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# Functorial comparisons of bitopology with topology and the case for redundancy of bitopology in lattice-valued mathematics

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Abstract. This paper studies various functors between (lattice-valued) topology and (lattice-valued) bitopology, including the expected "doubling" functor $E_d : L\text{-}\mathbf{Top} \to L\text{-}\mathbf{BiTop}$ and the "cross" functor $E_\times : L\text{-}\mathbf{BiTop} \to L^2\text{-}\mathbf{Top}$ introduced in this paper, both of which are extremely well-behaved strict, concrete, full embeddings. Given the greater simplicity of lattice-valued topology *vis-a-vis* lattice-valued bitopology and the fact that the class of $L^2$-**Top**'s is strictly smaller than the class of $L$-**Top**'s encompassing fixed-basis topology, the class of $E_\times$'s makes the case that lattice-valued bitopology is categorically redundant. As a special application, traditional bitopology as represented by **BiTop** is (isomorphic in an extremely well-behaved way to) a strict subcategory of **4-Top,** where **4** is the four element Boolean algebra; this makes the case that traditional bitopology is a special case of a much simpler fixed-basis topology.

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## 1. Introduction and preliminaries

1.1. **Motivation.** Bitopology has a long and distinguished history spanning five decades and a literature of some 700 papers [29] with traditional bitopology playing a wide range of roles in Baire spaces, homotopy and algebraic topology, generalizations of metric spaces, biframes, programming semantics, etc.

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First defined and used in [31, 32, 3, 4], a bitopological space was originally defined as a triple $((X, \mathfrak{T}), (X, \mathfrak{S}), e)$ with $(X, \mathfrak{T}), (X, \mathfrak{S})$ topological spaces and $e : (X, \mathfrak{T}) \to (X, \mathfrak{S})$ a continuous bijection—cf. [3]. But if we set

$$\mathfrak{T}' = \{e^{\to}(U) : U \in \mathfrak{T}\},$$

then $\mathfrak{T}'$ is a topology on $X$ and the continuity of $e$ insures that $id_X : (X, \mathfrak{T}') \to (X, \mathfrak{S})$ is continuous, i.e., that $\mathfrak{T}' \supset \mathfrak{S}$. It is therefore not surprising that almost immediately [4] the original definition was replaced by the simpler, equivalent definition that a bitopological space is a triple $(X, \mathfrak{T}, \mathfrak{S})$ with $\mathfrak{T}, \mathfrak{S}$ topologies on $X$ with $\mathfrak{T} \supset \mathfrak{S}$, $\mathfrak{T}$ being called the strong topology and $\mathfrak{S}$ the weak topology. Even in the broader lattice-valued topology setting, this definition plays a categorical role (Proposition 3.5 below).

Since a quasi-pseudo-metric $p$ on a set $X$ determines its conjugate quasi-pseudo-metric $q$, namely by $q(x, y) = p(y, x)$, quasi-pseudo-metrics necessarily occur in conjugate pairs which generate pairs of topologies that need not be related. Thus the definition of a traditional bitopological space was generalized in [22] to its modern form to be an ordered triple $(X, \mathfrak{T}, \mathfrak{S})$ with $\mathfrak{T}, \mathfrak{S}$ topologies on $X$ (and no relationship assumed between $\mathfrak{T}$ and $\mathfrak{S}$). Further, a bicontinuous mapping $f : (X, \mathfrak{T}_1, \mathfrak{T}_2) \to (Y, \mathfrak{S}_1, \mathfrak{S}_2)$ is a mapping $f : X \to Y$ satisfying

$$\mathfrak{T}_1 \supset (f^{\leftarrow})^{\to}(\mathfrak{S}_1),\ \mathfrak{T}_2 \supset (f^{\leftarrow})^{\to}(\mathfrak{S}_2),$$

i.e., $f : (X, \mathfrak{T}_1) \to (Y, \mathfrak{S}_1)$ and $f : (X, \mathfrak{T}_2) \to (Y, \mathfrak{S}_2)$ are both continuous. With the composition and identities of **Set**, one has the category **BiTop**, which is a topological construct and hence strongly complete and strongly cocomplete along with many other properties.

There is a voluminous literature for **BiTop** concerning separation, compactness, connectedness, completion, connections to uniform and quasi-uniform spaces, homotopy groups and algebraic topology, relationships to bilocales [2], a recently emerging role in programming semantics [25], etc. A significant part of the recent literature on bitopology is in lattice-valued mathematics [30, 27, 50, 51]. Letting $L$ be a us-quantale (Subsection 1.2 below) and $X$ a set, the triple $(X, \tau, \sigma)$ is an $L$-bitopological space if $\tau, \sigma$ are $L$-topologies on $X$ (Subsection 1.5); and such spaces with $L$-bicontinuous mappings comprise the category $L$-**BiTop**. This category is a topological construct, strongly complete, strongly cocomplete, and so on. The schemum $\{L\text{-}\mathbf{BiTop} : L \in |\mathbf{USQuant}|\}$ essentially includes **BiTop** via its functorial isomorph **2-BiTop**.

This paper studies functorial relationships between (lattice-valued) bitopology and (lattice-valued) topology in Sections 2–3. The expected functor $E_d$ strictly embeds $L$-**Top** into $L$-**BiTop**, a functor we dub the "doubling" functor; and to fully study $E_d$, it is necessary to construct several functors from $L$-**BiTop** to $L$-**Top** whose relationships with $E_d$ lead us to conclude that $E_d$ is extremely well-behaved. But on the other hand, for each $L \in |\mathbf{USQuant}|$, the direct product $L^2 \in |\mathbf{USQuant}|$ and there is an embedding $E_\times$ of $L$-**BiTop** into $L^2$-**Top** (3.4.1) which is extremely well-behaved (Subsubsections 3.4.2, 3.4.3) if $L$ is a u-quantale (Subsection 1.2) and a strict embedding if $L$

is consistent (Subsubsection 3.4.1). Given that this embedding is strict (for consistent $L$) and that the $L^2$'s form a proper subclass of **USQuant**—which means (lattice-valued) bitopology is properly "contained" in the proper subclass $\left\{L^2\text{-}\mathbf{BiTop} : L \in |\mathbf{USQuant}|\right\}$ of $\{L\text{-}\mathbf{Top} : L \in |\mathbf{USQuant}|\}$, it follows (lattice-valued) topology (twice) strictly generalizes bitopology. In Section 4 we summarize some metamathematical facts: given that lattice-valued topology is fundamentally simpler than lattice-valued bitopology—a membership lattice and *one* topology *vis-a-vis* a membership lattice and *two* topologies, it follows that topology and the class of embeddings $E_\times$'s make lattice-valued bitopology categorically redundant; and as a special application, traditional bitopology **BiTop** strictly embeds in an extremely well behaved way into **4-Top**, the latter being lattice-valued topology based on the four-element Boolean algebra **4**, so that traditional bitopology both is a strictly special case of the simpler lattice-valued topology and demonstrates the necessity of lattice-valued topology. On the other hand, this last fact points the way for bringing over into lattice-valued topology successful ideas from the extensive literature of traditional bitopology; in particular, traditional bicompactness mandates, via the embedding of **BiTop** into **4-Top**, the compactness of [5] for lattice-valued topology (Corollary 4.7).

1.2. **Lattice theoretics.** A **semi-quantale** $(L, \le, \otimes)$ (s-quantale) is a complete lattice $(L, \le)$ equipped with a binary operation $\otimes : L \times L \to L$, with no additional assumptions, called a **tensor product**; an **ordered semi-quantale** (os-quantale) is an s-quantale in which $\otimes$ is isotone in both variables; a **complete quasi-monoidal lattice** (cqml) [20, 41] is an os-quantale for which $\top$ is an idempotent element for $\otimes$; a **unital semi-quantale** (us-quantale) is an s-quantale in which $\otimes$ has an identity element $e \in L$ called the **unit** [33]—units are unique; a **quantale** is an s-quantale with $\otimes$ associative and distributing across arbitrary $\bigvee$ from both sides (implying $\bot$ is a two-sided **zero**) [20, 33, 49]; and a **unital quantale** (u-quantale) is a us-quantale which is a quantale; and a **strictly two-sided quantale** (st-quantale) is a u-quantale for which $e = \top$ [20]. All quantales are os-quantales. The notions of s-quantales, os-quantales, and us-quantales are from [45, 46].

**SQuant** comprises all semi-quantales together with mappings preserving $\otimes$ and arbitrary $\bigvee$; **OSQuant** is the full subcategory of **SQuant** of all os-quantales; **USQuant** is a subcategory of **SQuant** comprising all us-quantales together with all mappings preserving arbitrary $\bigvee$, $\otimes$, and $e$; **Quant** is the full subcategory of **OSQuant** of all quantales; and **UQuant** is the full subcategory of **UOSQuant** of all unital quantales. Note uos-quantales for which $\otimes = \wedge$ (binary) are **semiframes** and **SFrm** is the full subcategory of **UOSQuant** of all semiframes; and u-quantales for which $\otimes = \wedge$ (binary) are **frames**—in which case $e = \top$—and **Frm** is the full subcategory of **UQuant** of all frames. Semiframes equipped with an order-reversing involution are **complete DeMorgan algebras**; and s-quantales equipped with a semi-polarity

[16] ($\forall \alpha, \beta \in L, \alpha \leq \beta \Rightarrow \beta' \leq \alpha'$ and $\alpha \leq (\alpha')'$) are **complete semi-DeMorgan s-quantales**.

Throughout this paper, the requirement of us-quantale [u-quantale] can be relaxed to s-quantale [quantale, resp.] if one wishes to consider the relationships between q-topology and q-bitopology ([46] and Subsection 1.5 below).

Justifying the above lattice-theoretic notions is a wealth of examples (see [17, 20, 21, 23, 33, 35, 39, 40, 41, 44, 46] and their references). The lattice $\mathbf{2} = \{\bot, \top\}$ with $\bot \neq \top$; and a lattice is **consistent** if it contains $\mathbf{2}$ and **inconsistent** if it is singleton (with $\bot = \top$).

1.3. **Powerset operators.** Let $X \in |\mathbf{Set}|$ and $L \in |\mathbf{SQuant}|$. Then $L^X$ is the **$L$-powerset of** $X$ of all **$L$-subsets** of $X$. The **constant** $L$-subset member of $L^X$ having value $\alpha$ is denoted $\underline{\alpha}$. All order-theoretic operations (e.g., $\bigvee, \bigwedge$) and algebraic operations (e.g., $\otimes$) on $L$ lift point-wise to $L^X$ and are denoted by the same symbols. In the case $L \in |\mathbf{USQuant}|$, the unit $e$ lifts to the constant map $\underline{e}$, which is the unit of $\otimes$ as lifted to $L^X$.

The operator $\wp_\varnothing : |\mathbf{Set}| \to |\mathbf{Set}|$ is useful in this paper, where $\wp_\varnothing(X)$ denotes the poset of all the nonempty subsets of $X$.

Let $L \in |\mathbf{SQuant}|$, $X, Y \in |\mathbf{Set}|$, and $f : X \to Y$ be in $\mathbf{Set}$. Then the standard (traditional) image and preimage operators $f^\to : \wp(X) \to \wp(Y)$, $f^\leftarrow : \wp(X) \leftarrow \wp(Y)$ are

$$f^\to(A) = \{f(x) \in Y : x \in A\}, \quad f^\leftarrow(B) = \{x \in X : f(x) \in B\},$$

and the Zadeh image and preimage operators $f_L^\to : L^X \to L^Y$, $f_L^\leftarrow : L^X \leftarrow L^Y$ [53] are

$$f_L^\to(a)(y) = \bigvee \{a(x) : x \in f^\leftarrow(\{y\})\}, \quad f_L^\leftarrow(b) = b \circ f.$$

If $L$ is understood, it may be dropped providing the context distinguishes these operators from the traditional operators. It is observed that $f^\to$ and $f^\leftarrow$ are naturally isomorphic to $f_{\mathbf{2}}^\to$ and $f_{\mathbf{2}}^\leftarrow$, resp.

It is well-known [36, 37, 39, 40, 46] that each $f_L^\leftarrow$ preserves arbitrary $\bigvee$, arbitrary $\bigwedge$, $\otimes$, and all constant maps, as well as the unit $\underline{e}$ if $L \in |\mathbf{USQuant}|$; each $f_L^\to$ preserves arbitrary $\bigvee$;

$$f^\to \dashv f^\leftarrow, \quad f_L^\to \dashv f_L^\leftarrow;$$

$f^\to$ and $f_L^\to$ are left-inverses [right-inverses] of $f^\leftarrow$ and $f_L^\leftarrow$, resp., if $f$ is surjective [injective, resp.]; and $f^\to, f^\leftarrow, f_L^\to, f_L^\leftarrow$ are all order-isomorphisms if and only if $f$ is a bijection.

Powerset operators and the powerset theories underlying lattice-valued mathematics are studied extensively in [6, 14, 7, 8, 15, 10, 11, 36, 37, 39, 40, 46].

1.4. **Category theoretics.** The main reference for categorical notions is [1], to which we refer the reader for various properties of functors as well as various versions of the Adjoint Functor Theorem and related notions.

The proving of functorial adjunctions is done via lifting (or major) and naturality (or minor) diagrams in the manner of [28, 36, 37, 41].

1.5. **Topology and bitopology.** Given $L \in |\mathbf{USQuant}|$, the category $L$-**Top** comprises objects of the form $(X, \tau)$, where $\tau \subset L^X$ is closed under arbitrary $\bigvee$ and binary $\otimes$ and contains $\underline{e}$—so that $\tau$ is a sub-us-quantale of $L^X$, together with morphisms $f : (X, \tau) \to (Y, \sigma)$, where $f : X \to Y$ is a function and

$$\tau \supset \left(f_L^{\leftarrow}\right)^{\to}(\sigma)\,,$$

namely $f_L^{\leftarrow}(v) \in \tau$ for each $v \in \sigma$. The objects $(X, \tau)$ are called $L$-**topological spaces** and $\tau$ is an $L$-**topology** on $X$ comprising $L$-**subsets** of $X$; and the morphisms $f$ are called $L$-**continuous**. Cf. [20, 41, 46].

Similarly, the category $L$-**BiTop** comprises objects of the form $(X, \tau, \sigma)$, where $\tau, \sigma$ are $L$-topologies on $X$, together with morphisms $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $f : X \to Y$ is a function and

$$\tau_1 \supset \left(f_L^{\leftarrow}\right)^{\to}(\sigma_1)\,,\ \tau_2 \supset \left(f_L^{\leftarrow}\right)^{\to}(\sigma_2)\,.$$

The objects $(X, \tau, \sigma)$ are called $L$-**bitopological spaces** and $(\tau, \sigma)$ is an $L$-**bitopology** on $X$; and the morphisms $f$ are called $L$-**bicontinuous**. If the $L$ is clear in context, it may be dropped from the labels.

As noted in Subsection 1.1, the traditional category **BiTop** is isomorphic to **2-BiTop** (cf. 3.25 below) and embeds into each $L$-**BiTop**, and similarly **Top** is isomorphic to **2-Top** and embeds into each $L$-**Top**.

Each of $L$-**Top** and $L$-**BiTop** has the base $L$ of the category fixed and so is part of fixed-basis (lattice-valued) topology and fixed-basis (lattice-valued) bitopology, resp. The disciplines of fixed-basis topology and fixed-basis bitopology are encompassed by the respective classes

$$\{L\text{-}\mathbf{Top} : L \in |\mathbf{USQuant}|\}\,,\ \{L\text{-}\mathbf{BiTop} : L \in |\mathbf{USQuant}|\}\,.$$

Both $L$-**Top** and $L$-**BiTop** are topological over **Set** and have small fibres, hence are [co]complete and [co]well-powered, and hence are strongly [co]complete with many other nice properties (see 3.36 and its proof below). The categorical product for $L$-**Top** is given in, or adapted from, [12, 52] (cf. [20, 41]) and for $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ denoted by

$$\left(\prod_{\gamma \in \Gamma}(X_\gamma, \tau_\gamma)\,, \{\pi_\gamma\}_{\gamma \in \Gamma}\right), \quad \prod_{\gamma \in \Gamma}(X_\gamma, \tau_\gamma) \equiv \left(\times_{\gamma \in \Gamma} X_\gamma, \Pi_{\gamma \in \Gamma}\tau_\gamma\right),$$

where $\{\pi_\gamma : \gamma \in \Gamma\}$ are the projections. The binary $L$-topological product for two spaces $(X, \tau)\,, (Y, \sigma)$ is denoted $(X, \tau)\ \Pi\ (Y, \sigma)$ or $(X \times Y, \tau\, \Pi\, \sigma)$ with projections $\{\pi_1, \pi_2\}$. The categorical product for $L$-**BiTop** for $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is

$$\left(\prod_{\gamma \in \Gamma}(X_\gamma, \tau_\gamma, \sigma_\gamma)\,, \{\pi_\gamma\}_{\gamma \in \Gamma}\right),$$

$$\prod_{\gamma \in \Gamma}(X_\gamma, \tau_\gamma, \sigma_\gamma) \equiv \left(\times_{\gamma \in \Gamma} X_\gamma, \Pi_{\gamma \in \Gamma}\tau_\gamma, \Pi_{\gamma \in \Gamma}\sigma_\gamma\right),$$

where $\Pi_{\gamma \in \Gamma}\tau_\gamma, \Pi_{\gamma \in \Gamma}\sigma_\gamma$ are the $L$-topological product topologies in each slot and the projections are as above.

An $L$-topology $\tau$ is **weakly stratified** [20] if $\{\underline{\alpha} : \alpha \in L\} \subset \tau$, **non-stratified** if it is not weakly stratified, and **anti-stratified** [9, 35] if

$$\{\underline{\alpha} : \alpha \in L,\ \underline{\alpha} \in \tau\} = \{\underline{\perp}, \underline{e}\};$$

so a weakly stratified topology contains all constant $L$-subsets, while an anti-stratified topology contains precisely the constant $L$-subsets $\underline{\perp}$ and $\underline{e}$ (which are the same if $L$ is inconsistent with $\perp = \top$). An $L$-topological space is weakly stratified [anti-stratified] if its topology is weakly stratified [anti-stratified], and an $L$-bitopological space is weakly stratified [anti-stratified] if both topologies are weakly stratified [anti-stratified]. The inclusionist position that the axioms of a fixed-basis topology must allow for all types of stratification has recently received additional, emphatic confirmations from both lattice-valued frames [35] and topological systems in domain theory [9].

The following definition and proposition are needed in this paper.

**Definition 1.1.** *Let $X$ be a set and let $L$ be a us-quantale. Then the $L$-**topological fibre**, respectively, $L$-**bitopological fibre** on $X$ is*

$$\begin{aligned} L\text{-}T(X) &\equiv \left\{\tau \subset L^X : (X,\tau) \in |L\text{-}\mathbf{Top}|\right\}, \\ L\text{-}BT(X) &\equiv \{(\tau,\sigma) : (X,\tau,\sigma) \in |L\text{-}\mathbf{BiTop}|\}. \end{aligned}$$

**Proposition 1.2.** *Let $X$ be a set, let $L$ be a us-quantale, and recall $\wp_\varnothing$ from Subsection 1.3.*

(1) *$L$-$T(X)$ is a complete meet subsemilattice of $\wp\left(L^X\right)$; and since each $L$-topology is nonempty, $L$-$T(X) \subset \wp_\varnothing\left(L^X\right)$.*

(2) *$L$-$BT(X)$, ordered coordinate-wise by inclusion, is a complete meet subsemilattice of $\wp\left(L^X\right)\times\wp\left(L^X\right)$; and further, $L$-$BT(X) \subset \wp_\varnothing\left(L^X\right)\times \wp_\varnothing\left(L^X\right)$.*

*Proof.* The first part of (1) is well-known, and the second part of (1) is trivial. Now (2) follows from (1) since

$$L\text{-}BT(X) = L\text{-}T(X) \times L\text{-}T(X) \subset \wp_\varnothing\left(L^X\right) \times \wp_\varnothing\left(L^X\right).$$

□

Finally, we need the notion of a subbase of an $L$-topology $\tau$ on $X$ [41]. We say $\sigma \subset L^X$ is a **subbase** of $\tau$, written $\tau = \langle\langle\sigma\rangle\rangle$, if

$$\tau = \bigcap\{\tau' \in L\text{-}T(X) : \sigma \subset \tau'\},$$

the right-hand side always existing by Proposition 1.2(1), and we say $\beta \subset L^X$ is a **base** of $\tau$, written $\tau = \langle\beta\rangle$, if

$$\forall u \in \tau,\ \exists\, B_u \subset \beta,\ u = \bigvee B_u.$$

One can always pass from a subbase $\sigma$ to a topology $\tau$ through a base $\beta$ in the traditional way, written

$$\tau = \langle\beta\rangle = \langle\langle\sigma\rangle\rangle,$$

if and only if $\otimes$ is associative and distributes across arbitrary $\bigvee$, i.e., if and only if $L$ is a u-quantale.

## 2. Functorial interpretations of topology as bitopology

For each us-quantale $L$, this section records a simple (and expected) "doubling" embedding $E_d : L\text{-}\mathbf{Top} \to L\text{-}\mathbf{BiTop}$. The behavior of $E_d$ w.r.t. limits and colimits—it preserves, reflects, detects both—is examined completely in Subsections 3.1–3.2 below. It emerges that $E_d$ is an extremely well-behaved embedding.

**Proposition 2.1.** *Let $L$ be a us-quantale. Define $E_d : L\text{-}\mathbf{Top} \to L\text{-}\mathbf{BiTop}$ by the following correspondences:*

$$E_d\,(X,\tau) = (X,\tau,\tau)\,,\ E_d\,(f) = f.$$

*Then $E_d$ is a concrete, full, strict embedding; and so $L$-**Top** is isomorphic to a full subcategory of $L$-**BiTop**.*

*Proof.* All details are straightforward. □

## 3. Functorial interpretations of bitopology as topology

This section records several interpretations of bitopology as topology, the most important of which would seem to be the extremely well-behaved embedding $E_\times$ of Subsection 3.4.

### 3.1. $F_\mathfrak{l}, F_\mathfrak{r}, F_\wedge : L\text{-}\mathbf{BiTop} \to L\text{-}\mathbf{Top}$ and behavior of $E_d : L\text{-}\mathbf{Top} \to L\text{-}\mathbf{BiTop}$ w.r.t. limits.

This subsection constructs the concrete, faithful, full forgetful functors—the "left-forgetful" functor $F_\mathfrak{l} : L\text{-}\mathbf{BiTop} \to L\text{-}\mathbf{Top}$ and the "right-forgetful" $F_\mathfrak{r} : L\text{-}\mathbf{BiTop} \to L\text{-}\mathbf{Top}$—as well as the concrete, faithful "meet" functor $F_\wedge : L\text{-}\mathbf{BiTop} \to L\text{-}\mathbf{Top}$ and shows $F_\wedge$ is the left-adjoint of $E_d$ of the previous section and that each of $F_\mathfrak{l}, F_\mathfrak{r}$ is a left-adjoint of $E_d$ under certain restrictions.

**Proposition 3.1.** *Let $L$ be a us-quantale and define $F_\mathfrak{l}, F_\mathfrak{r} : L\text{-}\mathbf{BiTop} \to L\text{-}\mathbf{Top}$ as follows:*

$$\begin{aligned} F_\mathfrak{l}\,(X,\tau,\sigma) &= (X,\tau)\,,\ F_\mathfrak{l}\,(f) = f,\\ F_\mathfrak{r}\,(X,\tau,\sigma) &= (X,\sigma)\,,\ F_\mathfrak{r}\,(f) = f. \end{aligned}$$

*Then each of $F_\mathfrak{l}, F_\mathfrak{r}$ is a concrete, faithful, full, object-surjective functor, but need not be an embedding.*

*Proof.* We comment only on $F_\mathfrak{l}$. Trivially, $F_\mathfrak{l}$ is a concrete, faithful, object-surjective functor. As for fullness, let $f : (X,\tau) \to (Y,\sigma)$ in $L$-**Top**; then $f : (X,\tau,\tau) \to (Y,\sigma,\sigma)$ is $L$-bicontinuous, so is in $L$-**BiTop**, and maps to $f : (X,\tau) \to (Y,\sigma)$. Now suppose that either $[|X| \geq 1$ and $|L| \geq 3]$ or $[|X| \geq 2$ and $|L| \geq 2]$; then $\exists\, \tau,\sigma \in L\text{-}T\,(X)$ with $\tau \neq \sigma$, so that $F_\mathfrak{l}\,(X,\tau,\sigma) = (X,\tau) = F_\mathfrak{l}\,(X,\tau,\tau)$, and hence $F_\mathfrak{l}$ does not inject objects and is not an embedding. □

**Proposition 3.2.** *Let $L$ be a us-quantale and define $F_\wedge : L\text{-}\mathbf{BiTop} \to L\text{-}\mathbf{Top}$ as follows:*

$$F_\wedge\,(X,\tau,\sigma) = (X,\tau\cap\sigma)\,,\ F_\wedge\,(f) = f.$$

*Then $F_\wedge$ is a concrete, faithful, object-surjective functor, but need not be full nor an embedding.*

*Proof.* Since $L$-**Top** has complete fibres, $F_\wedge$ is well-defined on objects. Now let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $L$-bicontinuous. Since the image operator of the Zadeh preimage operator preserves $\subset$, it follows

$$\begin{aligned} \tau_1 &\supset (f_L^{\leftarrow})^{\rightarrow}(\sigma_1), \ \tau_2 \supset (f_L^{\leftarrow})^{\rightarrow}(\sigma_2) \ \Rightarrow \\ \tau_1 \cap \tau_2 &\supset (f_L^{\leftarrow})^{\rightarrow}(\sigma_1) \cap (f_L^{\leftarrow})^{\rightarrow}(\sigma_2) \supset (f_L^{\leftarrow})^{\rightarrow}(\sigma_1 \cap \sigma_2), \end{aligned}$$

so that $f : (X, \tau_1 \cap \tau_2) \to (Y, \sigma_1 \cap \sigma_2)$ is $L$-continuous. Immediately, $F_\wedge$ is a concrete, faithful functor which surjects objects.

Now let $L$ be a complete DeMorgan algebra (with $\otimes = \wedge$ (binary)) and consider each of the $L$-bitopological spaces $(\mathbb{R}(L), \tau_l(L), \tau_l(L))$ and $(\mathbb{R}(L), \tau_l(L), \tau(L))$, where $\mathbb{R}(L)$ is the $L$-fuzzy real line, $\tau_l(L)$ is the left-hand $L$-topology on $\mathbb{R}(L)$ determined by the $\mathfrak{L}_t$ operators and $\tau(L)$ is the standard $L$-topology on $\mathbb{R}(L)$ [43]. Then $(\mathbb{R}(L), \tau_l(L), \tau_l(L)) \neq (\mathbb{R}(L), \tau_l(L), \tau(L))$ and

$$\begin{aligned} F_\wedge(\mathbb{R}(L), \tau_l(L), \tau_l(L)) &= (\mathbb{R}(L), \tau_l(L)) \\ &= (\mathbb{R}(L), \tau_l(L) \cap \tau(L)) \\ &= F_\wedge(\mathbb{R}(L), \tau_l(L), \tau(L)), \end{aligned}$$

showing that $F_\wedge$ does not inject objects, so is not an embedding. Now letting $f : \mathbb{R}(L) \to \mathbb{R}(L)$ be $id_{\mathbb{R}(L)}$, we have

$$f : F_\wedge(\mathbb{R}(L), \tau_l(L), \tau(L)) \to F_\wedge(\mathbb{R}(L), \tau(L), \tau_l(L))$$

is $L$-continuous, but

$$f : (\mathbb{R}(L), \tau_l(L), \tau(L)) \to (\mathbb{R}(L), \tau(L), \tau_l(L))$$

cannot be $L$-bicontinuous (because of the first slot). The concreteness of $F_\wedge$ implies there exists no $g \in L$-**BiTop** with $F_\wedge(g) = f$, so $F_\wedge$ is not full. □

**Theorem 3.3.** *Let $L$ be a us-quantale. Then $F_\wedge \dashv E_d$, this adjunction is a monocoreflection, and $F_\wedge$ takes $L$-**BiTop** to a monocoreflective subcategory of $L$-**Top**. On the other hand, $E_d \not\dashv F_\wedge$.*

*Proof.* Let $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiTop}|$, choose

$$\eta = id : (X, \tau_1, \tau_2) \to E_d F_\wedge(X, \tau_1, \tau_2) = (X, \tau_1 \cap \tau_2, \tau_1 \cap \tau_2),$$

and note $\eta$ is an $L$-continuous injection. Now let $(Y, \sigma) \in |L\text{-}\mathbf{Top}|$, suppose $f : (X, \tau_1, \tau_2) \to E_d(Y, \sigma) = (Y, \sigma, \sigma)$ is $L$-bicontinuous, and note

$$\tau_1 \supset (f_L^{\leftarrow})^{\rightarrow}(\sigma), \ \tau_2 \supset (f_L^{\leftarrow})^{\rightarrow}(\sigma) \ \Rightarrow \ \tau_1 \cap \tau_2 \supset (f_L^{\leftarrow})^{\rightarrow}(\sigma),$$

making $f : F_\wedge(X, \tau_1, \tau_2) = (X, \tau_1 \cap \tau_2) \to (Y, \sigma)$ $L$-continuous. Then $\overline{f} = f$ is the unique choice making $f = \overline{f} \circ \eta$. The naturality diagram now follows by concreteness as do the other claims concerning $F_\wedge \dashv E_d$. Finally, given $F_\vee$ of Subsection 3.2 and $E_d \dashv F_\vee$ of 3.9 below, $E_d \not\dashv F_\wedge$ since $F_\wedge \ncong F_\vee$ and right-adjoints are essentially unique. □

**Definition 3.4.** *$L$-**BiTop** $(\subset)$ [$L$-**BiTop** $(\supset)$] is the full subcategory of $L$-**BiTop** of all spaces $(X, \tau, \sigma)$ in which $\tau \subset \sigma$ [$\tau \supset \sigma$].*

Note **BiTop** $(\subset)$ and **BiTop** $(\supset)$ (essentially setting $L = \mathbf{2}$) express the original sense of traditional bitopology [3, 4].

**Proposition 3.5.** *Let $L$ be a us-quantale. Then*

$$F_{\mathfrak{l}\ |\ L\text{-}\mathbf{BiTop}(\subset)} = F_{\wedge\ |\ L\text{-}\mathbf{BiTop}(\subset)}, \quad F_{\mathfrak{r}\ |\ L\text{-}\mathbf{BiTop}(\supset)} = F_{\wedge\ |\ L\text{-}\mathbf{BiTop}(\supset)}.$$

*Hence $F_{\mathfrak{l}\ |\ L\text{-}\mathbf{BiTop}(\subset)} \dashv E_d$ and $F_{\mathfrak{r}\ |\ L\text{-}\mathbf{BiTop}(\supset)} \dashv E_d$, but $E_d \not\dashv F_{\mathfrak{l}\ |\ L\text{-}\mathbf{BiTop}(\subset)}$ and $E_d \not\dashv F_{\mathfrak{r}\ |\ L\text{-}\mathbf{BiTop}(\supset)}$.*

*Proof.* The restricted forgetful functors obviously coincide with the meet functor. Observing that $E_d$ maps into each of $L$-**BiTop** $(\subset)$ and $L$-**BiTop** $(\supset)$, the claimed adjunctions are then immediate from 3.3. The claimed non-adjunctions follow from 3.10 below. □

**Corollary 3.6.** *Let $L$ be a us-quantale. The following hold:*

(1) *$E_d$ preserves all strong limits and $F_\wedge$ preserves all strong colimits.*
(2) *$F_{\mathfrak{l}}$ preserves the strong colimits of $L$-**BiTop** $(\subset)$, $F_{\mathfrak{r}}$ preserves the strong colimits of $L$-**BiTop** $(\supset)$, and $E_d$ preserves strong limits into each of $L$-**BiTop** $(\subset)$ and $L$-**BiTop** $(\supset)$.*

**Proposition 3.7.** *For each us-quantale $L$, $E_d : L$-**Top** $\to L$-**BiTop** reflects and detects all limits and hence lifts all limits and is transportable.*

*Proof.* The details are straightforward using 3.6 and Proposition 13.34 [1]. □

3.2. **$F_\vee : L$-BiTop $\to L$-Top and behavior of $E_d : L$-Top $\to L$-BiTop w.r.t. colimits.** This subsection constructs the concrete, faithful "join" functor $F_\vee : L$-**BiTop** $\to L$-**Top** and shows it is the right-adjoint of $E_d$ of the previous section.

**Proposition 3.8.** *Let $L$ be a us-quantale and define $F_\vee : L$-**BiTop** $\to L$-**Top** as follows:*

$$F_\vee (X, \tau, \sigma) = (X, \tau \vee \sigma), \ F_\vee (f) = f,$$

*where*

$$\tau \vee \sigma = \langle\langle \tau \cup \sigma \rangle\rangle$$

*Then $F_\vee$ is a concrete, faithful, object-surjective functor, but need not be full nor an embedding.*

*Proof.* Since $L$-**Top** has complete fibres, $F_\vee$ is well-defined on objects. Now let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $L$-bicontinuous. Since the image operator of the Zadeh preimage operator preserves unions, then

$$\tau_1 \supset (f_L^\leftarrow)^\to (\sigma_1), \ \tau_2 \supset (f_L^\leftarrow)^\to (\sigma_2) \ \Rightarrow$$

$$\tau_1 \vee \tau_2 \supset \tau_1 \cup \tau_2 \supset (f_L^\leftarrow)^\to (\sigma_1) \cup (f_L^\leftarrow)^\to (\sigma_2) = (f_L^\leftarrow)^\to (\sigma_1 \cup \sigma_2),$$

so that $f : (X, \tau_1 \vee \tau_2) \to (Y, \sigma_1 \vee \sigma_2)$ is $L$-subbasic continuous. By Theorem 3.2.6 of [41] as restricted to the fixed-basis case and then adapted to the us-quantalic case, $f : (X, \tau_1 \vee \tau_2) \to (Y, \sigma_1 \vee \sigma_2)$ is $L$-continuous. Immediately, $F_\vee$ is a concrete, faithful functor which surjects objects.

Now let $L$ be a complete DeMorgan algebra (with $\otimes = \wedge$ (binary)) and consider each of the $L$-bitopological spaces $(\mathbb{R}(L), \tau_l(L), \tau_r(L))$ and $(\mathbb{R}(L), \tau(L), \tau(L))$, where $\mathbb{R}(L)$ is the $L$-fuzzy real line, $\tau_l(L)$ is the left-hand $L$-topology on $\mathbb{R}(L)$ determined by the $\mathfrak{L}_t$ operators, $\tau_r(L)$ is the left-hand $L$-topology on $\mathbb{R}(L)$ determined by the $\mathfrak{R}_t$ operators, and $\tau(L)$ is the standard $L$-topology on $\mathbb{R}(L)$ [43]. Then $(\mathbb{R}(L), \tau_l(L), \tau_r(L)) \neq (\mathbb{R}(L), \tau(L), \tau(L))$ and

$$\begin{aligned} F_\vee(\mathbb{R}(L), \tau_l(L), \tau_r(L)) &= (\mathbb{R}(L), \tau_l(L) \vee \tau_r(L)) \\ &= (\mathbb{R}(L), \tau(L)) \\ &= F_\vee(\mathbb{R}(L), \tau(L), \tau(L)), \end{aligned}$$

showing that $F_\vee$ does not inject objects, so is not an embedding. Now letting $f : \mathbb{R}(L) \to \mathbb{R}(L)$ be $id_{\mathbb{R}(L)}$, we have

$$f : F_\vee(\mathbb{R}(L), \tau_l(L), \tau_r(L)) \to F_\vee(\mathbb{R}(L), \tau_r(L), \tau_l(L))$$

is $L$-continuous, but

$$f : (\mathbb{R}(L), \tau_l(L), \tau_r(L)) \to (\mathbb{R}(L), \tau_r(L), \tau_l(L))$$

cannot be $L$-bicontinuous. The concreteness of $F_\vee$ implies there exists no $g \in L$-**BiTop** with $F_\vee(g) = f$, so $F_\vee$ is not full. □

**Theorem 3.9.** *Let $L$ be a us-quantale. Then $E_d \dashv F_\vee$, this adjunction is an isoreflection, and $F_\vee$ takes $L$-**BiTop** to an isoreflective subcategory of $L$-**Top**. On the other hand, $F_\vee \not\dashv E_d$.*

*Proof.* Let $(X, \tau) \in |L\text{-}\mathbf{Top}|$, choose

$$\eta = id : (X, \tau) \to F_\vee E_d(X, \tau) = (X, \tau \vee \tau) = (X, \tau),$$

and note $\eta$ is an $L$-homeomorphism. Now let $(Y, \sigma_1, \sigma_2) \in |L\text{-}\mathbf{BiTop}|$, suppose $f : (X, \tau) \to F_\vee(Y, \sigma_1, \sigma_2) = (Y, \sigma_1 \vee \sigma_2)$ is $L$-continuous, and note

$$\tau \supset (f_L^{\leftarrow})^{\to}(\sigma_1 \vee \sigma_2) \supset (f_L^{\leftarrow})^{\to}(\sigma_1 \cup \sigma_2) \supset (f_L^{\leftarrow})^{\to}(\sigma_1), (f_L^{\leftarrow})^{\to}(\sigma_2),$$

making $f : E_d(X, \tau) = (X, \tau, \tau) \to (Y, \sigma_1, \sigma_2)$ $L$-bicontinuous. Then $\overline{f} = f$ is the unique choice making $f = \overline{f} \circ \eta$. The naturality diagram now follows by concreteness, as do the other claims concerning $E_d \dashv F_\vee$. Finally, given $F_\wedge$ of Subsection 3.1 and $F_\wedge \dashv E_d$ of 3.3 above, $F_\vee \not\dashv E_d$ since $F_\wedge \ncong F_\vee$ and left-adjoints are essentially unique. □

**Corollary 3.10.** *Let $L$ be a us-quantale. The following hold:*

(1) *$E_d$ preserves all strong colimits and $F_\vee$ preserves all strong limits.*
(2) *$E_d \not\dashv F_{\mathfrak{l} \mid L\text{-}\mathbf{BiTop}(\subset)}$ and $E_d \not\dashv F_{\mathfrak{r} \mid L\text{-}\mathbf{BiTop}(\supset)}$, and hence $E_d \not\dashv F_{\mathfrak{l}}$ and $E_d \not\dashv F_{\mathfrak{r}}$.*

*Proof.* (1) is immediate. As for (2), it is clear that $F_{\mathfrak{l}\ |\ L\text{-}\mathbf{BiTop}(\subset)}, F_{\mathfrak{r}\ |\ L\text{-}\mathbf{BiTop}(\supset)} \ncong F_{\vee\ |\ L\text{-}\mathbf{BiTop}(\subset)}, F_{\vee\ |\ L\text{-}\mathbf{BiTop}(\supset)}$, resp., implying $E_d \not\dashv F_{\mathfrak{l}\ |\ L\text{-}\mathbf{BiTop}(\subset)}$ and $E_d \not\dashv F_{\mathfrak{r}\ |\ L\text{-}\mathbf{BiTop}(\supset)}$ by the essential uniqueness of the right-adjoint in 3.9; and hence $E_d \not\dashv F_{\mathfrak{l}}$ and $E_d \not\dashv F_{\mathfrak{r}}$. □

**Proposition 3.11.** *For each us-quantale $L$, $E_d : L$-**Top** $\to L$-**BiTop** reflects and detects all colimits.*

*Proof.* The details are straightforward. □

3.3. $F_{\Pi} : L$**-BiTop** $\to L$**-Top.** This subsection constructs the non-concrete, faithful "product" functor $F_{\Pi} : L$-**BiTop** $\to L$-**Top** which, when appropriately restricted, is an embedding. It need not preserve finite products and hence lacks a left-adjoint.

**Proposition 3.12.** *Let $L$ be a us-quantale and define $F_{\Pi} : L$-**BiTop** $\to L$-**Top** as follows:*

$$F_{\Pi}\,(X, \tau, \sigma) = (X \times X, \tau\, \Pi\, \sigma)\,,\ F_{\Pi}\,(f) = f \times f,$$

*where $\tau\,\Pi\,\sigma$ is the $L$-product topology on $X \times X$ (Subsection 1.5). Then $F_{\Pi}$ is a non-concrete, faithful functor which need not be full nor object-surjective nor an embedding.*

*Proof.* Immediately $F_{\Pi}$ is well-defined on objects. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $L$-bicontinuous and let $v \in \sigma_1\,\Pi\,\sigma_2$ be a subbasic open set of the form $(\pi_1)_L^{\leftarrow}(s_1)$ with $s_1 \in \sigma_1$. Then given $(x_1, x_2) \in X \times X$,

$$\begin{aligned}(f \times f)_L^{\leftarrow}(v)\,(x_1, x_2) &= (\pi_1)_L^{\leftarrow}(s_1)\,(f(x_1), f(x_2)) \\ &= s_1\,(\pi_1\,(f(x_1), f(x_2))) \\ &= s_1\,(f(x_1)) \\ &= f_L^{\leftarrow}(s_1)\,(x_1) \\ &= f_L^{\leftarrow}(s_1)\,(\pi_1\,(x_1, x_2)) \\ &= (\pi_1)_L^{\leftarrow}(f_L^{\leftarrow}(s_1))\,(x_1, x_2)\,,\end{aligned}$$

so that $(f \times f)_L^{\leftarrow}(v) = (\pi_1)_L^{\leftarrow}(f_L^{\leftarrow}(s_1)) \in \tau_1\,\Pi\,\tau_2$; and similarly, if $v$ is a subbasic open set of the form $(\pi_2)_L^{\leftarrow}(s_2)$ with $s_2 \in \sigma_2$, $(f \times f)_L^{\leftarrow}(v) \in \tau_1\,\Pi\,\tau_2$. So $F_{\Pi}(f) : F_{\Pi}(X, \tau_1, \tau_2) \to F_{\Pi}(Y, \sigma_1, \sigma_2)$ is $L$-subbasic continuous and hence $L$-continuous (cf. Theorem 3.2.6 of [41]). It is easy to show $F_{\Pi}$ preserves composition and identities—and so is a functor—and is faithful and need not be full nor object-surjective.

To see that $F_{\Pi}$ need not inject objects, let $L = \{\bot, \alpha, \beta, \top\}$ be a chain with $\otimes = \wedge$ (binary), $X = \{x\}$, $\tau_1 = \{\underline{\bot}, \underline{\alpha}, \underline{\top}\}$, and $\tau_2 = \{\underline{\bot}, \underline{\beta}, \underline{\top}\}$. Then $(X, \tau_1, \tau_2) \neq (X, \tau_2, \tau_1)$, yet $F_{\Pi}(X, \tau_1, \tau_2) = F_{\Pi}(X, \tau_2, \tau_1)$. □

**Proposition 3.13.** *$F_{\Pi}$ does not preserve binary products and hence has no left-adjoint.*

*Proof.* Let $(X, \tau_1, \tau_2)\,, (Y, \sigma_1, \sigma_2)$ be given with $X \neq Y$. Then the carrier set of $F_{\Pi}[(X, \tau_1, \tau_2)\ \Pi\ (Y, \sigma_1, \sigma_2)]$ is $(X \times Y) \times (X \times Y)$ and the carrier set of $F_{\Pi}(X, \tau_1, \tau_2)\ \Pi\ F_{\Pi}(Y, \sigma_1, \sigma_2)$ is $(X \times X) \times (Y \times Y)$, clearly not the same. □

**Definition 3.14.** *Letting $L$ be a us-quantale, $L$-**NBiTop** is the full subcategory of all spaces $(X, \tau, \sigma)$ satisfying the condition that each open $L$-subset $u \neq \bot$ in each of $\tau, \sigma$ is $L$-**normalized**, i.e., has the property that*

$$\bigvee_{x \in X} u(x) = e.$$

*If $L$ is an st-quantale, then the notion of $L$-normalized subsets coincides with the usual notion, namely $\bigvee_{x \in X} u(x) = \top$.*

**Theorem 3.15.** *Let $L$ be a u-quantale. Then $F_{\Pi \mid L\text{-}\mathbf{NBiTop}} : L\text{-}\mathbf{NBiTop} \to L\text{-}$**Top** is an embedding. This embedding does not preserve binary products and hence has no left-adjoint.*

*Proof.* Because of 3.12, it suffices to show $F_\Pi$ as restricted injects objects. For two distinct objects, let us consider

$$(X, \tau_1, \sigma) \neq (X, \tau_2, \sigma)$$

with $\tau_1 \neq \tau_2$; all other cases are similar and left to the reader. Suppose W.L.O.G. there is $u \in \tau_1 - \tau_2$ and assume $\tau_1 \, \Pi \, \sigma = \tau_2 \, \Pi \, \sigma$ on $X \times X$. Then setting $\boxtimes \equiv \otimes \circ \times$,

$$\exists \, \{u_\gamma \boxtimes v_\gamma\}_{\gamma \in \Gamma} \subset \tau_2 \, \Pi \, \sigma$$

such that

$$(\pi_1)_L^{\leftarrow}(u) = \bigvee_{\gamma \in \Gamma} (u_\gamma \boxtimes v_\gamma).$$

Applying the surjectivity of $\pi_1$ and properties of Zadeh image operators (Subsection 1.3), we obtain the contradiction

$$\begin{aligned} u &= (\pi_1)_L^{\rightarrow}\left((\pi_1)_L^{\leftarrow}(u)\right) \\ &= (\pi_1)_L^{\rightarrow}\left(\bigvee_{\gamma \in \Gamma} (u_\gamma \boxtimes v_\gamma)\right) \\ &= \bigvee_{\gamma \in \Gamma} \left((\pi_1)_L^{\rightarrow}(u_\gamma \boxtimes v_\gamma)\right) \\ &= \bigvee_{\gamma \in \Gamma} u_\gamma \in \tau_2, \end{aligned}$$

where we have used the fact, for each $\gamma \in \Gamma$ and each $x \in X$, that

$$\begin{aligned} (\pi_1)_L^{\rightarrow}(u_\gamma \boxtimes v_\gamma)(x) &= \bigvee_{y \in X} (u_\gamma \boxtimes v_\gamma)(x, y) \\ &= \bigvee_{y \in X} (u_\gamma(x) \otimes v_\gamma(y)) \\ &= u_\gamma(x) \otimes \bigvee_{y \in X} v_\gamma(y) \\ &= u_\gamma(x) \otimes e \\ &= u_\gamma(x). \end{aligned}$$

The non-preservation of products follow for the restricted functor as in the proof of 3.13. □

**Corollary 3.16.** $F_{\Pi} \circ G_{\chi} : \mathbf{BiTop} \to \mathbf{Top}$ *is an embedding. This embedding does not preserve binary products and hence has no left-adjoint.*

*Proof.* The first statement is a corollary of 3.15 as follows: given any non-empty subset $A$ of set $X$, $\chi_A : X \to \mathbf{2}$ is normalized; $\mathbf{2}$**-NBiTop** $=$ $\mathbf{2}$**-BiTop**; and $G_{\chi} : \mathbf{BiTop} \to \mathbf{2}$**-BiTop** is a categorical isomorphism. The non-preservation of products follows for the composite functor as in the proof of 3.13. □

**Remark 3.17.** Corollary 3.16 furnishes an embedding of **BiTop** into **Top**; but this is not enough to say that **Top** may be categorically regarded as a generalization of **BiTop** since $F_{\Pi} \circ G_{\chi}$ is not sufficiently well-behaved. This motivates the search for a better behaved embedding of bitopology into topology conducted in the next subsection.

3.4. $E_{\times} : L$**-BiTop** $\to L^2$**-Top.** This subsection constructs the concrete, full, strict "cross" embedding $E_{\times} : L$-**BiTop** $\to L^2$-**Top**, establishes its behavior w.r.t. limits and colimits—for appropriate $L$, $E_{\times}$ preserves both and detects and reflects the former, and shows that $E_{\times}$ is essentially neutral w.r.t. stratification issues. It follows that $E_{\times}$ is an extremely well-behaved embedding.

3.4.1. *Construction of* $E_{\times} : L$-**BiTop** $\to L^2$-**Top**.

**Proposition 3.18** (cf. [16])**.** *Let $X$ be a set.*

(1) *For each set $L$ the mapping $\varphi_X : L^X \times L^X \to \left(L^2\right)^X$ given by*

$$\varphi_X\left(a_1, a_2\right) = a_1 \times a_2, \text{ i.e., } \varphi_X\left(a_1, a_2\right)(x) = \left(a_1(x), a_2(x)\right)$$

*is a bijection with inverse mapping $\varphi_X^{-1} : L^X \times L^X \leftarrow \left(L^2\right)^X$ given by*

$$\varphi_X^{-1}(a) = \left(\pi_1 \circ a, \pi_2 \circ a\right),$$

where $\pi_1, \pi_2$ *are the projections from $L^2$ to $L$.*

(2) *If $L$ is a poset, then $\varphi_X$ is an order-isomorphism.*

(3) *If $L$ is a semi-DeMorgan s-quantale, then $\varphi_X$ preserves semi-complements.*

(4) *If $L$ is an [u]s-quantale, then $\varphi_X$ is an [u]s-quantalic isomorphism (i.e., $\varphi_X$ also preserves tensor products [and the unit]).*

*Proof.* The details of $(1)-(3)$ are the same as, or analogous to, those of Lemma 4.4.1 of [16]. The details of (4) are straightforward. □

**Corollary 3.19.** $\varphi_X^{\rightarrow} : \wp\left(L^X \times L^X\right) \to \wp\left(\left(L^2\right)^X\right)$ *is an order-isomorphism.*

*Proof.* This is immediate from 3.18(1) using Subsection 1.3. □

**Proposition 3.20.** *Let $A, B$ be nonempty sets. Then $\zeta : \wp_{\varnothing}(A) \times \wp_{\varnothing}(B) \to \wp_{\varnothing}(A \times B)$ given by*

$$\zeta(C, D) = C \times D$$

*is an order-isomorphism onto its image, i.e., an order-embedding.*

*Proof.* Clearly $\zeta$ is well-defined. As for injectivity, let $(C_1, D_1) \neq (C_2, D_2)$. Then there are several cases, and a typical case is $C_1 \neq C_2$, $D_1 = D_2$. Then W.L.O.G. there is $x \in C_1 - C_2$. Since $D_1 = D_2 \neq \varnothing$, there is $y \in D_1 = D_2$. So $(x, y) \in (C_1 \times D_1) - (C_2 \times D_2)$; hence $\zeta(C_1, D_1) \neq \zeta(C_2, D_2)$. Since all orderings in question are coordinate-wise, it follows that both $\zeta$ and $\zeta^{-1}$ (on $\mathrm{Im}(\zeta)$) are isotone. □

**Proposition 3.21.** *Let $X$ be a set, $L$ be a us-quantale, and $\zeta$ denote any restriction of the $\zeta$ of 3.20.*

(1) $\zeta : \wp_{\varnothing}(L^X) \times \wp_{\varnothing}(L^X) \to \wp_{\varnothing}(L^X \times L^X)$ *is an order-isomorphism onto its image.*

(2) $\zeta : L\text{-}BT(X) \to \wp_{\varnothing}(L^X \times L^X)$ *is an order-isomorphism onto its image.*

*Proof.* Conjoin Proposition 1.2 and 3.20. □

**Lemma 3.22.** *Let $X$ be a set and $L$ be a us-quantale, and put $E_\times : L\text{-}BT(X) \to L^2\text{-}T(X)$ by*

$$E_\times = \varphi_X^{\to} \circ \zeta.$$

*Then $E_\times$ is an order-isomorphism onto its image.*

*Proof.* It must be first verified that $E_\times$ actually maps into $L^2\text{-}T(X)$. Let $(\tau_1, \tau_2) \in L\text{-}BT(X)$. Then $\tau_1, \tau_2$ are $L$-topologies on $X$ and hence sub-us-quantales of $L^X$. It is straightforward to check that as direct products,

$$\zeta(\tau_1, \tau_2) = \tau_1 \times \tau_2 \subset L^X \times L^X$$

and $\tau_1 \times \tau_2$ is a sub-us-quantale of $L^X \times L^X$. It follows

$$E_\times(\tau_1, \tau_2) = \varphi_X^{\to}(\zeta(\tau_1, \tau_2)) = \varphi_X^{\to}(\tau_1 \times \tau_2) \subset \wp\left(\left(L^2\right)^X\right)$$

and that $E_\times(\tau_1, \tau_2)$ is a sub-us-quantale of $\left(L^2\right)^X$, namely an $L^2$-topology on $X$. Hence $E_\times(\tau_1, \tau_2) \in L^2\text{-}T(X)$.

The remaining claims concerning $E_\times$ follow from 3.19 and 3.21. □

**Theorem 3.23.** *Let $L$ be a us-quantale, let*

$$f \in L\text{-}\mathbf{BiTop}((X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)),$$

*and put*

$$E_\times(X, \tau_1, \tau_2) = (X, E_\times(\tau_1, \tau_2)), \; E_\times(f) = f.$$

*Then $E_\times : L\text{-}\mathbf{BiTop} \to L^2\text{-}\mathbf{Top}$ is a concrete, full embedding; and hence $L$-**BiTop** is concretely isomorphic to a full subcategory of $L^2$-**Top**. Further, if $L$ is consistent, $E_\times$ is a strict embedding (not a functorial isomorphism).*

*Proof.* It is immediate from 3.22 that $E_\times$ is well-defined at the object-level into $L^2$-**Top**. It must be now checked that $E_\times$ is well-defined at the morphism-level, i.e., that $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $L$-bicontinuous implies $f : (X, E_\times(\tau_1, \tau_2)) \to (Y, E_\times(\sigma_1, \sigma_2))$ is $L^2$-continuous. To that end, let

$$v \in E_\times(\sigma_1, \sigma_2) = \varphi_Y^{\to}(\sigma_1 \times \sigma_2).$$

Then $\exists\,(v_1,v_2)\in\sigma_1\times\sigma_2$ with $v=\varphi_Y(v_1,v_2)$. Now let $x\in X$. Then

$$\begin{aligned} f_L^{\leftarrow}(v)(x) &= v(f(x)) \\ &= \varphi_Y(v_1,v_2)(f(x)) \\ &= (v_1(f(x)),v_2(f(x))) \\ &= (f_L^{\leftarrow}(v_1)(x),f_L^{\leftarrow}(v_2)(x)). \end{aligned}$$

Since $f$ is $L$-bicontinuous,

$$u_1\equiv f_L^{\leftarrow}(v_1)\in\tau_1,\quad u_2\equiv f_L^{\leftarrow}(v_2)\in\tau_2;$$

and so choosing

$$u=\varphi_X(u_1,u_2)\in E_\times(\tau_1,\tau_2),$$

we have

$$f_L^{\leftarrow}(v)=u,$$

finishing the proof that $f$ is $L^2$-continuous.

Since $E_\times$ is concrete (with respect to the usual forgetful functors), it is immediate that $E_\times$ is a functor and that $E_\times$ injects hom-sets. To verify that $E_\times$ is full, we show that $f:(X,E_\times(\tau_1,\tau_2))\to(Y,E_\times(\sigma_1,\sigma_2))$ is $L^2$-continuous implies $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is $L$-bicontinuous. Let $v\in\sigma_1$ and note $\underline{\perp}\in\sigma_2$. Then $(v,\underline{\perp})\in\sigma_1\times\sigma_2$, so that $\varphi_Y(v,\underline{\perp})\in E_\times(\sigma_1,\sigma_2)$. Hence $f_L^{\leftarrow}(\varphi_Y(v,\underline{\perp}))\in E_\times(\tau_1,\tau_2)$ by the $L^2$-continuity of $f$. It follows $\exists\,u\in E_\times(\tau_1,\tau_2)$, and hence $\exists\,(u_1,u_2)\in\tau_1\times\tau_2$, such that

$$f_L^{\leftarrow}(\varphi_Y(v,\underline{\perp}))=u=\varphi_X(u_1,u_2).$$

Now let $x\in X$. Then

$$\begin{aligned} (u_1(x),u_2(x)) &= \varphi_X(u_1,u_2)(x) \\ &= f_L^{\leftarrow}(\varphi_Y(v,\underline{\perp}))(x) \\ &= (v(f(x)),\perp) \\ &= (f_L^{\leftarrow}(v)(x),\perp), \end{aligned}$$

so that $u_1(x)=f_L^{\leftarrow}(v)(x)$. It follows $f_L^{\leftarrow}(v)=u_1\in\tau_1$. Similarly, it can be shown that if $v\in\sigma_2$, then $f_L^{\leftarrow}(v)\in\tau_2$. Hence $f$ is $L$-bicontinuous.

For $E_\times$ to be an embedding, it remains to show that $E_\times$ injects objects. To that end let $(X,\tau_1,\tau_2)\neq(Y,\sigma_1,\sigma_2)$. If $X\neq Y$, we are done. So suppose that $X=Y$ and that $(\tau_1,\tau_2)\neq(\sigma_1,\sigma_2)$. Then immediately by 3.22,

$$E_\times(\tau_1,\tau_2)\neq E_\times(\sigma_1,\sigma_2).$$

It follows that

$$E_\times(X,\tau_1,\tau_2)\neq E_\times(Y,\sigma_1,\sigma_2).$$

Finally, the strictness of $E_\times$, when $L$ is consistent, follows from 3.24 below. □

Many more properties of $E_\times$ are developed in the next three subsections which show that it is an extremely well-behaved embedding.

**Counterexample 3.24.** *If $E_\times$ were to surject objects, then $E_\times$ would be a functorial isomorphism. This however is usually not the case. Let $L$ be any consistent us-quantale, note $L \supset \mathbf{2} = \{\bot, e\}$, and consider the $L^2$-topological space $(X, \tau)$ with $X$ nonempty and $\tau$ the indiscrete $L^2$-topology*

$$\tau = \left\{ \underline{(\bot, \bot)}, \underline{(e, e)} \right\}.$$

*Suppose a space $(X, E_\times(\tau_1, \tau_2))$ from the image of $E_\times$ is $(X, \tau)$. This forces*

$$\varphi_X^{\rightarrow}(\tau_1 \times \tau_2) = E_\times(\tau_1, \tau_2) = \tau.$$

*Noting*

$$\{\underline{\bot}, \underline{e}\} \subset \tau_1, \ \{\underline{\bot}, \underline{e}\} \subset \tau_2,$$

*it follows*

$$\varphi_X(\underline{\bot}, \underline{e}) \in \varphi_X^{\rightarrow}(\tau_1 \times \tau_2), \quad \varphi_X(\underline{\bot}, \underline{e}) = \underline{(\bot, e)} \notin \tau,$$

*a contradiction. Hence the space $(X, \tau)$ is not in the image of $E_\times$. Hence, for consistent $L$ it is the case that $E_\times$ is not a functorial isomorphism, but a strict embedding. This justifies examining $E_\times$'s behavior w.r.t. limits and colimits in the next Subsubsections 3.4.2–3.4.3 as well as characterizing $\left|E_\times^{\rightarrow}(L\text{-}\mathbf{BiTop})\right|$ in 3.28.*

**Corollary 3.25.** *Let $\mathbf{4}$ be the 4-element Boolean algebra $\{\bot, \alpha, \beta, \top\}$with $\otimes = \wedge$ (binary). Then the traditional category* **BiTop** *of bitopological spaces and bicontinuous maps concretely, fully, strictly embeds into* **4-Top** *as a full monocoreflective subcategory that is closed under all limits and colimits.*

*Proof.* Consider the bitopological version $G_\chi : \mathbf{BiTop} \to \mathbf{2\text{-}BiTop}$ of the characteristic functor given by

$$G_\chi(\mathfrak{T}) = \{\chi_U : U \in \mathfrak{T}\}, \ G_\chi(\mathfrak{S}) = \{\chi_V : V \in \mathfrak{S}\},$$

$$G_\chi(X, \mathfrak{T}, \mathfrak{S}) = (X, G_\chi(\mathfrak{T}), G_\chi(\mathfrak{S})), \ G_\chi(f) = f.$$

Then this bitopological $G_\chi$ is a concrete functorial isomorphism. Now clearly by the direct product of us-quantales, $\mathbf{2}^2 \cong \mathbf{4}$, so by 3.23 and 3.24, **2-BiTop** concretely, fully, strictly embeds into **4-Top**. Hence via the composition

$$E_\times \circ G_\chi : \mathbf{BiTop} \hookrightarrow \mathbf{4\text{-}Top},$$

**BiTop** concretely, fully, strictly embeds into **4-Top**. For the monoreflectivity claim, see 3.26 below; and the claim regarding limits and colimits follows from Subsubsections 3.4.2–3.4.3 below, the limit claim needing the observation that **4** is a u-quantale with $\otimes = \wedge$. □

3.4.2. *Behavior of* $E_\times : L$-**BiTop** $\to L^2$-**Top** *w.r.t. colimits.* Since for all consistent $L$ the full concrete embedding $E_\times$ is not a functorial isomorphism, but only a strict embedding, it is worthwhile to investigate its behavior w.r.t. limits and colimits. This subsection shows for any us-quantale $L$ that the embedding $E_\times$ has a right-adjoint—and hence preserves colimits. The next subsection then shows step by step for $L$ a u-quantale that the Special Adjoint Functor Theorem constructs for $E_\times$ a left-adjoint—and hence $E_\times$ preserves limits; and further the next subsection shows for any us-quantale $L$ that the embedding $E_\times$ reflects and detects all limits and is transportable. Therefore, this subsection—in concert with the preceding and subsequent subsections—shows that $E_\times$ is an extremely well-behaved embedding.

**Theorem 3.26** ($E_\times \dashv F_\pi$)**.** *Let $L$ be a us-quantale and put the "projection" functor $F_\pi : L$-**BiTop** $\leftarrow L^2$-**Top** as follows:*

$$F_\pi (X, \tau) = (X, F_\pi (\tau)), \ F_\pi (f) = f,$$

*where the fibre level of $F_\pi$*

$$F_\pi (\tau) = (\pi_1 \circ \tau \equiv \{\pi_1 \circ u : u \in \tau\}, \pi_2 \circ \tau \equiv \{\pi_2 \circ u : u \in \tau\})$$

*uses the projections $\pi_1, \pi_2 : L \times L \to L$ for the us-quantalic (direct) product. Then the following hold:*

(1) *$F_\pi$ is a concrete embedding which is not full and does not lift limits.*
(2) *$E_\times \dashv F_\pi$,* so *$E_\times$ preserves all strong colimits and $F_\pi$ preserves all strong limits.*
(3) *$F_\pi$ need not detect limits nor be transportable.*
(4) *$F_\pi \circ E_\times = Id_{L\text{-}\mathbf{BiTop}}$.*
(5) *$L$-**BiTop** is isomorphic (via $E_\times$) to a full monocoreflective subcategory of $L^2$-**Top**.*
(6) *$L^2$-**Top** is isomorphic (via $F_\pi$) to an isoreflective subcategory of $L$-**BiTop**.*

*Proof. Ad(1).* Since us-quantalic projections preserve arbitrary joins, the tensor, and the unit, it follows that $F_\pi (\tau) \in L\text{-}BT(X)$; and hence $(X, F_\pi (\tau)) \in |L\text{-}\mathbf{BiTop}|$ and $F_\pi$ is well-defined at the object level. As for morphisms, let $f : (X, \tau) \to (Y, \sigma)$ be $L^2$-continuous in $L^2$-**Top**. Then, given $v \in \sigma$, the identities

$$f_L^{\leftarrow} (\pi_1 \circ v) = \pi_1 \circ f_L^{\leftarrow} (v), \quad f_L^{\leftarrow} (\pi_2 \circ v) = \pi_2 \circ f_L^{\leftarrow} (v)$$

are easily checked and immediately imply that $f : (X, F_\pi (\tau)) \to (Y, F_\pi (\sigma))$ is $L$-bicontinuous in $L$-**BiTop**. Now by the concreteness of $F_\pi$, it is immediately a concrete and faithful functor. To show that $F_\pi$ is an embedding, it remains to check that $F_\pi$ injects objects: but if $u, v : X \to L^2$ are distinct, there exists $x \in X$ such that W.L.O.G.

$$\pi_1 (u (x)) \neq \pi_1 (v (x));$$

which implies that if $\tau \neq \sigma$ as $L^2$-topologies on $X$, then $F_\pi (\tau) \neq F_\pi (\sigma)$ as $L$-bitopologies on $X$, showing that $F_\pi$ injects objects.

To see that $F_\pi$ need not be full, let $L = \mathbf{2}$, write the Boolean algebra $L^2 = \mathbf{4}$ as $\{(\perp,\perp),(\perp,\top),(\top,\perp),(\top,\top)\}$, let $X = \{x\}$, and choose

$$\tau = \left\{\underline{(\perp,\perp)}, \underline{(\top,\top)}\right\}, \quad \sigma = \left\{\underline{(\perp,\perp)}, \underline{(\perp,\top)}, \underline{(\top,\perp)}, \underline{(\top,\top)}\right\}.$$

Then it follows $id_X : (X,\tau) \to (X,\sigma)$ is *not* $L^2$-continuous (since $\sigma$ is not a subset of $\tau$). Now

$$\begin{aligned} \pi_1 \circ \underline{(\perp,\perp)} &= \pi_1 \circ \underline{(\perp,\top)} = \underline{\perp}, \quad \pi_1 \circ \underline{(\top,\perp)} = \pi_1 \circ \underline{(\top,\top)} = \underline{\top}, \\ \pi_2 \circ \underline{(\perp,\perp)} &= \pi_2 \circ \underline{(\top,\perp)} = \underline{\perp}, \quad \pi_2 \circ \underline{(\perp,\top)} = \pi_2 \circ \underline{(\top,\top)} = \underline{\top}, \end{aligned}$$

so that

$$\begin{aligned} F_\pi(\tau) &= (\pi_1 \circ \tau,\, \pi_2 \circ \tau) \\ &= (\{\underline{\perp}, \underline{\top}\}, \{\underline{\perp}, \underline{\top}\}) \\ &= (\pi_1 \circ \sigma,\, \pi_2 \circ \sigma) \\ &= F_\pi(\sigma), \end{aligned}$$

implying $id_X : (X, F_\pi(\tau)) \to (X, F_\pi(\sigma))$ is $L$-bicontinuous. The concreteness of $F_\pi$ implies there exists no $g \in L$-**Top** with $F_\pi(g) = id_X$, so $F_\pi$ is not full.

To see that $F_\pi$ need not lift limits, let the diagram in $L^2$-**Top** be the space $(X,\sigma)$ of the preceding paragraph. Then the image of this diagram is the space $(X, F_\pi(\sigma))$ in $L$-**BiTop**. Now the space $(X, F_\pi(\tau))$, together with the arrow $id_X : (X, F_\pi(\tau)) \to (X, F_\pi(\sigma))$, is a limit of the diagram $(X, F_\pi(\sigma))$: any $L$-bicontinuous $f : (Z, \upsilon_1, \upsilon_2) \to (X, F_\pi(\sigma))$ trivially factors uniquely through $id_X$. But as seen in the preceding paragraph, there is no $g \in L$-**Top** with $F_\pi(g) = id_X$, which means there is no limiting cone of $(X,\sigma)$ in $L^2$-**Top** which $F_\pi$ carries over to the limit $id_X : (X, F_\pi(\tau)) \to (X, F_\pi(\sigma))$ in $L$-**BiTop**. Hence $F_\pi$ need not lift limits.

*Ad(2).* Let $(X,\tau_1,\tau_2) \in |L\text{-}\mathbf{BiTop}|$ be given. Then

$$F_\pi E_\times (X,\tau_1,\tau_2) = F_\pi\left(X, \varphi_X^{\rightarrow}(\tau_1 \times \tau_2)\right) \equiv (X, \hat{\tau}_1, \hat{\tau}_2),$$

where it follows that

$$\begin{aligned} \hat{\tau}_1 &= \{\pi_1 \circ \varphi_X(u,v) : u \in \tau_1,\, v \in \tau_2\}, \\ \hat{\tau}_2 &= \{\pi_1 \circ \varphi_X(u,v) : u \in \tau_1,\, v \in \tau_2\}. \end{aligned}$$

We choose the right unit $\eta$ to be the identity mapping $id : X \to X$. Then for each $x \in X$,

$$\begin{aligned} (\pi_1 \circ \varphi_X(u,v))(x) &= \pi_1(u(x), v(x)) = u(x), \\ (\pi_2 \circ \varphi_X(u,v))(x) &= \pi_2(u(x), v(x)) = v(x), \end{aligned}$$

which immediately gives the $L$-bicontinuity of $\eta$.

For universality of the lifting, let $(X,\tau) \in \left|L^2\text{-}\mathbf{Top}\right|$ be given, along with an $L$-bicontinuous map $f : (X,\tau_1,\tau_2) \to (X, F_\pi(\tau))$. Choosing $\bar{f} = f$, we now check $\bar{f} : E_\times(X,\tau_1,\tau_2) \to (X,\tau)$ is an $L$-continuous map from $E_\times(X,\tau_1,\tau_2)$ to $(X,\tau)$ by letting $v \in \tau$ and $x \in X$. Then the $L$-bicontinuity of $f$ implies

$$f_L^{\leftarrow}(\pi_1 \circ u) \in \tau_1,\ f_L^{\leftarrow}(\pi_2 \circ u) \in \tau_2,$$

from which it follows

$$\varphi_X\left(f_L^{\leftarrow}\left(\pi_1 \circ u\right) \in \tau_1,\, f_L^{\leftarrow}\left(\pi_2 \circ u\right) \in \tau_2\right) \in \varphi_X^{\rightarrow}\left(\tau_1 \times \tau_2\right).$$

Further, we note

$$\begin{aligned} f_L^{\leftarrow}(u)(x) &= u(f(x)) = \left(\pi_1(u(x)), \pi_2(u(x))\right) \\ &= \left(f_L^{\leftarrow}\left(\pi_1 \circ u\right)(x), f_L^{\leftarrow}\left(\pi_2 \circ u\right)(x)\right) \\ &= \varphi_X\left(f_L^{\leftarrow}\left(\pi_1 \circ u\right) \in \tau_1,\, f_L^{\leftarrow}\left(\pi_2 \circ u\right) \in \tau_2\right)(x). \end{aligned}$$

Finally, it is immediate that $\bar{f}$ is the unique $L$-continuous map from $E_\times(X, \tau_1, \tau_2)$ to $(X, \tau)$ such that

$$f = \bar{f} \circ \eta,$$

completing the universality of the lifting. The naturality diagram now follows by concreteness.

*Ad(3).* This is an immediate consequence of (1), (2), and Proposition 13.34 [1].

*Ad(4).* Since $\hat{\tau}_1 = \tau_1$, $\hat{\tau}_2 = \tau_2$ in the proof of (2), it is immediate that $F_\pi \circ E_\times = Id_{L\text{-}\mathbf{BiTop}}$.

*Ad(5).* Using $F_\pi \circ E_\times = Id_{L\text{-}\mathbf{BiTop}}$, the components of the left unit (counit) of $E_\times \dashv F_\pi$ furnish the needed monocoreflection arrows to $L^2$-topological spaces from the $E_\times$ image of $L$-**BiTop**.

*Ad(6).* Using $F_\pi \circ E_\times = Id_{L\text{-}\mathbf{BiTop}}$, the components of the right unit of $E_\times \dashv F_\pi$ furnish the needed isocoreflection arrows to $L$-bitopological spaces from the $F_\pi$ image of $L^2$-**Top**. □

**Remark 3.27.** We collect some facts concerning $E_\times$, $F_\pi$, and their fibre-dependent constructions, where $L$ is a us-quantale:

(1) $F_\pi \not\dashv E_\times$ if $L$ is consistent. This is a consequence of 3.24.
(2) $E_\times \dashv F_\pi$ need not be a categorical equivalence. This follows from (1).
(3) For each $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiTop}|$,

$$\begin{aligned} F_\pi E_\times(\tau_1, \tau_2) &= F_\pi\left(\varphi_X^{\rightarrow}\left(\tau_1 \times \tau_2\right)\right) \\ &= \left(\pi_1 \circ \varphi_X^{\rightarrow}\left(\tau_1 \times \tau_2\right),\, \pi_2 \circ \varphi_X^{\rightarrow}\left(\tau_1 \times \tau_2\right)\right) \\ &= (\tau_1, \tau_2). \end{aligned}$$

(4) For each $(X, \tau) \in \left|L^2\text{-}\mathbf{Top}\right|$,

$$\text{“}E_\times\left(F_\pi(\tau)\right) = \varphi_X^{\rightarrow}\left(\left(\pi_1 \circ \tau\right) \times \left(\pi_2 \circ \tau\right)\right) \supset \tau\text{”}$$

always holds; but for $L$ consistent,

$$\text{“}E_\times\left(F_\pi(\tau)\right) = \varphi_X^{\rightarrow}\left(\left(\pi_1 \circ \tau\right) \times \left(\pi_2 \circ \tau\right)\right) \subset \tau\text{”}$$

need not hold. The latter statement is another version of (1).

**Theorem 3.28** (characterization of $\left|E_\times^{\rightarrow}(L\text{-}\mathbf{BiTop})\right|$)**.** *Let $L$ be a us-quantale and $(X, \tau) \in \left|L^2\text{-}\mathbf{Top}\right|$. Then $(X, \tau) \in \left|E_\times^{\rightarrow}(L\text{-}\mathbf{BiTop})\right|$ if and only if $E_\times\left(F_\pi(\tau)\right) = \tau$, i.e., both inequalities of 3.27(4) hold.*

3.4.3. *Behavior of* $E_\times : L$-**BiTop** $\to L^2$-**Top** *w.r.t. limits.* The question of a left-adjoint for $E_\times$ is open for general us-quantales $L$; and it is our conjecture is that for general us-quantales $L$, $E_\times$ would not preserve products or intersections and hence would not have a left-adjoint. But on the other hand, this section shows $E_\times$ has a left-adjoint (and therefore preserves all limits) for $L$ any u-quantale. We point out that our proof of this left-adjoint is existential (via the Special Adjoint Functor Theorem) and not constructive; and it is an additional open question whether there is a direct construction of this left adjoint not essentially factoring through our proof. It is further proved that $E_\times$ reflects and detects limits and is transportable.

**Lemma 3.29** (preservation of products)**.** *For each u-quantale $L$, $E_\times : L$-**BiTop** $\to L^2$-**Top** preserves arbitrary (small) products.*

**Sublemma 3.30.** *Let $L$ be a u-quantale and suppose $X$ is a set and $\tau_1, \tau_2$ are $L$-topologies on $X$ with respective subbases $\sigma_1, \sigma_2$, namely*

$$\tau_1 = \langle\langle \sigma_1 \rangle\rangle \,,\ \tau_2 = \langle\langle \sigma_2 \rangle\rangle \,,$$

*such that $\{\underline{\perp}, \underline{e}\} \subset \sigma_1 \cap \sigma_2$. Then*

$$(*) \qquad \varphi_X^{\rightarrow}\,(\tau_1 \times \tau_2) = \langle\langle \varphi_X^{\rightarrow}\,(\sigma_1 \times \sigma_2) \rangle\rangle \,.$$

*Proof.* To see that "$\supset$" holds in (*), note that

$$\sigma_1 \times \sigma_2 \subset \tau_1 \times \tau_2,$$

$$\varphi_X^{\rightarrow}\,(\sigma_1 \times \sigma_2) \subset \varphi_X^{\rightarrow}\,(\tau_1 \times \tau_2)\,,$$

$$\langle\langle \varphi_X^{\rightarrow}\,(\sigma_1 \times \sigma_2) \rangle\rangle \subset \varphi_X^{\rightarrow}\,(\tau_1 \times \tau_2)\,.$$

For "$\subset$" in (*), we first invoke the associativity of $\otimes$ and its infinite distributivity over $\bigvee$ to write members of $\tau_1, \tau_2$ as joins of tensor products of members of $\sigma_1, \sigma_2$, respectively. More precisely, consider these typical members

$$\bigvee_{\alpha \in A_1} \left( \bigotimes_{\beta \in B_1} u_{\alpha\beta} \right), \quad \bigvee_{\alpha \in A_2} \left( \bigotimes_{\beta \in B_2} v_{\alpha\beta} \right)$$

of $\tau_1, \tau_2$, respectively, where $A_1, A_2$ are arbitrary indexing sets, $B_1, B_2$ are arbitrary finite indexing sets, each $u_{\alpha\beta} \in \sigma_1$, each $v_{\alpha\beta} \in \sigma_2$, and where W.L.O.G. we assume

$$A_1 \cap A_2 = \varnothing = B_1 \cap B_2.$$

Next, we augment the $u_{\alpha\beta}$'s and $v_{\alpha\beta}$'s as follows, using the assumption that $\{\underline{\perp}, \underline{e}\} \subset \sigma_1 \cap \sigma_2$:

$$\alpha \in A_1,\ \beta \in B_2,\ u_{\alpha\beta} \equiv \underline{e},$$

$$\alpha \in A_2,\ \beta \in B_1 \cup B_2,\ u_{\alpha\beta} \equiv \underline{\perp},$$

$$\alpha \in A_2,\ \beta \in B_1,\ v_{\alpha\beta} \equiv \underline{e},$$

$$\alpha \in A_1,\ \beta \in B_1 \cup B_2,\ v_{\alpha\beta} \equiv \underline{\perp}.$$

It follows that as maps from $X$ to $L$ that

$$\bigvee_{\alpha \in A_1 \cup A_2} \left( \bigotimes_{\beta \in B_1 \cup B_2} u_{\alpha\beta} \right) = \bigvee_{\alpha \in A_1} \left( \bigotimes_{\beta \in B_1} u_{\alpha\beta} \right),$$

$$\bigvee_{\alpha \in A_1 \cup A_2} \left( \bigotimes_{\beta \in B_1 \cup B_2} v_{\alpha\beta} \right) = \bigvee_{\alpha \in A_2} \left( \bigotimes_{\beta \in B_2} v_{\alpha\beta} \right).$$

We thus have that a typical member

$$\left( \bigvee_{\alpha \in A_1} \left( \bigotimes_{\beta \in B_1} u_{\alpha\beta} \right), \bigvee_{\alpha \in A_2} \left( \bigotimes_{\beta \in B_2} v_{\alpha\beta} \right) \right)$$

of $\tau_1 \times \tau_2$ may be rewritten as

$$\left( \bigvee_{\alpha \in A_1 \cup A_2} \left( \bigotimes_{\beta \in B_1 \cup B_2} u_{\alpha\beta} \right), \bigvee_{\alpha \in A_1 \cup A_2} \left( \bigotimes_{\beta \in B_1 \cup B_2} v_{\alpha\beta} \right) \right)$$

$$= \bigvee_{\alpha \in A_1 \cup A_2} \left( \bigotimes_{\beta \in B_1 \cup B_2} (u_{\alpha\beta}, v_{\alpha\beta}) \right),$$

the latter being the form of a typical member of $\langle\langle \sigma_1 \times \sigma_2 \rangle\rangle$. To complete the proof of "$\subset$", we invoke the fact that $\varphi_X^{\rightarrow}$ is an order-isomorphism preserving all tensor products (3.18(4)) to conclude that

$$\varphi_X^{\rightarrow}(\tau_1 \times \tau_2) \subset \varphi_X^{\rightarrow}\langle\langle \sigma_1 \times \sigma_2 \rangle\rangle = \langle\langle \varphi_X^{\rightarrow}(\sigma_1 \times \sigma_2) \rangle\rangle .$$

□

*Proof of 3.29.* Recall the categorical products in $L$-**BiTop** use the categorical product of $L$-**Top** in each slot as well as the usual projections for the morphisms of the product (Subsection 1.5), and let $\{(X_\gamma, (\tau_1^\gamma, \tau_2^\gamma))\}_{\gamma \in \Gamma} \subset |L\text{-}\mathbf{BiTop}|$. Because of the concreteness of $E_\times$, the validity of

$$E_\times \left( \prod_{\gamma \in \Gamma} (X_\gamma, (\tau_1^\gamma, \tau_2^\gamma)), \{\pi_\gamma\}_{\gamma \in \Gamma} \right) = \left( \prod_{\gamma \in \Gamma} E_\times (X_\gamma, (\tau_1^\gamma, \tau_2^\gamma)), \{\pi_\gamma\}_{\gamma \in \Gamma} \right)$$

holds if and only if we have the equality of topologies

(**) $$\varphi_{\times_{\gamma \in \Gamma} X_\gamma}^{\rightarrow} (\Pi_{\gamma \in \Gamma}\, \tau_1^\gamma \times \Pi_{\gamma \in \Gamma}\, \tau_2^\gamma) = \Pi_{\gamma \in \Gamma}\, \varphi_{X_\gamma}^{\rightarrow} (\tau_1^\gamma \times \tau_2^\gamma),$$

where "$\times$" denotes as usual the direct product of us-quantales. For convenience, "LHS" and "RHS" respectively denote the left-hand side and right-hand side of (**). Let a subbasic open subset $W$ be given from RHS. Then $W$ may be written as follows:

$$W = (\pi_\beta)_L^{\leftarrow} \left( \varphi_{X_\beta} \left( t_1^\beta, t_2^\beta \right) \right),$$

where $\left(t_1^\beta, t_2^\beta\right) \in \tau_1^\beta \times \tau_2^\beta$ for a fixed index $\beta \in \Gamma$. Given $\{x_\gamma\}_{\gamma\in\Gamma} \in \times_{\gamma\in\Gamma} X_\gamma$, then

$$\begin{aligned}
W\left(\{x_\gamma\}_{\gamma\in\Gamma}\right) &= \varphi_{X_\beta}\left(t_1^\beta, t_2^\beta\right)\left(\pi_\beta\left(\{x_\gamma\}_{\gamma\in\Gamma}\right)\right) \\
&= \varphi_{X_\beta}\left(t_1^\beta, t_2^\beta\right)(x_\beta) \\
&= \left(t_1^\beta(x_\beta), t_2^\beta(x_\beta)\right) \\
&= \left(\left[(\pi_\beta)_L^{\leftarrow}\left(t_1^\beta\right)\right]\left(\{x_\gamma\}_{\gamma\in\Gamma}\right), \left[(\pi_\beta)_L^{\leftarrow}\left(t_2^\beta\right)\right]\left(\{x_\gamma\}_{\gamma\in\Gamma}\right)\right).
\end{aligned}$$

This shows $W$ is in LHS, LHS contains a subbasis of RHS, and so LHS contains RHS.

For the reverse direction, let $Z$ be in LHS. Then $\exists\,(u_1, u_2) \in \Pi_{\gamma\in\Gamma}\tau_1^\gamma \times \Pi_{\gamma\in\Gamma}\tau_2^\gamma$ with $Z = \varphi_{\times_{\gamma\in\Gamma}X_\gamma}(u_1, u_2)$. Since the $\tau_1^\gamma$'s and $\tau_2^\gamma$'s contain $\{\underline{\perp}, \underline{e}\}$ and since these $L$-subsets are preserved by the Zadeh preimage operators of all the projection maps, the usual subbasis for each of $\Pi_{\gamma\in\Gamma}\tau_1^\gamma$ and $\Pi_{\gamma\in\Gamma}\tau_2^\gamma$ contains $\{\underline{\perp}, \underline{e}\}$. Thus 3.30 applies to say it suffices to let $u_1, u_2$ be subbasic in their respective $L$-product topologies $\prod_{\gamma\in\Gamma}\tau_1^\gamma$, $\prod_{\gamma\in\Gamma}\tau_2^\gamma$; so we may write

$$u_1 = (\pi_\alpha)_L^{\leftarrow}(t_1^\alpha),\ u_2 = (\pi_\beta)_L^{\leftarrow}\left(t_2^\beta\right),$$

where $t_1^\alpha \in \tau_1^\alpha$, $t_2^\beta \in \tau_2^\beta$ for fixed indices $\alpha, \beta \in \Gamma$. Let $\{x_\gamma\}_{\gamma\in\Gamma} \in \times_{\gamma\in\Gamma} X_\gamma$. Then recalling that $L$ has a unit $e$ for $\otimes$ and that $\underline{e}$ is the corresponding unit for $\otimes$ lifted to $L^X$, we have

$$\begin{aligned}
Z\left(\{x_\gamma\}_{\gamma\in\Gamma}\right) &= \varphi_{\times_{\gamma\in\Gamma}X_\gamma}(u_1, u_2)\left(\{x_\gamma\}_{\gamma\in\Gamma}\right) \\
&= \left(u_1\left(\{x_\gamma\}_{\gamma\in\Gamma}\right), u_2\left(\{x_\gamma\}_{\gamma\in\Gamma}\right)\right) \\
&= \left((\pi_\alpha)_L^{\leftarrow}(t_1^\alpha)\left(\{x_\gamma\}_{\gamma\in\Gamma}\right), (\pi_\beta)_L^{\leftarrow}\left(t_2^\beta\right)\left(\{x_\gamma\}_{\gamma\in\Gamma}\right)\right) \\
&= \left(t_1^\alpha\left(\pi_\alpha\left(\{x_\gamma\}_{\gamma\in\Gamma}\right)\right), t_2^\beta\left(\pi_\beta\left(\{x_\gamma\}_{\gamma\in\Gamma}\right)\right)\right) \\
&= \left(t_1^\alpha(x_\alpha), t_2^\beta(x_\beta)\right) \\
&= \left(t_1^\alpha(x_\alpha) \otimes e, e \otimes t_2^\beta(x_\beta)\right) \\
&= \left(t_1^\alpha(x_\alpha), \underline{e}(x_\alpha)\right) \otimes \left(\underline{e}(x_\beta), t_2^\beta(x_\beta)\right) \\
&= \left(\begin{array}{c} \left[(\pi_\alpha)_L^{\leftarrow}\left(\varphi_{X_\alpha}\left(t_1^\alpha, \underline{e}\right)\right)\right] \otimes \\ \left[(\pi_\beta)_L^{\leftarrow}\left(\varphi_{X_\beta}\left(\underline{e}, t_2^\beta\right)\right)\right] \end{array}\right)\left(\{x_\gamma\}_{\gamma\in\Gamma}\right),
\end{aligned}$$

the last line being the evaluation at $\{x_\gamma\}_{\gamma\in\Gamma}$ by a tensor of open subsets of RHS and hence of an open subset of RHS. Thus $Z$ is in RHS, so LHS is contained in RHS, completing the proof of the theorem. □

**Lemma 3.31.** *For each us-quantale $L$, $E_\times$ : $L$-**BiTop** $\to$ $L^2$-**Top** preserves equalizers.*

**Sublemma 3.32.** *Let $L$ be a us-quantale,* $(X,\tau,\sigma) \in |L\text{-}\mathbf{BiTop}|$, $Z \subset X$, and $\tau(Z), \sigma(Z), E_\times(\tau,\sigma)(Z)$*be the $L$-subspace topologies on $Z$ given by*

$$\tau(Z) = \left\{u_{|Z} : u \in \tau\right\},$$

$$\sigma(Z) = \left\{v_{|Z} : v \in \sigma\right\},$$

$$E_\times(\tau,\sigma)(Z) = \varphi_X^{\rightarrow}(\tau \times \sigma)(Z)$$

*(cf.* [41]*). Then*

$$E_\times(\tau,\sigma)(Z) = E_\times(\tau(Z), \sigma(Z)).$$

*Restated, $E_\times$ respects subspace topologies.*

*Proof.* Let $u \in \tau$, $v \in \sigma$, $z \in Z$. Then

$$\varphi_X(u,v)_{|Z}(z) = (u(z), v(z)) = \left(u_{|Z}(z), v_{|Z}(z)\right) = \varphi_X\left(u_{|Z}, v_{|Z}\right)(z).$$

This implies

$$E_\times(\tau,\sigma)(Z) = \varphi_X^{\rightarrow}(\tau\times\sigma)(Z) = \varphi_X^{\rightarrow}(\tau(Z)\times\sigma(Z)) = E_\times(\tau(Z),\sigma(Z)).$$

□

*Proof of 3.31.* A categorical proof based upon the concreteness of $E_\times, F$ and $F_\pi E_\times = Id_{L\text{-}\mathbf{BiTop}}$ (3.26) does not work since it would generally require that $E_\times F_\pi(\tau) \subset \tau$, which need not be true by (3.27(4)). It is necessary to look at the actual construction of equalizers in each of $L$-**BiTop** and $L^2$-**Top** and show that $E_\times$ carries the former into the latter. It can be checked that the equalizer of $f, g : (X,\tau_1,\tau_2) \rightrightarrows (Y,\sigma_1,\sigma_2)$ in $L$-**BiTop** is given by $((Z,\tau_1(Z),\tau_2(Z)), \hookrightarrow)$, where

$$Z = \{x \in X : f(x) = g(x)\},$$

and that the equalizer of $f, g : E_\times(X,\tau_1,\tau_2) \rightrightarrows E_\times(Y,\sigma_1,\sigma_2)$ in $L^2$-**Top** is given by $((Z, E_\times(\tau_1,\tau_2)(Z)), \hookrightarrow)$ using the same $Z$. Because of the concreteness of $E_\times$, the issue is whether $E_\times(\tau_1,\tau_2)(Z)$ is the same as $E_\times(\tau_1(Z),\tau_2(Z))$, and this is settled in 3.32. □

**Corollary 3.33.** *For each u-quantale $L$, $E_\times : L$-**BiTop** $\to L^2$-**Top** preserves all small limits. In particular, for each frame $L$, $E_\times$ preserves all small limits.*

*Proof.* It is not difficult to show that $L$-**BiTop** is topological over **Set** w.r.t. the usual forgetful functor; and since **Set** is complete, it follows that $L$-**BiTop** is complete (Theorem 21.16 [1]). Conjoin 3.29 and 3.31 to get that $E_\times$ preserves equalizers and (all) products; and then apply Proposition 13.4 [1] to finish the proof. □

**Lemma 3.34.** *For each u-quantale $L$, $E_\times : L$-**BiTop** $\to L^2$-**Top** preserves all intersections.*

*Proof.* As in the proof of 3.31, it is necessary to look at the actual construction of intersections in each of $L$-**BiTop** and $L^2$-**Top** and show that $E_\times$ carries the former into the latter. Since this is trivially the case if the indexing class of the intersection is empty, we assume *sequens* that the indexing class is nonempty.

To describe intersections in $L$-**BiTop**, let $\{((X_\gamma, \tau_1^\gamma, \tau_2^\gamma), m_\gamma)\}_{\gamma \in \Gamma}$ be a class of subobjects of $(Y, \sigma_1, \sigma_2)$—by the well-poweredness of $L$-**BiTop** (Subsection 1.5), this class is not proper, i.e., we may take $\Gamma$ as a set; form the product

$$\left((\times_{\gamma \in \Gamma} X_\gamma, \Pi_{\gamma \in \Gamma}\, \tau_1^\gamma, \Pi_{\gamma \in \Gamma}\, \tau_2^\gamma), \{\pi_\gamma\}_{\gamma \in \Gamma}\right)$$

of these subobjects in $L$-**BiTop**; let

$$X \equiv \left\{\{x_\gamma\}_{\gamma \in \Gamma} : \forall \beta, \delta \in \Gamma,\ m_\beta(x_\beta) = m_\delta(x_\delta)\right\} \subset \times_{\gamma \in \Gamma} X_\gamma;$$

fix $\zeta \in \Gamma$; and put

$$m \equiv m_\zeta \circ \pi_\zeta \circ \hookrightarrow\ : X \to Y.$$

Then equipping $X$ with the $L$-subspace topologies

$$[\Pi_{\gamma \in \Gamma}\, \tau_1^\gamma](X),\ [\Pi_{\gamma \in \Gamma}\, \tau_2^\gamma](X),$$

respectively, it can be shown that

$$((X,\ [\Pi_{\gamma \in \Gamma}\, \tau_1^\gamma](X), [\Pi_{\gamma \in \Gamma}\, \tau_2^\gamma](X)), m)$$

is the required intersection in $L$-**BiTop**.

We now consider in $L^2$-**Top** the image

$$\{(E_\times(X_\gamma, \tau_1^\gamma, \tau_2^\gamma), m_\gamma)\}_{\gamma \in \Gamma} = \{((X_\gamma, E_\times(\tau_1^\gamma, \tau_2^\gamma)), m_\gamma)\}_{\gamma \in \Gamma},$$

under $E_\times$ of the family $\{((X_\gamma, \tau_1^\gamma, \tau_2^\gamma), m_\gamma)\}_{\gamma \in \Gamma}$, which image by the functoriality of $E_\times$ is a sink of subobjects for $E_\times(Y, \sigma_1, \sigma_2)$. Using the $X$ and $m$ of the preceding paragraph, it can be shown that

$$((X,\ [\Pi_{\gamma \in \Gamma}\, E_\times(\tau_1^\gamma, \tau_2^\gamma)](X)), m)$$

is the required intersection in $L^2$-**Top**.

To show that $E_\times$ takes the $L$-**BiTop** intersection to the $L^2$-**Top** intersection, we note

$$\begin{aligned} & E_\times([\Pi_{\gamma \in \Gamma}\, \tau_1^\gamma](X), [\Pi_{\gamma \in \Gamma}\, \tau_2^\gamma](X)) \\ = \quad & E_\times(\Pi_{\gamma \in \Gamma}\, \tau_1^\gamma, \Pi_{\gamma \in \Gamma}\, \tau_2^\gamma)(X) \quad \text{(by 3.32)} \\ = \quad & [\Pi_{\gamma \in \Gamma}\, E_\times(\tau_1^\gamma, \tau_2^\gamma)](X) \quad \text{(by proof of 3.29 (**))}, \end{aligned}$$

which shows

$$E_\times(X,\ [\Pi_{\gamma \in \Gamma}\, \tau_1^\gamma](X), [\Pi_{\gamma \in \Gamma}\, \tau_2^\gamma](X)) = (X,\ [\Pi_{\gamma \in \Gamma}\, E_\times(\tau_1^\gamma, \tau_2^\gamma)](X)).$$

□

**Theorem 3.35.** *For each u-quantale $L$, $E_\times$ : $L$-**BiTop** $\to L^2$-**Top** preserves all strong limits.*

*Proof.* This follows from 3.33, 3.34, and Definition 13.1(3) [1]. □

**Theorem 3.36.** *For each u-quantale $L$, $E_\times$ : $L$-**BiTop** $\to L^2$-**Top** has a left adjoint.*

*Proof.* First, $L$-**BiTop** has small fibres and is a topological construct (proof of 3.33); hence, $L$-**BiTop** is complete and well-powered with coseparators by Corollary 21.17 [1]. Second, Proposition 12.5 [1] now gives $L$-**BiTop** is strongly complete. Third, since $E_\times$ preserves all strong limits (3.35), the Special Adjoint Functor Theorem 18.17 [1] now implies $E_\times$ is a right-adjoint. Finally, apply Proposition 18.9 [1]. □

**Proposition 3.37.** *For each us-quantale $L$, $E_\times : L$-**BiTop** $\to L^2$-**Top** reflects and detects all limits and hence lifts all limits and is transportable.*

*Proof.* The details are straightfoward using the preservation of limits by $F_\pi$, $F_\pi \circ E_\times = Id_{L\text{-}\mathbf{BiTop}}$, 3.36, and Proposition 13.34 [1]. □

3.4.4. *Behavior of $E_\times : L$-**BiTop** $\to L^2$-**Top** w.r.t. stratification issues.* This subsubsection shows $E_\times$ is essentially neutral w.r.t. stratification issues.

**Lemma 3.38.** *Let $L$ be a us-quantale, $(X,\tau,\sigma) \in |L\text{-}\mathbf{BiTop}|$, and $(\gamma,\delta) \in L^2$. Then $\underline{(\gamma,\delta)} \in E_\times(\tau,\sigma)$ if and only if $\underline{\gamma} \in \tau$ and $\underline{\delta} \in \sigma$.*

*Proof.* It is straightforward to check that

$$\begin{aligned} \underline{(\gamma,\delta)} \in \varphi_X^{\rightarrow}(\tau\times\sigma) \quad &\Leftrightarrow \quad \exists\, u \in \tau,\ \exists\, v \in \sigma,\ \forall x \in X,\ (u(x), v(x)) = (\gamma,\delta) \\ &\Leftrightarrow \quad \underline{\gamma} \in \tau,\ \underline{\delta} \in \sigma. \end{aligned}$$

□

**Theorem 3.39.** *Let $L$ be a us-quantale,$(X,\mathfrak{T}) \in |\mathbf{Top}|$, $(X,\tau) \in |L\text{-}\mathbf{Top}|$,* and *$(X,\tau,\sigma) \in |L\text{-}\mathbf{BiTop}|$. The following hold:*

1. *$E_\times(X,\tau,\sigma)$ always has $\underline{(\bot,\bot)}, \underline{(\bot,e)}, \underline{(e,\bot)}, \underline{(e,e)}$ as open subsets.*
2. *$E_\times(X,\tau,\sigma)$ is anti-stratified if and only if $L$ is inconsistent.*
3. *For $L = \mathbf{2}$, $E_\times(X,\tau,\sigma)$is weakly stratified.*
4. *$E_\times G_\chi(X,\mathfrak{T})$ is weakly stratified for $|L| = 2$ and non-stratified for $|L| > 2$.*
5. *$E_\times(X,\tau,\sigma)$ is weakly stratified if and only if $(X,\tau,\sigma)$ is weakly stratified.*
6. *$E_\times(X,\tau,\sigma)$ is non-stratified if and only if $(X,\tau,\sigma)$ is non-stratified.*
7. *Statements (1–3, 5–6) with $E_\times(X,\tau,\sigma)$* replaced with *$E_\times F_d(X,\tau)$ and $(X,\tau,\sigma)$ replaced with $(X,\tau)$.*

*Proof.* (1) follows from 3.38 given that $\{\underline{\bot},\underline{e}\} \subset \tau\cap\sigma$; $(2,3,4)$ follow from (1) and the fact that $\mathbf{4}$ may be taken as precisely $\{(\bot,\bot),(\bot,e),(e,\bot),(e,e)\}$; (5) follows from 3.38; (6) contraposes (5); and (7) is immediate from the other statements. □

## 4. Summary

This paper surveys the relationship between (lattice-valued) bitopology and (lattice-valued) topology by examing a variety of functorial relationships —$E_d, F_\mathfrak{l}, F_\mathfrak{r}, F_\wedge, F_\vee, F_\Pi, E_\times, F_\pi$—when $L$ is a us-quantale. From this overview of these functors and their properties, the following metamathematical conclusions emerge:

(1) If it were assumed that the underlying lattice $L$ of membership values is *not* allowed to change, then this survey would support the following viewpoint:
   (a) (lattice-valued) bitopology is strictly more general than (lattice-valued) topology in an extremely well-behaved way—justified by $E_d$; and
   (b) (lattice-valued) topology is not more general than (lattice-valued) bitopology—justified by $F_{\mathfrak{l}}, F_{\mathfrak{r}}, F_{\wedge}, F_{\vee}, F_{\Pi}$ in comparison with $E_d$, though the variety of ways in which bitopological spaces may be interpreted as topological spaces is rather striking.

(2) If it were assumed that the underlying lattice $L$ of membership values *is* allowed to change (e.g., to the direct s-quantalic product $L^2$), then this survey would support the following viewpoint:
   (a) (lattice-valued) topology is strictly more general than (lattice-valued) bitopology in an extremely well-behaved way—justified by $E_{\times}$; and
   (b) (lattice-valued) bitopology is not more general than (lattice-valued) topology—justified by $F_{\pi}$ in comparison with $E_{\times}$, though $F_{\pi}$ is a rather interesting interpretation of topological spaces as bitopological spaces.

(3) This paper supports viewpoint (2) against viewpoint (1) for the following reasons:
   (a) We are in fact allowed to choose whatever underlying lattice of membership values we wish, so in fact the underlying assumption of (1) is false and the underlying assumption of (2) is true. The class of embeddings $E_{\times}$ stands and must be reckoned with.
   (b) Topology (lattice-valued) is fundamentally simpler than bitopology (lattice-valued):
      (i) An $L$-bitopological space $(X, \tau, \sigma)$ adds to the ground object $X$ *three* parameters—$L, \tau, \sigma$; while an $M$-topological space $(X, \tau)$ adds to the ground object $X$ *two* parameters—$M, \tau$.
      (ii) When passing (via $E_{\times}$) from the $L$-bitopological space $(X, \tau, \sigma)$ to the $L^2$-topological space, the complexity of *two* topologies is isolated in the underlying lattice of membership values, leaving behind *one* topology.
   (c) Topology (lattice-valued) is strictly more general than bitopology (lattice-valued) in each of *two* ways:
      (i) For each $L \in |\mathbf{USQuant}|$, the direct product $L^2 \in |\mathbf{USQuant}|$ and $L$-**BiTop** embeds as a *strict* subcategory of $L^2$-**Top** (via $E_{\times}$), which is extremely well-behaved if $L \in |\mathbf{UQuant}|$.
      (ii) The class

$$\left\{L^2\text{-}\mathbf{Top} : L \in |\mathbf{USQuant}|\right\}$$

representing the field of fixed-basis bitopology using us-quantales is a *strictly proper* subclass of the class

$$\{L\text{-}\mathbf{Top} : L \in |\mathbf{USQuant}|\}$$

representing the field of fixed-basis topology using us-quantales (and not every us-quantale is a direct square of another us-quantale), and this strictness holds if the class is indexed by $|\mathbf{UQuant}|$.

(iii) Thus when one proves a theorem in fixed-basis topology, it is strictly more general w.r.t. coverage of categories *and* coverage of objects in each category in which bitopological spaces are embedded.

(d) The upshot of (a, b, c) is that (lattice-valued) bitopology is categorically redundant, particularly for underlying unital quantales: (lattice-valued) topology is fundamentally simpler and strictly more general. Fixed-basis bitopology is a complicated version of restricted subcategories of categories from a restricted class of categories of fixed-basis topological spaces. For lattice-theoretic bases larger than **2**, workers in lattice-valued bitopology should now be working in lattice-valued topology.

(4) The above arguments apply to traditional bitopology in a more subtle way. On the one hand, traditional bitopology is isomorphic—in an extremely well-behaved way—to a *strictly proper*, extremely well-behaved subcategory of the much simpler **4**-topology (**BiTop** embeds into **4-Top**: 3.25 above); restated, traditional bitopology is a restricted subcase of a particular kind of fuzzy topology (namely **4**-topology) and therefore traditional bitopology is categorically redundant *vis-a-vis* fixed-basis lattice-valued topology. On the other hand, the crisp lattice **2** underlying **BiTop** is so extremely simple that it is really a question of two topologies in **BiTop** *vis-a-vis* the lattice **4** and one topology in **4-Top**; restated, moving from $(X, \mathfrak{T}, \mathfrak{S})$ to $(X, E_\times(\mathfrak{T}, \mathfrak{S}))$ means moving from the parameters $(\mathbf{2}, \mathfrak{T}, \mathfrak{S})$ to the parameters $(\mathbf{4}, E_\times(\mathfrak{T}, \mathfrak{S}))$, with the increased complexity in going from **2** to **4** offset by going from the *two* topologies $\mathfrak{T}, \mathfrak{S}$ to the *one* **4**-topology $E_\times(\mathfrak{T}, \mathfrak{S})$, noting that each of $\mathfrak{T}, \mathfrak{S}$ is more complex than **4**. At the very least, workers in traditional bitopology should consider working in **4**-topology.

(5) The above arguments for redundancy in some sense are even stronger than those used in [16] to show that various versions of "intuitionistic" topologies or topologies comprising double subsets are redundant and a categorically special case of fixed-basis topology since the $E_\times$'s of this paper are strict embeddings and not functorial isomorphisms (when $L$ is consistent) as in [16].

(6) The rich history and literature of traditional bitopology, including interesting separation and compactness axioms which "mix" together the two topologies, are now immediately part of the literature of **4-Top**

since the functorial embedding $E_\times \circ G_\chi$ is an embedding at the powerset and fibre levels in which these axioms are formulated. The precise shape of these axioms as packaged by $E_\times \circ G_\chi$ in **4-Top** is, however, an open question. Answering this question may teach us how to use successful axioms of traditional bitopology to formulate successful axioms for fixed-basis topology.

We illustrate (6) by showing that from traditional bicompactness $E_\times$ induces the compactness of [5] for lattice-valued topology and by discussing the relationship between the respective Tihonov Theorems for the two categories **BiTop** and **4-Top**. As repeatedly shown in [36, 37, 38, 42, 34], Chang's original axiom of compactness [5] for lattice-valued topology, dubbed **localic compactness** in [38] and simply **compactness** in [19, 42], has been extraordinarily successful and justified with regard to classes of representations of $L$-spatial locales, $L$-coherent locales, distributive lattices, Boolean algebras, traditional compact Hausdorff spaces, classes of Stone-Čech compactifications, classes of Stone-Weierstraß theorems [42], etc; indeed, for $L$ a frame, only this compactness axiom (and the very closely related axiom of [20]) has an unrestricted compactification reflector for *all* of $L$-**Top**. Further, its Tihonov Theorem, namely the Goguen-Tihonov Theorem [12], is one of the few Tihonov Theorems in the fuzzy literature which does not need the classical theorem in its proof; and hence it generalizes and explains both the statement and the proof of the classical theorem. We need the statement of this theorem.

Let $L$ be any complete lattice and let $\kappa$ be a cardinal. We say $\top$ is $\kappa$-**isolated** [12] in $L$ if for each $A \subset L - \{\top\}$ with $|A| \le \kappa$, $\bigvee A < \top$.

**Theorem 4.1** (Goguen-Tihonov [12])**.** *Let $L$ be a complete lattice and $\Gamma$ be an indexing set. Then $\top$ is $|\Gamma|$-isolated in $L$ if and only if each collection $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\} \subset L$-**Top** of compact spaces (in the sense of* [5]*) yields a compact product $\prod_{\gamma \in \Gamma} (X_\gamma, \tau_\gamma)$.*

**Corollary 4.2.** *The traditional Tihonov Theorem holds: for any indexing set $\Gamma$, $\prod_{\gamma \in \Gamma} (X_\gamma, \mathcal{T}_\gamma)$ is compact if and only if each $(X_\gamma, \mathcal{T}_\gamma)$ is compact.*

*Proof.* The forward direction—the easier direction—can be given the usual proof. As for the backward direction—the harder direction, we proceed as follows. First, the backward direction transfers directly, via the functorial isomorphism $G_\chi$ : **Top** $\to$ **2-Top**, to the claim that each collection $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\} \subset$ **2-Top** of compact spaces (in the sense of [5]) yields a compact product $\prod_{\gamma \in \Gamma} (X_\gamma, \tau_\gamma)$; and this claim holds immediately from 4.1 since in the lattice **2**, $\top$ is $\kappa$-isolated in **2** for each cardinal $\kappa$, and so the claim holds for each indexing set $\Gamma$. □

A traditional **bicompact** bitopological space $(X, \mathfrak{T}, \mathfrak{S})$ is defined by saying that $X$ is compact w.r.t. each of the topologies $\mathfrak{T}, \mathfrak{S}$. Given the construction of products in **BiTop** (Subsection 1.5), we immediately have the usual Tihonov Theorem for traditional bitopology.

**Corollary 4.3.** *For any indexing set* $\Gamma$, $\prod_{\gamma\in\Gamma}(X_\gamma, \mathfrak{T}_\gamma, \mathfrak{S}_\gamma)$ *is bicompact if and only if each* $(X_\gamma, \mathfrak{T}_\gamma, \mathfrak{S}_\gamma)$ *is bicompact.*

**Corollary 4.4.** *Let* $\Gamma$ *be an indexing set. Then each collection* $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\} \subset$ **4-Top** *of compact spaces (in the sense of* [5]*) yields a compact product* $\prod_{\gamma\in\Gamma}(X_\gamma, \tau_\gamma)$ *if and only if* $|\Gamma| = 0$ *or* 1.

*Proof.* Letting **4** be written as $\{\bot, a, b, \top\}$ with $a, b$ unrelated, this is immediate from 4.1 since $\top$ is $\kappa$-isolated in **4** if and only if $\kappa \leq 1$. □

The plot thickens with the next definition, theorem, and corollary.

**Definition 4.5.** *Let* $L \in |\mathbf{USQuant}|$. *An* $L$*-bitopological space* $(X, \tau_1, \tau_2)$ *is* ($L$-)***bicompact*** *if* $X$ *is compact (in the sense of* [5]*) w.r.t. each of* $\tau_1$ *and* $\tau_2$.

**Theorem 4.6.** *For each* $L \in |\mathbf{USQuant}|$, $E_\times : L\text{-}\mathbf{BiTop} \to L^2\text{-}\mathbf{Top}$ *preserves bicompactness to compactness in the sense of* [5].

*Proof.* Let a bicompact $L$-topological space $(X, \tau_1, \tau_2)$ be given and let

$$\{u_\gamma \times v_\gamma : \gamma \in \Gamma\}$$

be a cover of $X$ from the $L^2$-topology $E_\times(\tau_1, \tau_2)$. If $\Gamma$ is finite, then this cover is its own finite subcover; so we assume $\Gamma$ is not finite. Now

$$(\underline{\top}, \underline{\top}) = \underline{(\top, \top)} = \bigvee_{\gamma\in\Gamma}(u_\gamma \times v_\gamma) = \bigvee_{\gamma\in\Gamma} u_\gamma \times \bigvee_{\gamma\in\Gamma} v_\gamma,$$

forcing each of $\{u_\gamma : \gamma \in \Gamma\}$ and $\{v_\gamma : \gamma \in \Gamma\}$ to be covers of $X$ from $\tau_1$ and $\tau_2$, respectively. The bicompactness yields two finite subcovers which we may respectively write as follows:

$$\{u_i : i = 1, ..., m\}, \quad \{v_i : i = m+1, ..., m+n\}.$$

Then $|\Gamma| \geq m + n$ and

$$\bigvee_{i=1}^{m+n}(u_i \times v_i) = \bigvee_{i=1}^{m+n} u_i \times \bigvee_{i=1}^{m+n} v_i \geq \bigvee_{i=1}^{m} u_i \times \bigvee_{i=m+1}^{m+n} v_i = \underline{\top} \times \underline{\top} = \underline{(\top, \top)},$$

showing that $\{u_i \times v_i : i = 1, ..., m+n\}$ is the needed subcover of $X$. □

**Corollary 4.7.** *The functorial embedding* $E_\times \circ G_\chi : \mathbf{BiTop} \to \mathbf{4\text{-}Top}$ *preserves bicompactness to compactness in the sense of* [5].

*Proof.* Since $G_\chi : \mathbf{BiTop} \to \mathbf{4\text{-}BiTop}$ preserves traditional bicompactness to the bicompactness of 4.5, the corollary follows from 4.6. □

We close this discussion of (6) above with a few comments. First, traditional bicompactness mandates the compactness of [5] for lattice-valued topology (4.7). Second, we note $(E_\times G_\chi)^{\rightarrow}(\mathbf{BiTop})$ is isomorphic to **BiTop** and closed under all products (in **4-Top**) (3.23, 3.29): this means that the cardinality unrestricted Tihonov Theorem for **BiTop** (4.3) transfers to a cardinality unrestricted Tihonov Theorem for the subcategory $(E_\times G_\chi)^{\rightarrow}(\mathbf{BiTop})$ of **4-Top** w.r.t. the compactness of [5]. Third, it now follows (4.4, 4.7) that $E_\times$

is not object-onto (already known) and that the special cardinality restriction of the Goguen-Tihonov Theorem for **4-Top** resides outside the subcategory $(E_\times G_\chi)^\rightarrow(\mathbf{BiTop})$.

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