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The diagonal of a first countable paratopological group, submetrizability, and related results

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Abstract. We discuss some properties stronger than G_δ-diagonal. Among other things, we prove that any first countable paratopological group has a G_δ-diagonal of infinite rank and hence also a regular G_δ-diagonal. This answer a question recently asked by Arhangel'skii and Burke.

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A semitopological group is a group with a topology such that the multiplication is separately continuous.

A paratopological group is a group with a topology such that the multiplication is jointly continuous.

In the sequel, all spaces are assumed to be Hausdorff. For notations and undefined notions we refer to [5].

The starting point of the present note was to answer Problem 25 in [1]. In that paper the authors proved that a first-countable Abelian paratopological group has a regularG_δ-diagonal, raising the question of whether the Abelianity of the group could be dropped.

A topological space X has a regular G_δ-diagonal if there exists a countable family $\{U_n : n < \omega\}$ of open subsets of $X \times X$ such that $\Delta(X) = \bigcap\{\overline{U_n} : n < \omega\}$. Here $\Delta(X)$ denotes the diagonal $\{(x, x) : x \in X\}$ of X. The star of a collection $\mathcal{U}$ with respect to a set A is the set $st(\mathcal{U}, A) = \bigcup\{U : U \in \mathcal{U}$ and $U \cap A \neq \varnothing\}$. When $A = \{x\}$, we simply write $st(\mathcal{U}, x)$. We put $st^1(\mathcal{U}, x) = st(\mathcal{U}, x)$ and recursively define $st^{n+1}(\mathcal{U}, x) = st(\mathcal{U}, st^n(\mathcal{U}, x))$.

We say that a space X has a G_δ-diagonal of rank n if there exists a countable collection $\{\mathcal{U}_k : k < \omega\}$ of open covers of$X$ such that $\bigcap\{st^n(\mathcal{U}_k, x) : k < \omega\} = \{x\}$ for each $x \in X$. If a space has a G_δ-diagonal of any possible rank, then we say that it has a G_δ-diagonal of infinite rank.

Zenor has pointed out in [9] that a G_δ-diagonal of rank 3 is always regular. For the reader's benefit, we provide here the simple proof.

Proposition 1. *A topological space X with a G_δ-diagonal of rank 3 has also a regular G_δ-diagonal.*

Proof. Let $\{\mathcal{U}_n : n < \omega\}$ be a sequence of open covers of X witnessing the rank 3 of the diagonal and put $V_n = \bigcup\{U \times U : U \in \mathcal{U}_n\}$. We will check that $\Delta(X) = \bigcap\{\overline{V_n} : n < \omega\}$. Let $(x, y) \in X \times X \setminus \Delta(X)$ and choose n in such a way that $y \notin st^3(\mathcal{U}_n, x)$. Then, select $U_x, U_y \in \mathcal{U}_n$ such that $x \in U_x$ and $y \in U_y$. We must have $U_x \times U_y \cap V_n = \varnothing$, otherwise there would exist some $U \in \mathcal{U}_n$ such that $U_x \cap U \neq \varnothing \neq U_y \cap U$ and this in turn would imply that $y \in st^3(\mathcal{U}_n, x)$. $\square$

We would like to mention that the requirement for the diagonal to be an intersection of closed neighbourhoods implies a stronger separation axiom.

Proposition 2. *If the diagonal of the space X is the intersection of a collection of closed neighbourhoods, then Xsatisfies the Urysohn separation axiom.*

Proof. Let $x, y \in X$ and $x \neq y$. Fix an open set $U \subseteq X \times X$ such that $\Delta(X) \subseteq U$ and $(x, y) \notin \overline{U}$. Clearly, we may find open sets $V, W \subseteq X$ such that $(x, y) \in V \times W \subseteq X \times X \setminus U$. Let us assume that there exists a point $z \in \overline{V} \cap \overline{W}$ and fix an open set $O \subseteq X$ such that $(z, z) \in O \times O \subseteq U$. As we have $V \times W \cap O \times O \neq \varnothing$, we reach a contradiction and so $\overline{V} \cap \overline{W} = \varnothing$. $\square$

Corollary 1. *Every space with a regular G_δ-diagonal is a Urysohn space.*

The usual Ψ-space over the integers is a pseudocompact space with a G_δ-diagonal of rank 2 which does not have a regular G_δ-diagonal. The latter fact maybe derived from a result of Mc Arthur [7], stating that any pseudocompact space with a regular G_δ-diagonal is compact.At the moment, we do not know the answer to the following:

Problem 1. *Is every regular G_δ diagonal always of rank 2?*

We are interested in the above problem also because a positive answer would allow us to deduce a recent remarkable result of Buzyakova [4], stating that a ccc-space with a regular G_δ-diagonal has cardinality at most $\mathfrak{c}$, from an old result of Bella [3], saying the same for a G_δ-diagonal of rank 2. If we assume a diagonal of higher rank, we can weaken the hypothesis on the cellularity in the last mentioned result.

Theorem 1. *If X is a space with a G_δ-diagonal of rank 4 and cellularity at most $\mathfrak{c}$, then $|X| \leq \mathfrak{c}$.*

Proof. Let $\{\mathcal{U}_n : n < \omega\}$ be a collection of open covers of X satisfying the formula $\bigcap\{st^4(\mathcal{U}_n, x) : n < \omega\} = \{x\}$ for each $x \in X$. Let $A_n \subseteq X$ be maximal with respect to the property that$st^2(\mathcal{U}_n, x) \cap A_n = \{x\}$. As the family $\{st(\mathcal{U}_n, x) : x \in A_n\}$ consists of pairwise disjoint sets, we have$|A_n| \leq \mathfrak{c}$. Moreover, $\{st^2(\mathcal{U}_n, x) : x \in A_n\}$ is a cover of X. For any $x \in X$ and any $n < \omega$, choose$x_n \in A_n$ such that $x \in st^2(\mathcal{U}_n, x_n)$ and let $\phi(x) = \{st^2(\mathcal{U}_n, x_n) : n < \omega\}$. Since we are assuming that $\bigcap\{st^4(\mathcal{U}_n, x) : n < \omega\} = \{x\}$, we have $\bigcap \phi(x) = \{x\}$ and so the map $x \mapsto \phi(x)$ is one-to-one. Now, an easy counting argument shows that $|X| \leq \mathfrak{c}$. □

As it is well-known [8], there are ccc-spaces withG_δ -diagonal and arbitrarily large cardinality. Nevertheless, the following question remains open.

Problem 2. *Is Theorem 1 still true if we assume the diagonal to be of rank* 2 *or* 3*?*

Now we move on to the case of a space with a group structure.

Let G be a semitopological group. A local π-base $\mathcal{P}$ at the neutral element e is called T-linked (here T stands for translation) if $Px \cap xP \neq \varnothing$ for any $P \in \mathcal{P}$ and $x \in G$. This notion is instrumental to the following:

Lemma 1. *Let G be a semitopological group and $\mathcal{P}$ be a T-linked local π-base at the neutral element. Then, for any $P \in \mathcal{P}$, the collection $\mathcal{U}(P) = \{Px \cap xP : x \in G\}$ is a cover of G.*

Proof. Fix $P \in \mathcal{P}$ and $x \in G$. Since $Px \cap xP \neq \varnothing$, we may find $p_1, p_2 \in P$ such that$p_1 x = x p_2$. Observe that $Pp_1^{-1}x \ni p_1 p_1^{-1} x = x$ and $p_1^{-1}xP \ni p_1^{-1} x p_2 = p_1^{-1} p_1 x = x$ and so $x \in Pp_1^{-1}x \cap p_1^{-1}xP$. □

Lemma 2. *If G is a paratopological group, then for any pair of distinct points $y, z \in G$ and any integer n there exists a neighbourhood P of the neutral element e such that$P^n y \cap zP^n = \varnothing$.*

Theorem 2. *If G is a paratopological group with a countable T-linked local π-base at the neutral element, then G has a G_δ-diagonal of infinite rank.*

Proof. For any $P \subseteq G$ and any $x \in G$ we put$P[x] = xP \cap Px$. Let $\mathcal{P}$ be a countable T-linked local π-base at the neutral element e.For any $P \in \mathcal{P}$, let $\mathcal{U}(P) = \{xP \cap Px : x \in G\} = \{P[x] : x \in G\}$. By Lemma 1, each $\mathcal{U}(P)$ is a cover of G. We will show that the collection $\Gamma = \{\mathcal{U}(P) : P \in \mathcal{P}\}$ witnesses the infinite rank of the diagonal. Towards this end, fix an integer n and a pair of distinct points $y, z \in G$. We have to check that there is some $P \in \mathcal{P}$ such that

$$* \quad z \notin st^n(\mathcal{U}(P), y)$$

By Lemma 2, we may fix $P \in \mathcal{P}$ in such a way that $P^n y \cap zP^n = \varnothing$. The failure of (*) for this P means that there exist $x_1, \ldots, x_{n-1} \in G$ such that$y \in P[x_1]$, $z \in P[x_{n-1}]$ and $P[x_i] \cap P[x_{i+1}] \neq \varnothing$ for $1 \leq i \leq n-1$. The previous conditions are equivalent to the existence of point s $p_1, \ldots, p_n, q_0, \ldots, q_{n-1} \in P$ satisfying $y = x_1 q_0$, $z = p_n x_n$ and $p_i x_i = x_{i+1} q_i$. From the equality$x_i = p_i^{-1} x_{i+1} q_i$, we may easily arrive at the formula $y = p_1^{-1} p_2^{-1} \cdots p_n^{-1} z q_{n-1} q_{n-2} \cdots q_1 q_0$. It follows that $y p_n p_{n-1} \cdots p_2 p_1 y = z q_{n-1} q_{n-2} \cdots q_1 q_0$. Hence, $P^n y \cap zP^n$ is non-empty, a contradiction. □

As a local base is always T-linked, we have:

Corollary 2. *Every first countable paratopological group has a G_δ-diagonal of infinite rank.*

As a local π-base in an Abelian group is clearly T-linked, we have:

Corollary 3. *Every Abelian paratopological group of countable π-character has a G_δ-diagonal of infinite rank.*

Obviously, Corollaries 2 and 3 suggest the following:

Problem 3. *Does a (Hausdorff, regular, Tychonoff) paratopological group of countable π-character have a G_δ-diagonal of rank n for each integer n?*

From Corollary 2 and Proposition 1 we immediately obtain the following recent result of C. Liu [6], answering Problem 25 in [1]:

Corollary 4. *Any first countable paratopological group has a regular G_δ-diagonal.*

Similarly, from Corollary 3 and Proposition 1 we obtain:

Corollary 5. *Every Abelian paratopological group of countable π-character has a regular G_δ-diagonal.*

Corollary 6. *Any first countable paratopological group with cellularity at most $\mathfrak{c}$ has cardinality at most $\mathfrak{c}$.*

Observe that in any first countable paratopological group there is one countable family of open covers witnessing that G has a G_δ-diagonal of rank n for each $n < \omega$. We finish this note with a sufficient condition for the submetrizability of a semitopological group, which slightly generalizes Theorem 28 in [1]. A semitopological group G is said to be ω *-narrow* if for every open neighbourhood U of the neutral element e of G there is a countable subset $A \subseteq G$ such that $AU = G$.

Theorem 3. *Every separable semitopological group is ω-narrow.*

Proof. Fix a countable subset A of G such that A^{-1} is dense in G, and let U be any open neighbourhood of the neutral element $e$in G.Let us show that $AU = G$. Take any $b \in G$. There is an open set$W$ such that$b^{-1} \in W$ and $Wb \subseteq U$. Since A^{-1} is dense in G, there is $a \in A$ such that $a^{-1} \in W$. Then $a^{-1}b \in U$ and hence, $b \in aU$. Thus, $AU = G$. □

Lemma 3. *Suppose that G is a semitopological group, and that y, z are any two distinct elements of G. Then there is an open neighbourhood U of the neutral element e in G such that for the family $\gamma_U = \{xU : x \in G\}$ we have $y \notin St(z, \gamma_U)$.*

Proof. Clearly, we may assume that $z = e$. Since G is Hausdorff, there is an open neighbourhood U of e such that $U \cap Uy = \varnothing$. Then $y \notin U^{-1}U$.We now show that U is the neighbourhood we are looking for. Indeed, take any $x \in G$ such that $e \in xU$. Then $x \in U^{-1}$ and hence, $xU \subseteq U^{-1}U$. It follows that y is not in xU. Thus, $y \notin St(z, \gamma_U)$. □

Theorem 4. *Suppose that G is a Tychonoff ω-narrow semitopological group of countable π-character. Then the space G is submetrizable.*

Proof. Let $\mathcal{P}$ be a countable local π-base at the neutral element e. Since the space G is Tychonoff, we may assume that every $U \in \mathcal{P}$ is a cozero-set. For each $U \in \mathcal{P}$, put $\gamma_U = \{xU : x \in G\}$. Since G is ω-narrow, there is a countable subcover $\eta_U \subseteq \gamma_U$ of G. Put $\mathcal{B} = \cup\{\eta_U : U \in \mathcal{P}\}$. Then $\mathcal{B}$ is a countable family of cozero-sets in G. The family $\mathcal{B}$ is T_1-separating. Indeed, fix any distinct y and z in G. By Lemma 2, there is $U \in \mathcal{P}$ such that for $\gamma_U = \{xU : x \in G\}$ we have $y \notin St(z, \gamma_U)$. By the choice of η_U, there is $V \in \eta_U$ such that $z \in V$. Then $y \notin V \in \mathcal{B}$. Thus, the family $\mathcal{B}$ is T_1-separating.It remains to make a standard step: for every $V \in \mathcal{B}$ fix a continuous real-valued function f_V on G such that $V = \{x \in G : f_V(x) \neq 0\}$, and take the diagonal product of functions f_V where V runs over the countable set $\mathcal{B}$. The resulting function is the desired one-to-one continuous mapping of the space G onto a separable metrizable space. $\square$

Problem 4. *Is every (Hausdorff, regular) semitopological (paratopological) group with countable Souslin number ω-narrow?*

Problem 5. *Let G be a paratopological (semitopological) (Hausdorff, regular) group of countable extent. Must G be ω-narrow?*

Problem 6. *Let G be a paratopological (semitopological) (Hausdorff, regular) group of countable extent. Must G be submetrizable*

In connection with the last open question, we have the following partial result:

Theorem 5. *If G is a first countable normal (weakly M-normal) paratopological group of the countable extent, then G can be condensed onto a separable metrizable space (hence, G is submetrizable).*

Proof. Indeed, by Theorem 2, G has a rank 5-diagonal. Besides, G is star-Lindelöf. It remains to apply a result from [2]. $\square$

Clearly, "Countable extent" can be replaced by "star-Lindelöf" in Theorem 5.

Another open problem, already formulated in [1], is whether every first countable regular paratopological group is submetrizable.

In connection with Problem 4, observe that if the answer is "yes", then every Tychonoff first countable paratopological group with countable Souslin number is submetrizable.

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