$  — on  $X$  making  $f$  continuous. If  $\tau$  is topological, so is  $f^- \tau$ . In contrast,  $f\xi$  can be non topological even when  $\xi$  is topological.

**Theorem 3.8.** *Let  $\mathbb{D}$  be an  $\mathbb{F}_1$ -composable class of filters. Let  $f : (X, \xi) \rightarrow (Y, \tau)$  be a surjection. The following are equivalent:*

- (1)  $f : (X, \xi) \rightarrow (Y, \tau)$  is  $\mathbb{D}$ -quotient;
- (2)  $\tau \geq \text{Adh}_{\mathbb{D}} f\xi$ ;
- (3)  $f : (X, f^{-\tau}) \rightarrow (Y, f\xi)$  is a  $\mathbb{D}$ -compact relation.

*Proof.* The equivalence  $(1 \iff 2)$  is [4, Theorem 1.2].

$(1 \iff 3)$ . Assume  $f$  is  $\mathbb{D}$ -quotient and let  $x \in \lim_{f^{-\tau}} \mathcal{F}$ . Then  $f(x) \in \lim_{\tau} f(\mathcal{F})$ , so that  $f(x) \in \text{adh}_{\tau} \mathcal{D}$  whenever  $\mathcal{D} \in \mathbb{D}(Y)$  and  $\mathcal{D} \# f(\mathcal{F})$ . By (3.1),  $f^{-}(f(x)) \cap \text{adh}_{\xi} f^{-}\mathcal{D} \neq \emptyset$  so that  $f(x) \in f(\text{adh}_{\xi} f^{-}\mathcal{D})$ . In view of [6, Lemma 2.1],  $f(x) \in \text{adh}_{f\xi} \mathcal{D}$ .

Conversely, assume that  $f : (X, f^{-\tau}) \rightarrow (Y, f\xi)$  is  $\mathbb{D}$ -compact and let  $y \in \text{adh}_{\tau} \mathcal{D}$ . There exists  $\mathcal{G} \# \mathcal{D}$  such that  $y \in \lim_{\tau} \mathcal{G}$ . By definition of  $f^{-\tau}$ , the filter  $f^{-}\mathcal{G}$  is converging to every point of  $f^{-}y$  for  $f^{-\tau}$ . In view of Proposition 3.1, the filter  $ff^{-}\mathcal{G}$  is  $\mathbb{D}$ -compact at  $\{y\}$  for  $f\xi$ . Since  $f$  is surjective,  $ff^{-}\mathcal{G} \approx \mathcal{G}$  and  $\mathcal{G} \# \mathcal{D}$  so that  $y \in \text{adh}_{f\xi} \mathcal{D} = f(\text{adh}_{\xi} f^{-}\mathcal{D})$ , by [6, Lemma 2.1]. Therefore,  $f^{-}(y) \cap \text{adh}_{\xi} f^{-}\mathcal{D} \neq \emptyset$ .  $\square$

Notice that even if the map has topological range and domain, the notions need to be extended to convergence spaces to obtain such a characterization.

#### 4. COMPACTLY MESHABLE FILTERS AND RELATIONS

In view of the characterizations above of various types of maps as  $\mathbb{D}$ -compact relations, results of stability of  $\mathbb{D}$ -compactness of filters under product would particularize to product theorems for  $\mathbb{D}$ -compact spaces, but also for various types of quotient maps, for variants of perfect and closed maps, for usc and usco maps. Product theorems for  $\mathbb{D}$ -compact filters and their applications is the purpose of the companion paper [15]. A (complicated but extremely useful) notion fundamental to this study of products is the following:

A filter  $\mathcal{F}$  is  $\mathbb{M}$ -compactly  $\mathbb{J}$  to  $\mathbb{D}$  meshable at  $A$ , or  $\mathcal{F}$  is an  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})_{\#}$ -filter at  $A$ , if

$$\mathcal{F} \in \mathbb{J}, \mathcal{F} \# \mathcal{F} \implies \exists \mathcal{D} \in \mathbb{D}, \mathcal{D} \# \mathcal{F} \text{ and } \mathcal{D} \text{ is } \mathbb{M}\text{-compact at } A.$$

While the importance of this concept will be best highlighted by how it is used in the companion paper [15], I show here that the notion of an  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})_{\#}$ -filter is instrumental in characterizing a large number of classical concepts.

The notion of total countable compactness was first introduced by Z. Frolík [10] for a study of product of countably compact and pseudocompact spaces and rediscovered under various names by several authors (see [19, p. 212]). A topological space  $X$  is *totally countably compact* if every countably based filter has a finer (equivalently, meshes a) compact countably based filter. The name comes from *total* nets of Pettis. Obviously, a topological space is totally countably compact if and only if  $\{X\}$  is compactly  $\mathbb{F}_{\omega}$  to  $\mathbb{F}_{\omega}$  meshable. In [19], J. Vaughan studied more generally under which condition a product of  $\mathbb{D}$ -compact spaces is  $\mathbb{D}$ -compact, under mild conditions on the class of filters

$\mathbb{D}$ . He used in particular the concept of a totally  $\mathbb{D}$ -compact space  $X$ , which amounts to  $\{X\}$  being a compactly  $(\mathbb{D}/\mathbb{D})_{\#}$ -filter.

On the other hand, Theorem 2.1 can be completed by the following immediate rephrasing of the notion of  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})_{\#}$ -filters relative to a singleton in convergence theoretic terms.

**Proposition 4.1.** *Let  $\mathbb{D}$ ,  $\mathbb{J}$  and  $\mathbb{M}$  be three classes of filters, and let  $\xi$  and  $\theta$  be two convergences on  $X$ . The following are equivalent:*

- (1)  $\theta \geq \text{Adh}_{\mathbb{J}} \text{Base}_{\mathbb{D}} \text{Adh}_{\mathbb{M}} \xi$ ;
- (2)  $\mathcal{F}$  is an  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})_{\#}$ -filter at  $\{x\}$  in  $\xi$  whenever  $x \in \lim_{\theta} \mathcal{F}$ .  
*In particular,  $\xi = \text{Adh}_{\mathbb{M}} \xi$  is  $(\mathbb{J}/\mathbb{D})$ -accessible if and only if  $\mathcal{F}$  is an  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})_{\#}$ -filter at  $\{x\}$  whenever  $x \in \lim \mathcal{F}$ .*

In view of Table 1, this applies to a variety of classical local topological properties.

A relation  $R : (X, \xi) \rightrightarrows (Y, \tau)$  is  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})$ -meshable if

$$\mathcal{F} \rightarrow x \implies R(\mathcal{F}) \text{ is } \mathbb{M}\text{-compactly } (\mathbb{J}/\mathbb{D})\text{-meshable at } Rx \text{ in } \tau.$$

**Theorem 4.2.** *Let  $\mathbb{M} \subset \mathbb{J}$ , let  $\tau = \text{Adh}_{\mathbb{M}} \tau$  and let  $f : (X, \xi) \rightarrow (Y, \tau)$  be a continuous surjection. The map  $f$  is  $\mathbb{M}$ -quotient with  $(\mathbb{J}/\mathbb{D})$ -accessible range if and only if  $f : (X, f^{-\tau}) \rightarrow (Y, f\xi)$  is an  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})$ -meshable relation.*

*Proof.* Assume that  $f$  is  $\mathbb{M}$ -quotient with  $(\mathbb{J}/\mathbb{D})$ -accessible range and let  $x \in \lim_{f^{-\tau}} \mathcal{F}$ . Then  $y = f(x) \in \lim_{\tau} f(\mathcal{F})$ . Let  $\mathcal{J}$  be a  $\mathbb{J}$ -filter such that  $\mathcal{J} \# f(\mathcal{F})$ . Since  $y \in \text{adh}_{\tau} \mathcal{J}$  and  $\tau$  is  $(\mathbb{J}/\mathbb{D})$ -accessible, there exists a  $\mathbb{D}$ -filter  $\mathcal{D} \# \mathcal{J}$  such that  $y \in \lim_{\tau} \mathcal{D}$ . To show that  $f(\mathcal{F})$  is  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})$ -meshable at  $y$  in  $f\xi$ , it remains to show that  $\mathcal{D}$  is  $\mathbb{M}$ -compact at  $\{y\}$  for  $f\xi$ , that is, that  $y \in \lim_{\text{Adh}_{\mathbb{M}} f\xi} \mathcal{D}$ , which follows from the  $\mathbb{M}$ -quotientness of  $f$ .

Conversely, assume that  $f : (X, f^{-\tau}) \rightarrow (Y, f\xi)$  is an  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})$ -meshable relation, and let  $y \in \lim_{\tau} \mathcal{G}$ . Then  $f^{-}(\mathcal{G})$  converges to any point  $x \in f^{-}y$  for  $f^{-\tau}$ . Therefore,  $f(f^{-}\mathcal{G})$  is  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})$ -meshable at  $\{y\}$  in  $f\xi$ . Because  $f$  is a surjection,  $f(f^{-}\mathcal{G}) = \mathcal{G}$ . Consider  $\mathcal{M} \in \mathbb{M} \subset \mathbb{J}$  such that  $\mathcal{M} \# \mathcal{G}$ . There exists a  $\mathbb{D}$ -filter  $\mathcal{D} \# \mathcal{M}$  which is  $\mathbb{M}$ -compact at  $\{y\}$  in  $f\xi$ . Hence,  $y \in \text{adh}_{f\xi} \mathcal{M}$ , so that  $y \in \lim_{\text{Adh}_{\mathbb{M}} f\xi} \mathcal{G}$ . Therefore,  $f$  is  $\mathbb{M}$ -quotient. Moreover, if  $y \in \text{adh}_{\tau} \mathcal{J}$  for a  $\mathbb{J}$ -filter  $\mathcal{J}$ , then there exists  $\mathcal{G} \# \mathcal{J}$  such that  $y \in \lim_{\tau} \mathcal{G}$ . By the previous argument,  $\mathcal{G}$  is  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})$ -meshable at  $\{y\}$  in  $f\xi$ . In particular, there exists a  $\mathbb{D}$ -filter  $\mathcal{D} \# \mathcal{J}$  which is  $\mathbb{M}$ -compact at  $\{y\}$  in  $f\xi$ . In other words,  $y \in \lim_{\text{Adh}_{\mathbb{M}} f\xi} \mathcal{D}$ . Since  $f : (X, \xi) \rightarrow (Y, \tau)$  is continuous,  $\tau \leq f\xi$  so that  $\text{Adh}_{\mathbb{M}} \tau = \tau \leq \text{Adh}_{\mathbb{M}} f\xi$ . Hence  $y \in \lim_{\tau} \mathcal{D}$  and  $\tau$  is  $(\mathbb{J}/\mathbb{D})$ -accessible.  $\square$

$\mathbb{M}$	$\mathbb{J}$	$\mathbb{D}$	map $f$ as in Theorem 4.2
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}_1$	hereditarily quotient with finitely generated range
$\mathbb{F}_1$	$\mathbb{F}_1$	$\mathbb{F}_\omega$	hereditarily quotient with Fréchet range
$\mathbb{F}_1$	$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	hereditarily quotient with strongly Fréchet range
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}_\omega$	hereditarily quotient with bisequential range
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}$	hereditarily quotient
$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	$\mathbb{F}_1$	countably biquotient with finitely generated range
$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	countably biquotient with strongly Fréchet range
$\mathbb{F}_\omega$	$\mathbb{F}$	$\mathbb{F}_\omega$	countably biquotient with bisequential range
$\mathbb{F}_\omega$	$\mathbb{F}$	$\mathbb{F}$	countably biquotient
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_1$	biquotient with finitely generated range
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_\omega$	biquotient with bisequential range
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	biquotient

**Theorem 4.3.** *Let  $\mathbb{M} \subset \mathbb{J}$  and  $\mathbb{D}$  be three classes of filters, where  $\mathbb{J}$  and  $\mathbb{D}$  are  $\mathbb{F}_1$ -composable. Let  $\tau = \text{Adh}_{\mathbb{M}} \tau$  and let  $\xi$  be a  $P$ -diagonal convergence such that  $\text{adh}_{\xi}^{\natural}(\mathbb{M}) \subset \mathbb{M}$ . Let  $f : (X, \xi) \rightarrow (Y, \tau)$  be a continuous surjection. The map  $f$  is  $\mathbb{M}$ -perfect with  $(\mathbb{J}/\mathbb{D})$ -accessible range if and only if  $f^{-} : (Y, \tau) \rightrightarrows (X, \xi)$  is an  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})$ -meshable relation.*

*Proof.* Assume that  $f$  is  $\mathbb{M}$ -perfect with  $(\mathbb{J}/\mathbb{D})$ -accessible range and let  $y \in \lim_{\tau} \mathcal{G}$ . Consider a  $\mathbb{J}$ -filter  $\mathcal{J} \# f^{-} \mathcal{G}$ . By  $\mathbb{F}_1$ -composability,  $f(\mathcal{J})$  is a  $\mathbb{J}$ -filter. Moreover  $f(\mathcal{J}) \# \mathcal{G}$  so that  $y \in \text{adh}_{\tau} f(\mathcal{J})$ . Since  $\tau$  is  $(\mathbb{J}/\mathbb{D})$ -accessible, there exists a  $\mathbb{D}$ -filter  $\mathcal{D} \# f(\mathcal{J})$  such that  $y \in \lim_{\tau} \mathcal{D}$ . In view of Corollary 3.5,  $f^{-} : (Y, \tau) \rightrightarrows (X, \xi)$  is  $\mathbb{M}$ -compact because  $f$  is  $\mathbb{M}$ -perfect. Therefore,  $f^{-} \mathcal{D}$  is  $\mathbb{M}$ -compact at  $f^{-} y$  in  $\xi$ . Moreover,  $f^{-} \mathcal{D} \in \mathbb{D}$  because  $\mathbb{D}$  is  $\mathbb{F}_1$ -composable and  $f^{-} \mathcal{D} \# \mathcal{J}$ . Hence,  $f^{-} \mathcal{G}$  is  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})$ -meshable at  $f^{-} y$  in  $\xi$ .

Conversely, assume that  $f^{-} : (Y, \tau) \rightrightarrows (X, \xi)$  is an  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})$ -meshable relation. It is in particular an  $\mathbb{M}$ -compact relation because  $\mathbb{M} \subset \mathbb{J}$ . In view of Corollary 3.5,  $f$  is  $\mathbb{M}$ -perfect. Now assume that  $y \in \text{adh}_{\tau} \mathcal{J}$  where  $\mathcal{J} \in \mathbb{J}$ . There exists  $\mathcal{G} \# \mathcal{J}$  such that  $y \in \lim_{\tau} \mathcal{G}$ . Therefore,  $f^{-} \mathcal{G}$  is  $\mathbb{M}$ -compactly  $(\mathbb{J}/\mathbb{D})$ -meshable at  $f^{-} y$  in  $\xi$ . The filter  $f^{-} \mathcal{J}$  meshes with  $f^{-} \mathcal{G}$  because  $f$  is surjective, and is a  $\mathbb{J}$ -filter because  $\mathbb{J}$  is  $\mathbb{F}_1$ -composable. Hence, there exists a  $\mathbb{D}$ -filter  $\mathcal{D} \# f^{-} \mathcal{J}$  which is  $\mathbb{M}$ -compact at  $f^{-} y$  in  $\xi$ . By continuity of  $f$ , the filter  $f(\mathcal{D})$  is  $\mathbb{M}$ -compact at  $\{y\}$  in  $\tau$  (Corollary 3.2). In view of Proposition 2.2,  $y \in \lim_{\text{Adh}_{\mathbb{M}} \tau} f(\mathcal{D})$ . Moreover, the filter  $f(\mathcal{D})$  meshes with  $\mathcal{J}$  and is a  $\mathbb{D}$ -filter, by  $\mathbb{F}_1$ -composability of  $\mathbb{D}$ . Since  $\tau = \text{Adh}_{\mathbb{M}} \tau$ , we conclude that  $y \in \lim_{\tau} f(\mathcal{D})$  and  $\tau$  is  $(\mathbb{J}/\mathbb{D})$ -accessible.  $\square$

$\mathbb{M}$	$\mathbb{J}$	$\mathbb{D}$	map $f$ as in Theorem 4.3
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}_1$	closed with finitely generated range
$\mathbb{F}_1$	$\mathbb{F}_1$	$\mathbb{F}_\omega$	closed with Fréchet range
$\mathbb{F}_1$	$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	closed with strongly Fréchet range
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}_\omega$	closed with bisequential range
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}$	closed
$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	$\mathbb{F}_1$	countably perfect with finitely generated range
$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	countably perfect with strongly Fréchet range
$\mathbb{F}_\omega$	$\mathbb{F}$	$\mathbb{F}_\omega$	countably perfect with bisequential range
$\mathbb{F}_\omega$	$\mathbb{F}$	$\mathbb{F}$	countably perfect
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_1$	perfect with finitely generated range
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_\omega$	perfect with bisequential range
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	perfect

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