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Applications of pre-open sets

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Abstract. Using the concept of pre-open set, we introduce and study topological properties of pre-limit points, pre-derived sets, pre-interior and pre-closure of a set, pre-interior points, pre-border, pre-frontier and pre-exterior. The relations between pre-derived set (resp. pre-limit point, pre-interior (point), pre-border, pre-frontier, and pre-exterior) and α-derived set (resp. α-limit point, α-interior (point), α-border, α-frontier, and α-exterior) are investigated.

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1. Introduction

The notion of α-open set was introduced by Njåstad [14]. Since then it has been widely investigated in several literatures (see [1, 3, 4, 5, 6, 7, 9, 10, 12, 15]). In [2], Caldas introduced and studied topological properties of α-derived, α-border, α-frontier, and α-exterior of a set by using the concept of α-open sets. The notion of pre-open set was introduced by Mashhour et al. [8]. In this paper, we introduce the notions of pre-limit points, pre-derived sets, pre-interior and pre-closure of a set, pre-interior points, pre-border, pre-frontier and pre-exterior by using the concept of pre-open sets, and study their topological properties. We provide relations between pre-derived set (resp. pre-limit point, pre-interior (point), pre-border, pre-frontier, and pre-exterior) and α-derived set (resp. α-limit point, α-interior (point), α-border, α-frontier, and α-exterior).

2. Preliminaries

Through this paper, $(X, \mathscr{T})$ and $(Y, \mathscr{K})$ (simply X and Y) always mean topological spaces. A subset A of X is said to be *pre-open* [11] (respectively, *α-open* [14] and *semi-open* [13]) if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$ (respectively, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ and $A \subset \mathrm{Cl}(\mathrm{Int}(A))$). The complement of a pre-open set (respectively, an α-open set and a semi-open set) is called a *pre-closed set* (respectively, an *α-closed set* and a *semi-closed set*). The intersection of all pre-closed sets (respectively, α-closed sets and semi-closed sets) containing A is called the *pre-closure* (respectively, *α-closure* and *semi-closure*) of A, denoted by $\mathrm{Cl}_p(A)$ (respectively, $\mathrm{Cl}_\alpha(A)$ and $\mathrm{Cl}_s(A)$). A subset A is also pre-closed (respectively, α-closed and semi-closed) if and only if $A = \mathrm{Cl}_p(A)$ (respectively, $A = \mathrm{Cl}_\alpha(A)$ and $A = \mathrm{Cl}_s(A)$). We denote the family of pre-open sets (respectively, α-open sets and semi-open sets) of $(X, \mathscr{T})$ by $\mathscr{T}^p$ (respectively, $\mathscr{T}^\alpha$ and $\mathscr{T}^s$). Obviously, we have the following relations.

$$\begin{tikzcd} & \text{open set (closed set)} \arrow[d] & \\ & \alpha\text{-open set (α-closed set)} \arrow[dl] \arrow[dr] & \\ \text{pre-open set (pre-closed set)} & & \text{semi-open set (semi-closed set)} \end{tikzcd}$$

None of these implications is reversible in general.

3. Pre-open sets and α-open sets

Definition 3.1 ([11, 14]). A subset A of X is said to be *pre-open* (respectively, *α-open*) if $A \subseteq \mathrm{Int}(\mathrm{Cl}A)$ (respectively, $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}A))$).

The complement of a pre-open set (respectively, an α-open set) is called a *pre-closed set* (respectively, an *α-closed set*).

The intersection of all pre-closed sets (respectively, α-closed sets) containing A is called the *pre-closure* (respectively, *α-closure*) of A, denoted by $\mathrm{Cl}_p(A)$ (respectively, $\mathrm{Cl}_\alpha(A)$).

A subset A is also pre-closed (respectively, α-closed) if and only if $A = \mathrm{Cl}_p(A)$ (respectively, $A = \mathrm{Cl}_\alpha(A)$). We denote the family of pre-open sets (respectively, α-open sets) of $(X, \mathscr{T})$ by $\mathscr{T}^p$ (respectively, $\mathscr{T}^\alpha$).

Example 3.2. Let $\mathscr{T} = \{\varnothing, X, \{a\}, \{c, d\}, \{a, c, d\}\}$ be a topology on $X = \{a, b, c, d, e\}$. Then we have

$$\mathscr{T}^\alpha = \mathscr{T} \cup \{\{a, b, c, d\}, \{a, c, d, e\}\},$$

$$\begin{aligned} \mathscr{T}^p = \mathscr{T} \cup \{&\{c\}, \{d\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, e\}, \\ &\{a, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}\}. \end{aligned}$$

4. Applications of pre-open sets

Definition 4.1. Let A be a subset of a topological space $(X, \mathscr{T})$. A point $x \in X$ is said to be *pre-limit point* (resp. *α-limit point*) of A if it satisfies the following assertion:

$$(\forall G \in \mathscr{T}^p(\text{ resp. } \mathscr{T}^\alpha))\,(x \in G \Rightarrow G \cap (A \setminus \{x\}) \neq \varnothing).$$

The set of all pre-limit points (resp. α-limit points) of A is called the *pre-derived set* (resp. *α-derived set*) of A and is denoted by $D_p(A)$ (resp. $D_\alpha(A)$). Denote by $D(A)$ the derived set of A.

Note that for a subset A of X, a point $x \in X$ is not a pre-limit point of A if and only if there exists a pre-open set G in X such that

$$x \in G \quad \text{and} \quad G \cap (A \setminus \{x\}) = \varnothing$$

or, equivalently,

$$x \in G \quad \text{and} \quad G \cap A = \varnothing \quad \text{or} \quad G \cap A = \{x\}$$

or, equivalently,

$$x \in G \quad \text{and} \quad G \cap A \subseteq \{x\}.$$

Example 4.2. Let $X = \{a, b, c\}$ with topology $\mathscr{T} = \{X, \varnothing, \{a\}\}$. Then we have the followings:

(i) $\mathscr{T}^p = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}\} = \mathscr{T}^\alpha$.

(ii) If $A = \{c\}$, then $D(A) = \{b\}$ and $D_\alpha(A) = D_p(A) = \varnothing$.

(iii) If $B = \{a\}$ and $C = \{b, c\}$, then $D_p(B) = \{b, c\}$, $D_p(C) = \varnothing$ and $D_p(B \cup C) = \{b, c\}$.

Theorem 4.3. *If a topology $\mathscr{T}$ on a set X contains only $\varnothing$, X, and $\{a\}$ for a fixed $a \in X$, then $\mathscr{T}^p = \mathscr{T}^\alpha$.*

Proof. Let $a \in X$ and let A be an element of $\mathscr{T}^p$. Then $a \in A$. In fact, if not then $A \nsubseteq \mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Int}(\{a\}^c) = \varnothing$. Hence $A \notin \mathscr{T}^p$, a contradiction. Now since $\mathrm{Int}(A) = \{a\}$, we have

$$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \mathrm{Int}(\mathrm{Cl}(\{a\})) = \mathrm{Int}(X) = X$$

which contains A, that is, $A \in \mathscr{T}^\alpha$. Note that $\mathscr{T}^\alpha \subseteq \mathscr{T}^p$. Thus $\mathscr{T}^\alpha = \mathscr{T}^p$. $\square$

Example 4.4. Let $X = \{a, b, c, d, e\}$ with topology

$$\mathscr{T} = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}.$$

Then

$$\begin{aligned} \mathscr{T}^p = \; & \{X, \varnothing, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \\ & \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}, \\ & \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \\ & \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\} \end{aligned}$$

and

$$\begin{aligned}\mathscr{T}^{\alpha} &= \{X, \varnothing, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \{c,d,e\},\\ &\quad \{a,b,c,d\}, \{a,c,d,e\}, \{b,c,d,e\}\}.\end{aligned}$$

Consider subsets $A = \{a, b, c\}$ and $B = \{b, d\}$ of X. Then

$$\begin{array}{ll} D(A) = \{b,d,e\}, & D_p(A) = \varnothing, \\ \mathrm{Int}(A) = \{a\}, & \mathrm{Int}_p(A) = A, \\ \mathrm{Int}_\alpha(A) = \{a\}, & \mathrm{Cl}_p(A) = A, \\ \mathrm{Cl}_\alpha(A) = X, & \mathrm{Cl}_p(B) = B, \\ \mathrm{Cl}_\alpha(B) = \{b,c,d,e\}, & \mathrm{Int}(B) = \varnothing, \\ \mathrm{Int}_p(B) = B, & \mathrm{Int}_\alpha(B) = \varnothing. \end{array}$$

Example 4.5. Consider a topology

$$\mathscr{T} = \{X, \varnothing, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}\}$$

on $X = \{a, b, c, d, e\}$. Then

$$\begin{aligned}\mathscr{T}^{p} &= \{X, \varnothing, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,b,c\},\\ &\quad \{a,b,d\}, \{a,b,e\}, \{a,c,d\}, \{a,c,e\}, \{a,d,e\}\\ &\quad \{a,b,c,d\}, \{a,b,c,e\}, \{a,b,d,e\}, \{a,c,d,e\}\}\\ &= \mathscr{T}^{\alpha}.\end{aligned}$$

For subsets $A = \{c, d, e\}$ and $B = \{b\}$ of X, we have

$$\begin{array}{ll} D(A) = \{c,d\} & D(B) = \{e\}. \\ D_p(A) = \varnothing & D_p(B) = \varnothing. \\ D_\alpha(A) = \varnothing & D_\alpha(B) = \varnothing. \\ \mathrm{Int}(A) = \varnothing & \mathrm{Int}(B) = \varnothing, \\ \mathrm{Int}_p(A) = \varnothing, & \mathrm{Int}_p(B) = \varnothing, \\ \mathrm{Int}_\alpha(A) = \varnothing, & \mathrm{Int}_\alpha(B) = \varnothing, \\ \mathrm{Cl}_p(A) = \{c,d,e\}, & \mathrm{Cl}_p(B) = \{b\}, \\ \mathrm{Cl}_\alpha(A) = \{c,d,e\}, & \mathrm{Cl}_\alpha(B) = \{b\}, \\ \mathrm{Cl}_p(\{b,d\}) = \{b,d\}, & \mathrm{Cl}_\alpha(\{b,d\}) = \{b,d\}, \\ \mathrm{Int}(\{b,d\}) = \varnothing, & \mathrm{Int}_p(\{b,d\}) = \varnothing, \\ \mathrm{Int}_\alpha(\{b,d\}) = \varnothing. & \end{array}$$

Lemma 4.6. *If there exists $a \in X$ such that $\{a\}$ is the smallest element of $(\mathscr{T} \setminus \{\varnothing\}, \subseteq)$, then every non-empty pre-open set contains $\bigcap\{G_i \mid G_i \in \mathscr{T} \setminus \{\varnothing\}; i = 1, 2, 3, \cdots\}$.*

Proof. If $\{a\}$ is the smallest element of $(\mathscr{T} \setminus \{\varnothing\}, \subseteq)$, then

$$\bigcap\{G_i \mid G_i \in \mathscr{T} \setminus \{\varnothing\}; i = 1, 2, 3, \cdots\} = \{a\}.$$

Let A be a non-empty pre-open set in X. If $a \notin A$, then $\mathrm{Cl}(A) \subseteq \{a\}$ and so

$$A \nsubseteq \mathrm{Int}(\mathrm{Cl}(A)) \subseteq \mathrm{Int}(\{a\}^c) = \varnothing$$

which is a contradiction. Hence $a \in A$, and so the desired result is valid. $\square$

Theorem 4.7. *Let $\mathscr{T}$ be a topology on a set X. If there exists $a \in X$ such that $\{a\}$ is the smallest element of $(\mathscr{T} \setminus \{\varnothing\}, \subseteq)$, then $\mathscr{T}^{\alpha} = \mathscr{T}^{p}$.*

Proof. It is sufficient to show that $\mathscr{T}^p \subseteq \mathscr{T}^{\alpha}$. Let $A \in \mathscr{T}^p$. If $A = \varnothing$, then clearly $A \in \mathscr{T}^{\alpha}$. Assume that $A \neq \varnothing$. Then $a \in A$ by Lemma 4.6. Since $\{a\} \subseteq \mathrm{Int}(A)$, it follows that $X = \mathrm{Cl}(\{a\}) \subseteq \mathrm{Cl}(\mathrm{Int}(A))$ so that

$$A \subseteq X = \mathrm{Int}(X) \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))).$$

Hence A is an α-open set. □

Theorem 4.8. *Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be topologies on X such that $\mathscr{T}_1^p \subseteq \mathscr{T}_2^p$. For any subset A of X, every pre-limit point of A with respect to $\mathscr{T}_2$ is a pre-limit point of A with respect to $\mathscr{T}_1$.*

Proof. Let x be a pre-limit point of A with respect to $\mathscr{T}_2$. Then $(G \cap A) \setminus \{x\} \neq \varnothing$ for every $G \in \mathscr{T}_2^p$ such that $x \in G$. But $\mathscr{T}_1^p \subseteq \mathscr{T}_2^p$, so, in particular, $(G \cap A) \setminus \{x\} \neq \varnothing$ for every $G \in \mathscr{T}_1^p$ such that $x \in G$. Hence x is a pre-limit point of A with respect to $\mathscr{T}_1$. □

The converse of Theorem 4.8 is not true in general as seen in the following example.

Example 4.9. Consider topologies $\mathscr{T}_1 = \{X, \varnothing, \{a\}\}$ and

$$\mathscr{T}_2 = \{X, \varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}$$

on a set $X = \{a, b, c, d\}$. Then

$$\mathscr{T}_1^p = \mathscr{T}_1 \cup \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$$

and

$$\mathscr{T}_2^p = \mathscr{T}_2 \cup \{\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}.$$

Note that $\mathscr{T}_1^p \subseteq \mathscr{T}_2^p$ and c is a pre-limit point of $A = \{a, b\}$ with respect to $\mathscr{T}_1$, but it is not a pre-limit point of A with respect to $\mathscr{T}_2$.

Lemma 4.10. *If $\{A_i \mid i \in \Lambda\}$ is a family of pre-open sets in X, then $\bigcup_{i \in \Lambda} A_i$ is a pre-open set in X where Λ is any index set.*

Proof. Straightforward. □

In Example 3.2, we see that

$$\{a, b, c, e\} \cap \{a, b, d, e\} = \{a, b, e\} \notin \mathscr{T}^p,$$

which shows that the intersection of two pre-open sets is not pre-open in general. Thus we know that for any topology $\mathscr{T}$ on a set X, $\mathscr{T}^p$ may not be a topology on X.

Proposition 4.11. *If $\mathscr{I}$* (resp. *$\mathscr{D}$*) *is the indiscrete* (resp. *discrete*) *topology on a set X, then $\mathscr{I}^p$* (resp. *$\mathscr{D}^p$*) *is a topology on X.*

Proof. Straightforward. □

Theorem 4.12. *For any subsets A and B of $(X, \mathscr{T})$, the following assertions are valid:*

(1) $D_p(A) \subseteq D_\alpha(A)$.
(2) *If* $A \subseteq B$, *then* $D_p(A) \subseteq D_p(B)$.
(3) $D_p(A) \cup D_p(B) \subseteq D_p(A \cup B)$ *and* $D_p(A \cap B) \subseteq D_p(A) \cap D_p(B)$.
(4) $D_p(D_p(A)) \setminus A \subseteq D_p(A)$.
(5) $D_p(A \cup D_p(A)) \subseteq A \cup D_p(A)$.

Proof. (1) It suffices to observe that every α-open set is pre-open.

(2) Let $x \in D_p(A)$ and let $G \in \mathscr{T}^p$ with $x \in G$. Then $(G \cap A) \setminus \{x\} \neq \varnothing$. Since $A \subseteq B$, it follows that $(G \cap B) \setminus \{x\} \neq \varnothing$ so that $x \in D_p(B)$.

(3) Straightforward by (2).

(4) Let $x \in D_p(D_p(A)) \setminus A$ and let $G \in \mathscr{T}^p$ with $x \in G$. Then $G \cap (D_p(A) \setminus \{x\}) \neq \varnothing$. Let $y \in G \cap (D_p(A) \setminus \{x\})$. Then $y \in G$ and $y \in D_p(A)$, and so $G \cap (A \setminus \{y\}) \neq \varnothing$. If we take $z \in G \cap (A \setminus \{y\})$, then $x \neq z$ because $x \notin A$. Hence $(G \cap A) \setminus \{x\} \neq \varnothing$. Therefore $x \in D_p(A)$.

(5) Let $x \in D_p(A \cup D_p(A))$. If $x \in A$, the result is obvious. Assume that $x \notin A$. Then $G \cap ((A \cup D_p(A)) \setminus \{x\}) \neq \varnothing$ for all $G \in \mathscr{T}^p$ with $x \in G$. Hence $(G \cap A) \setminus \{x\} \neq \varnothing$ or $G \cap (D_p(A) \setminus \{x\}) \neq \varnothing$. The first case implies $x \in D_p(A)$. If $G \cap (D_p(A) \setminus \{x\}) \neq \varnothing$, then $x \in D_p(D_p(A))$. Since $x \notin A$, it follows similarly from (4) that $x \in D_p(D_p(A)) \setminus A \subseteq D_p(A)$. Therefore (5) is valid. $\square$

In general, in Theorem 4.12, the reverse inclusion of (1), (4) and (5), and the converse of (2) may not be true, and the equality in (3) does not hold as seen in the following example.

Example 4.13. (1) Consider the topology $\mathscr{T}$ on $X = \{a, b, c, d, e\}$ described in Example 3.2. For a subset $A = \{b, c, d\}$ of X, we have $D_\alpha(A) = \{b, c, d, e\}$ and $D_p(A) = \{b, e\}$. This shows that the reverse inclusion of Theorem 4.12(1) is not true. Now let $X = \{a, b, c, d\}$ with a topology

$$\mathscr{T} = \{X, \varnothing, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}\}.$$

Then $\mathscr{T}^p = \mathscr{T}$. For two subsets $A = \{a, c\}$ and $B = \{a, b, d\}$ of X, we get

$$D_p(A) = \{b\} \subseteq \{b, c\} = D_p(B),$$

but $A \nsubseteq B$. This shows that the converse of Theorem 4.12(2) is not valid. Now consider two subsets $A = \{a, b\}$ and $B = \{b, c, d\}$ of X in Example 3.2. Then $D_p(A) = \{b, e\} = D_p(B)$, and so $D_p(A \cap B) = \varnothing \subseteq D_p(A) \cap D_p(B)$. Thus the equality in Theorem 4.12(3) is not valid.

(2) Consider a topology $\mathscr{T} = \{X, \varnothing, \{b, c\}, \{b, c, d\}, \{a, b, c\}\}$ on $X = \{a, b, c, d\}$. Then

$$\begin{aligned} \mathscr{T}^p = \; & \{X, \varnothing, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \\ & \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}. \end{aligned}$$

Let $A = \{a, b\}$ and $B = \{a, c\}$ be subsets of X. Then $D_p(A) = \varnothing = D_p(B)$, and so $D_p(A) \cup D_p(B) = \varnothing \subset \{a, d\} = D_p(A \cup B)$. For a subset $A = \{a, b, c\}$ of X, we have $D_p(D_p(A)) = D_p(\{a, d\}) = \varnothing$, $D_p(D_p(A)) \setminus A = \varnothing \subseteq D_p(A) = \{a, d\}$, and so the equality in Theorem 4.12(4) is not valid. Now for a subset $B = \{b, c\}$ of X, we get $D_p(B) = \{a, d\}$, and so $B \cup D_p(B) = X$ and $D_p(X) = \{a, d\} \subseteq X$. This shows that $D_p(B \cup D_p(B)) \neq B \cup D_p(B) = X$. Hence the equality in Theorem 4.12(5) is not valid.

Theorem 4.14. *Let A be a subset of X and $x \in X$. Then the following are equivalent:*

(i) $(\forall G \in \mathscr{T}^p)\ (x \in G \Rightarrow A \cap G \neq \varnothing)$.

(ii) $x \in \mathrm{Cl}_p(A)$.

Proof. (i) $\Rightarrow$ (ii) If $x \notin \mathrm{Cl}_p(A)$, then there exists a pre-closed set F such that $A \subseteq F$ and $x \notin F$. Hence $X \setminus F$ is a pre-open set containing x and $A \cap (X \setminus F) \subseteq A \cap (X \setminus A) = \varnothing$. This is a contradiction, and hence (ii) is valid.

(ii) $\Rightarrow$ (i) Straightforward. □

Corollary 4.15. *For any subset A of X, we have $D_p(A) \subseteq \mathrm{Cl}_p(A)$.*

Proof. Straightforward. □

Theorem 4.16. *For any subset A of X, $\mathrm{Cl}_p(A) = A \cup D_p(A)$.*

Proof. Let $x \in \mathrm{Cl}_p(A)$. Assume that $x \notin A$ and let $G \in \mathscr{T}^p$ with $x \in G$. Then $(G \cap A) \setminus \{x\} \neq \varnothing$, and so $x \in D_p(A)$. Hence $\mathrm{Cl}_p(A) \subseteq A \cup D_p(A)$. The reverse inclusion is by $A \subseteq \mathrm{Cl}_p(A)$ and Corollary 4.15. □

Theorem 4.17. *Let A and B be subsets of X. If $A \in \mathscr{T}^p$ and $\mathscr{T}^p$ is a topology on X, then $A \cap \mathrm{Cl}_p(B) \subseteq \mathrm{Cl}_p(A \cap B)$.*

Proof. Let $x \in A \cap \mathrm{Cl}_p(B)$. Then $x \in A$ and $x \in \mathrm{Cl}_p(B) = B \cup D_p(B)$. If $x \in B$, then $x \in A \cap B \subseteq \mathrm{Cl}_p(A \cap B)$. If $x \notin B$, then $x \in D_p(B)$ and so $G \cap B \neq \varnothing$ for all pre-open set G containing x. Since $A \in \mathscr{T}^p$, $G \cap A$ is also a pre-open set containing x. Hence $G \cap (A \cap B) = (G \cap A) \cap B \neq \varnothing$, and consequently $x \in D_p(A \cap B) \subseteq \mathrm{Cl}_p(A \cap B)$. Therefore $A \cap \mathrm{Cl}_p(B) \subseteq \mathrm{Cl}_p(A \cap B)$. □

Example 4.18. Let $\mathscr{T} = \{X, \varnothing, \{b\}, \{b, c\}, \{b, c, d\}\}$ be a topology on a set $X = \{a, b, c, d\}$. Then

$$\mathscr{T}^p = \{X, \varnothing, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$$

which is a topology on X. Let $A = \{a, b\}$ and $B = \{b, c\}$ be subsets of X. Then $A \cap \mathrm{Cl}_p(B) = \{a, b\} \neq X = \mathrm{Cl}_p(A \cap B)$. This shows that the equality in Theorem 4.17 is not true in general.

Example 4.19. Consider $\mathscr{T}$ and $\mathscr{T}^p$ which are given in Example 4.13(2). Note that $\mathscr{T}^p$ is not a topology on X. For subsets $A = \{a, b\}$ and $B = \{b, c\}$ of X, we have $A \cap \mathrm{Cl}_p(B) = \{a, b\} \nsubseteq \{b\} = \mathrm{Cl}_p(A \cap B)$. This shows that if $\mathscr{T}^p$ is not a topology on X then the result in Theorem 4.17 is not true in general.

Theorem 4.20. *Let A and B subsets of X. If A is pre-closed, then*

$$\mathrm{Cl}_p(A \cap B) \subseteq A \cap \mathrm{Cl}_p(B).$$

Proof. If A is pre-closed, then $\mathrm{Cl}_p(A) = A$ and so

$$\mathrm{Cl}_p(A \cap B) \subseteq \mathrm{Cl}_p(A) \cap \mathrm{Cl}_p(B) = A \cap \mathrm{Cl}_p(B)$$

which is the desired result. □

Lemma 4.21. *A subset A of X is pre-open if and only if there exists an open set H in X such that $A \subseteq H \subseteq \mathrm{Cl}(A)$.*

Proof. Straightforward. □

Lemma 4.22. *The intersection of an open set and a pre-open set is a pre-open set.*

Proof. Let A be an open set in X and B a pre-open set in X. Then there exists an open set G in X such that $B \subseteq G \subseteq \mathrm{Cl}(B)$. It follows that

$$A \cap B \subseteq A \cap G \subseteq A \cap \mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \cap B).$$

Now since $A \cap G$ is open, it follows from Lemma 4.21 that $A \cap B$ is pre-open. □

Theorem 4.23. *Let A and B be subsets of X. If A is open, then*

$$A \cap \mathrm{Cl}_p(B) \subseteq \mathrm{Cl}_p(A \cap B).$$

Proof. It is by Theorem 4.17 and Lemma 4.22. □

Theorem 4.24. *If A is a subset of a discrete topological space X, then $D_p(A) = \varnothing$.*

Proof. Let x be any element of X. Recall that every subset of X is open, and so pre-open. In particular, the singleton set $G := \{x\}$ is pre-open. But $x \in G$ and $G \cap A = \{x\} \cap A \subseteq \{x\}$. Hence x is not a pre-limit point of A, and so $D_p(A) = \varnothing$. □

Theorem 4.25. *For every subset A of X, we have*

$$A \text{ is pre-closed if and only if } D_p(A) \subseteq A.$$

Proof. Assume that A is pre-closed. Let $x \notin A$, i.e., $x \in X \setminus A$. Since $X \setminus A$ is pre-open, x is not a pre-limit point of A, i.e., $x \notin D_p(A)$, because $(X \setminus A) \cap (A \setminus \{x\}) = \varnothing$. Hence $D_p(A) \subseteq A$. The reverse implication is by Theorem 4.16. □

Theorem 4.26. *Let A be a subset of X. If F is a pre-closed superset of A, then $D_p(A) \subseteq F$.*

Proof. By Theorem 4.12(2) and Theorem 4.25, $A \subseteq F$ implies $D_p(A) \subseteq D_p(F) \subseteq F$. □

Theorem 4.27. *Let A be a subset of X. If a point $x \in X$ is a pre-limit point of A, then x is also a pre-limit point of $A \setminus \{x\}$.*

Proof. Straightforward. □

Definition 4.28 ([2])**.** Let A be a subset of a topological space X. A point $x \in X$ is called an *α-interior point* of A if there exists an α-open set G containing x such that $G \subseteq A$. The set of all α-interior points of A is called the *α-interior* of A and is denoted by $\mathrm{Int}_\alpha(A)$.

Based on the above definition, we give the notion of a pre-interior point.

Definition 4.29. Let A be a subset of a topological space X. A point $x \in X$ is called a *pre-interior point* of A if there exists a pre-open set G such that $x \in G \subseteq A$. The set of all pre-interior points of A is called the *pre-interior* of A and is denoted by $\mathrm{Int}_p(A)$.

Example 4.30. Let $(X, \mathscr{T})$ be a topological space which is given in Example 4.4. We know that a is the only pre-interior point of $A = \{a, b, e\}$, i.e., $\mathrm{Int}_p(A) = \{a\}$.

Theorem 4.31. *Let A be a subset of X. Then every α-interior point of A is a pre-interior point of A, i.e.,* $\mathrm{Int}_\alpha(A) \subseteq \mathrm{Int}_p(A)$.

Proof. If x is an α-interior point of A, then there exists an α-open set G containing x such that $G \subseteq A$. Since every α-open set is pre-open, it follows that x is a pre-interior point of A. □

The following example shows that there exists a pre-interior point of A which is not an α-interior point of A.

Example 4.32. In Example 4.4, $\mathrm{Int}_\alpha(A) = \{a\}$ and $\mathrm{Int}_p(A) = \{a, b, c\}$. Hence b and c are pre-interior points of A. But they are not α-interior points of A.

Proposition 4.33. *For subsets A and B of X, the following assertions are valid.*

(1) $\mathrm{Int}_p(A)$ *is the union of all pre-open subsets of* A;
(2) *A is pre-open if and only if* $A = \mathrm{Int}_p(A)$;
(3) $\mathrm{Int}_p(\mathrm{Int}_p(A)) = \mathrm{Int}_p(A)$;
(4) $\mathrm{Int}_p(A) = A \setminus D_p(X \setminus A)$.
(5) $X \setminus \mathrm{Int}_p(A) = \mathrm{Cl}_p(X \setminus A)$.
(6) $X \setminus \mathrm{Cl}_p(A) = \mathrm{Int}_p(X \setminus A)$.
(7) $A \subseteq B \Rightarrow \mathrm{Int}_p(A) \subseteq \mathrm{Int}_p(B)$.
(8) $\mathrm{Int}_p(A) \cup \mathrm{Int}_p(B) \subseteq \mathrm{Int}_p(A \cup B)$.
(9) $\mathrm{Int}_p(A \cap B) \subseteq \mathrm{Int}_p(A) \cap \mathrm{Int}_p(B)$.

Proof. (1) Let $\{G_i \mid i \in \Lambda\}$ be a collection of all pre-open subsets of A. If $x \in \mathrm{Int}_p(A)$, then there exists $j \in \Lambda$ such that $x \in G_j \subseteq A$. Hence $x \in \bigcup_{i\in\Lambda} G_i$, and so $\mathrm{Int}_p(A) \subseteq \bigcup_{i\in\Lambda} G_i$. On the other hand, if $y \in \bigcup_{i\in\Lambda} G_i$, then $y \in G_k \subseteq A$ for some $k \in \Lambda$. Thus $y \in \mathrm{Int}_p(A)$, and $\bigcup_{i\in\Lambda} G_i \subseteq \mathrm{Int}_p(A)$. Accordingly, $\mathrm{Int}_p(A) = \bigcup_{i\in\Lambda} G_i$.

(2) Straightforward.

(3) It follows from (1) and (2).

(4) If $x \in A \setminus D_p(X \setminus A)$, then $x \notin D_p(X \setminus A)$ and so there exists a pre-open set G containing x such that $G \cap (X \setminus A) = \varnothing$. Thus $x \in G \subseteq A$ and hence $x \in \mathrm{Int}_p(A)$. This shows that $A \setminus D_p(X \setminus A) \subseteq \mathrm{Int}_p(A)$. Now let $x \in \mathrm{Int}_p(A)$. Since $\mathrm{Int}_p(A) \in \mathscr{T}^p$ and $\mathrm{Int}_p(A) \cap (X \setminus A) = \varnothing$, we have $x \notin D_p(X \setminus A)$. Therefore $\mathrm{Int}_p(A) = A \setminus D_p(X \setminus A)$.

(5) Using (4) and Theorem 4.16, we have

$$X \setminus \mathrm{Int}_p(A) = X \setminus (A \setminus D_p(X \setminus A)) = (X \setminus A) \cup D_p(X \setminus A) = \mathrm{Cl}_p(X \setminus A).$$

(6) Using (4) and Theorem 4.16, we get

$$\mathrm{Int}_p(X \setminus A) = (X \setminus A) \setminus D_p(A) = X \setminus (A \cup D_p(A)) = X \setminus \mathrm{Cl}_p(A).$$

(7) Straightforward.

(8) and (9) They are by (7). □

The converse of (7) in Proposition 4.33 is not true in general as seen in the following example.

Example 4.34. Consider a topological space $(X, \mathscr{T})$ which is described in Example 4.4. Let $A = \{a, b\}$ and $B = \{a, c, d\}$ be subsets of X. Then $\mathrm{Int}_p(A) = \{a\} \subseteq \mathrm{Int}_p(B) = \{a, c, d\}$.

Definition 4.35 ([2])**.** For any subset A of X, the set

$$b_\alpha(A) := A \setminus \mathrm{Int}_\alpha(A)$$

is called the *α-border* of A, and the set

$$\mathrm{Fr}_\alpha(A) := \mathrm{Cl}_\alpha(A) \setminus \mathrm{Int}_\alpha(A)$$

is called the *α-frontier* of A.

Definition 4.36. For any subset A of X, the set

$$b_p(A) := A \setminus \mathrm{Int}_p(A)$$

is called the *pre-border* of A, and the set

$$\mathrm{Fr}_p(A) := \mathrm{Cl}_p(A) \setminus \mathrm{Int}_p(A)$$

is called the *pre-frontier* of A.

Note that if A is a pre-closed subset of X, then $b_p(A) = \mathrm{Fr}_p(A)$.

Example 4.37. (1) Let $(X, \mathscr{T})$ be the topological space which is described in Example 4.4. Let $A = \{a, b, e\}$ be a subset of X. Then $\mathrm{Int}_p(A) = \{a\}$, and so $b_p(A) = \{b, e\}$. Since $A = \{a, b, e\}$ is pre-closed, $\mathrm{Cl}_p(A) = \{a, b, e\}$ and thus $\mathrm{Fr}_p(A) = \{b, e\}$.

(2) Consider the topological space $(X, \mathscr{T})$ which is given in Example 3.2. For a subset $A = \{b, c, d\}$ of X, we have $\mathrm{Int}_p(A) = \{c, d\}$ and $\mathrm{Cl}_p(A) = \{b, c, d, e\}$. Hence $b_p(A) = \{b\}$ and $\mathrm{Fr}_p(A) = \{b, e\}$.

Proposition 4.38. *For a subset A of X, the following statements hold:*

(1) $b_p(A) \subseteq b_\alpha(A)$.
(2) $A = \mathrm{Int}_p(A) \cup b_p(A)$.
(3) $\mathrm{Int}_p(A) \cap b_p(A) = \varnothing$.
(4) *A is a pre-open set if and only if* $b_p(A) = \varnothing$.
(5) $b_p(\mathrm{Int}_p(A)) = \varnothing$.
(6) $\mathrm{Int}_p(b_p(A)) = \varnothing$.
(7) $b_p(b_p(A)) = b_p(A)$.
(8) $b_p(A) = A \cap \mathrm{Cl}_p(X \setminus A)$.
(9) $b_p(A) = A \cap D_p(X \setminus A)$.

Proof. (1) Since $\mathrm{Int}_\alpha(A) \subseteq \mathrm{Int}_p(A)$, we have

$$b_p(A) = A \setminus \mathrm{Int}_p(A) \subseteq A \setminus \mathrm{Int}_\alpha(A) = b_\alpha(A).$$

(2) and (3). Straightforward.

(4) Since $\mathrm{Int}_p(A) \subseteq A$, it follows from Proposition 4.33(2) that

$$A \text{ is pre-open} \quad \Leftrightarrow \quad A = \mathrm{Int}_p(A) \Leftrightarrow b_p(A) = A \setminus \mathrm{Int}_p(A) = \varnothing.$$

(5) Since $\mathrm{Int}_p(A)$ is pre-open, it follows from (4) that $b_p(\mathrm{Int}_p(A)) = \varnothing$.

(6) If $x \in \mathrm{Int}_p(b_p(A))$, then $x \in b_p(A) \subseteq A$ and $x \in \mathrm{Int}_p(A)$ since $\mathrm{Int}_p(b_p(A)) \subseteq \mathrm{Int}_p(A)$. Thus $x \in b_p(A) \cap \mathrm{Int}_p(A) = \varnothing$, which is a contradiction. Hence $\mathrm{Int}_p(b_p(A)) = \varnothing$.

(7) Using (6), we get

$$b_p(b_p(A)) = b_p(A) \setminus \mathrm{Int}_p(b_p(A)) = b_p(A).$$

(8) Using Proposition 4.33(6), we have

$$b_p(A) = A \setminus \mathrm{Int}_p(A) = A \setminus (X \setminus \mathrm{Cl}_p(X \setminus A)) = A \cap \mathrm{Cl}_p(X \setminus A).$$

(9) Applying (8) and Theorem 4.16, we have

$$b_p(A) = A \cap \mathrm{Cl}_p(X \setminus A) = A \cap ((X \setminus A) \cup D_p(X \setminus A)) = A \cap D_p(X \setminus A).$$

This completes the proof. □

Lemma 4.39. *For a subset A of X,*

$$A \textit{ is pre-closed if and only if } \mathrm{Fr}_p(A) \subseteq A.$$

Proof. Assume that A is pre-closed. Then

$$\mathrm{Fr}_p(A) = \mathrm{Cl}_p(A) \setminus \mathrm{Int}_p(A) = A \setminus \mathrm{Int}_p(A) \subseteq A.$$

Conversely suppose that $\mathrm{Fr}_p(A) \subseteq A$. Then $\mathrm{Cl}_p(A) \setminus \mathrm{Int}_p(A) \subseteq A$, and so $\mathrm{Cl}_p(A) \subseteq A$ since $\mathrm{Int}_p(A) \subseteq A$. Noticing that $A \subseteq \mathrm{Cl}_p(A)$, we have $A = \mathrm{Cl}_p(A)$. Therefore A is pre-closed. □

Theorem 4.40. *For a subset A of X, the following assertions are valid:*

(1) $\mathrm{Fr}_p(A) \subseteq \mathrm{Fr}_\alpha(A)$.
(2) $\mathrm{Cl}_p(A) = \mathrm{Int}_p(A) \cup \mathrm{Fr}_p(A)$.
(3) $\mathrm{Int}_p(A) \cap \mathrm{Fr}_p(A) = \varnothing$.
(4) $b_p(A) \subseteq \mathrm{Fr}_p(A)$.

(5) $\mathrm{Fr}_p(A) = b_p(A) \cup (D_p(A) \setminus \mathrm{Int}_p(A))$.
(6) *A is a pre-open set if and only if* $\mathrm{Fr}_p(A) = b_p(X \setminus A)$.
(7) $\mathrm{Fr}_p(A) = \mathrm{Cl}_p(A) \cap \mathrm{Cl}_p(X \setminus A)$.
(8) $\mathrm{Fr}_p(A) = \mathrm{Fr}_p(X \setminus A)$.
(9) $\mathrm{Fr}_p(A)$ *is pre-closed.*
(10) $\mathrm{Fr}_p(\mathrm{Fr}_p(A)) \subseteq \mathrm{Fr}_p(A)$.
(11) $\mathrm{Fr}_p(\mathrm{Int}_p(A)) \subseteq \mathrm{Fr}_p(A)$.
(12) $\mathrm{Fr}_p(\mathrm{Cl}_p(A)) \subseteq \mathrm{Fr}_p(A)$.
(13) $\mathrm{Int}_p(A) = A \setminus \mathrm{Fr}_p(A)$.

Proof. (1) Since $\mathrm{Cl}_p(A) \subseteq \mathrm{Cl}_\alpha(A)$ and $\mathrm{Int}_\alpha(A) \subseteq \mathrm{Int}_p(A)$, it follows that

$$\mathrm{Fr}_p(A) = \mathrm{Cl}_p(A) \setminus \mathrm{Int}_p(A) \subseteq \mathrm{Cl}_\alpha(A) \setminus \mathrm{Int}_p(A) \subseteq \mathrm{Cl}_\alpha(A) \setminus \mathrm{Int}_\alpha(A) = \mathrm{Fr}_\alpha(A).$$

(2) Straightforward.
(3) $\mathrm{Int}_p(A) \cap \mathrm{Fr}_p(A) = \mathrm{Int}_p(A) \cap (\mathrm{Cl}_p(A) \setminus \mathrm{Int}_p(A)) = \varnothing$.
(4) Since $A \subseteq \mathrm{Cl}_p(A)$, we have

$$b_p(A) = A \setminus \mathrm{Int}_p(A) \subseteq \mathrm{Cl}_p(A) \setminus \mathrm{Int}_p(A) = \mathrm{Fr}_p(A).$$

(5) Using Theorem 4.16, we obtain

$$\begin{aligned} \mathrm{Fr}_p(A) &= \mathrm{Cl}_p(A) \setminus \mathrm{Int}_p(A) \\ &= (A \cup D_p(A)) \cap (X \setminus \mathrm{Int}_p(A)) \\ &= (A \setminus \mathrm{Int}_p(A)) \cup (D_p(A) \setminus \mathrm{Int}_p(A)) \\ &= b_p(A) \cup (D_p(A) \setminus \mathrm{Int}_p(A)). \end{aligned}$$

(6) Assume that A is pre-open. Then

$$\begin{aligned} \mathrm{Fr}_p(A) &= b_p(A) \cup (D_p(A) \setminus \mathrm{Int}_p(A)) \\ &= \varnothing \cup (D_p(A) \setminus A) \\ &= D_p(A) \setminus A \\ &= b_p(X \setminus A) \end{aligned}$$

by using (5), Proposition 4.38(4), Proposition 4.33(2) and Proposition 4.38(9). Conversely suppose that $\mathrm{Fr}_p(A) = b_p(X \setminus A)$. Then

$$\begin{aligned} \varnothing &= \mathrm{Fr}_p(A) \setminus b_p(X \setminus A) \\ &= (\mathrm{Cl}_p(A) \setminus \mathrm{Int}_p(A)) \setminus ((X \setminus A) \setminus \mathrm{Int}_p(X \setminus A)) \\ &= A \setminus \mathrm{Int}_p(A) \end{aligned}$$

by (4) and (5) of Proposition 4.33, and so $A \subseteq \mathrm{Int}_p(A)$. Since $\mathrm{Int}_p(A) \subseteq A$ in general, it follows that $\mathrm{Int}_p(A) = A$ so from Proposition 4.33(2) that A is pre-open.

(7) Using Proposition 4.33(5), we have

$$\mathrm{Cl}_p(A) \cap \mathrm{Cl}_p(X \setminus A) = \mathrm{Cl}_p(A) \cap (X \setminus \mathrm{Int}_p(A)) = \mathrm{Cl}_p(A) \setminus \mathrm{Int}_p(A) = \mathrm{Fr}_p(A).$$

(8) It follows from (7).

(9) we have

$$\begin{aligned}\mathrm{Cl}_p(\mathrm{Fr}_p(A)) &= \mathrm{Cl}_p(\mathrm{Cl}_p(A)\cap \mathrm{Cl}_p(X\setminus A))\\ &\subseteq \mathrm{Cl}_p(\mathrm{Cl}_p(A))\cap \mathrm{Cl}_p(\mathrm{Cl}_p(X\setminus A))\\ &= \mathrm{Cl}_p(A)\cap \mathrm{Cl}_p(X\setminus A)\\ &= \mathrm{Fr}_p(A).\end{aligned}$$

Obviously $\mathrm{Fr}_p(A)\subseteq \mathrm{Cl}_p(\mathrm{Fr}_p(A))$, and so $\mathrm{Fr}_p(A)=\mathrm{Cl}_p(\mathrm{Fr}_p(A))$. Hence $\mathrm{Fr}_p(A)$ is pre-closed.

(10) This is by (9) and Lemma 4.39.

(11) Using Proposition 4.33(3), we get

$$\begin{aligned}\mathrm{Fr}_p(\mathrm{Int}_p(A)) &= \mathrm{Cl}_p(\mathrm{Int}_p(A))\setminus \mathrm{Int}_p(\mathrm{Int}_p(A))\\ &\subseteq \mathrm{Cl}_p(A)\setminus \mathrm{Int}_p(A)\\ &= \mathrm{Fr}_p(A).\end{aligned}$$

(12) We obtain

$$\begin{aligned}\mathrm{Fr}_p(\mathrm{Cl}_p(A)) &= \mathrm{Cl}_p(\mathrm{Cl}_p(A))\setminus \mathrm{Int}_p(\mathrm{Cl}_p(A))\\ &\subseteq \mathrm{Cl}_p(A)\setminus \mathrm{Int}_p(A)\\ &= \mathrm{Fr}_p(A).\end{aligned}$$

(13) We get

$$\begin{aligned}A\setminus \mathrm{Fr}_p(A) &= A\setminus (\mathrm{Cl}_p(A)\setminus \mathrm{Int}_p(A))\\ &= A\cap ((X\setminus \mathrm{Cl}_p(A))\cup \mathrm{Int}_p(A))\\ &= \varnothing\cup (A\cup \mathrm{Int}_p(A))\\ &= \mathrm{Int}_p(A).\end{aligned}$$

This completes the proof. □

The converses of (1) and (4) of Theorem 4.40 are not true in general as seen in the following example.

Example 4.41. In Example 3.2, let $A=\{a,b,c\}$. Then $\mathrm{Fr}_p(A)=\{e\}\subsetneq \{b,c,d,e\}=\mathrm{Fr}_\alpha(A)$, which shows that the reverse inclusion of Theorem 4.40(1) is not valid. Also, Example 4.37(2) shows that the reverse inclusion of Theorem 4.40(4) is not valid in general.

Definition 4.42 ([2])**.** For a subset A of X, $\mathrm{Ext}_\alpha(A)=\mathrm{Int}_\alpha(X\setminus A)$ is said to be an *α-exterior* of A.

Definition 4.43. For a subset A of X, the semi-interior of $X\setminus A$ is called the *pre-exterior* of A, and is denoted by $\mathrm{Ext}_p(A)$, that is,

$$\mathrm{Ext}_p(A)=\mathrm{Int}_p(X\setminus A).$$

Example 4.44. Let $(X,\mathscr{T})$ be a topological space in Example 4.4. For subsets $A=\{a,b,c\}$ and $B=\{b,d\}$ of X, we have $\mathrm{Ext}_p(A)=\{d,e\}$ and $\mathrm{Ext}_p(B)=\{a,c,e\}$.

Theorem 4.45. *For subsets A and B of X, the following assertions are valid.*

(1) $\mathrm{Ext}_\alpha(A) \subseteq \mathrm{Ext}_p(A)$.
(2) $\mathrm{Ext}_p(A)$ *is pre-open.*
(3) $\mathrm{Ext}_p(A) = X \setminus \mathrm{Cl}_p(A)$.
(4) $\mathrm{Ext}_p(\mathrm{Ext}_p(A)) = \mathrm{Int}_p(\mathrm{Cl}_p(A)) \supseteq \mathrm{Int}_p(A)$.
(5) $A \subseteq B \Rightarrow \mathrm{Ext}_p(B) \subseteq \mathrm{Ext}_p(A)$.
(6) $\mathrm{Ext}_p(A \cup B) \subseteq \mathrm{Ext}_p(A) \cap \mathrm{Ext}_p(B)$.
(7) $\mathrm{Ext}_p(A \cap B) \supseteq \mathrm{Ext}_p(A) \cup \mathrm{Ext}_p(B)$.
(8) $\mathrm{Ext}_p(X) = \varnothing$, $\mathrm{Ext}_p(\varnothing) = X$.
(9) $\mathrm{Ext}_p(A) = \mathrm{Ext}_p(X \setminus \mathrm{Ext}_p(A))$.
(10) $X = \mathrm{Int}_p(A) \cup \mathrm{Ext}_p(A) \cup \mathrm{Fr}_p(A)$.

Proof. (1) Using Theorem 4.31, we have

$$\mathrm{Ext}_\alpha(A) = \mathrm{Int}_\alpha(X \setminus A) \subset \mathrm{Int}_p(X \setminus A) = \mathrm{Ext}_p(A).$$

(2) It follows from Lemma 4.10 and Proposition 4.33(1).
(3) It is straightforward by Proposition 4.33(6).
(4) Applying (5) and (7) of Proposition 4.33, we get

$$\begin{aligned} \mathrm{Ext}_p(\mathrm{Ext}_p(A)) &= \mathrm{Ext}_p(\mathrm{Int}_p(X \setminus A)) \\ &= \mathrm{Int}_p(X \setminus \mathrm{Int}_p(X \setminus A)) \\ &= \mathrm{Int}_p(\mathrm{Cl}_p(A)) \supset \mathrm{Int}_p(A). \end{aligned}$$

(5) Assume that $A \subset B$. Then

$$\mathrm{Ext}_p(B) = \mathrm{Int}_p(X \setminus B) \subseteq \mathrm{Int}_p(X \setminus A) = \mathrm{Ext}_p(A)$$

by using Proposition 4.33(7).
(6) Applying Proposition 4.33(9), we get

$$\begin{aligned} \mathrm{Ext}_p(A \cup B) &= \mathrm{Int}_p(X \setminus (A \cup B)) \\ &= \mathrm{Int}_p((X \setminus A) \cap (X \setminus B)) \\ &\subseteq \mathrm{Int}_p(X \setminus A) \cap \mathrm{Int}_p(X \setminus B) \\ &= \mathrm{Ext}_p(A) \cap \mathrm{Ext}_p(B). \end{aligned}$$

(7) Using Proposition 4.33(8), we obtain

$$\begin{aligned} \mathrm{Ext}_p(A \cap B) &= \mathrm{Int}_p(X \setminus (A \cap B)) \\ &= \mathrm{Int}_p((X \setminus A) \cup (X \setminus B)) \\ &\supseteq \mathrm{Int}_p(X \setminus A) \cup \mathrm{Int}_p(X \setminus B) \\ &= \mathrm{Ext}_p(A) \cup \mathrm{Ext}_p(B). \end{aligned}$$

(8) Straightforward.
(9) Using Proposition 4.33(3), we have

$$\mathrm{Ext}_p(X \setminus \mathrm{Ext}_p(A)) = \mathrm{Ext}_p(X \setminus \mathrm{Int}_p(X \setminus A)) = \mathrm{Int}_p(X \setminus A) = \mathrm{Ext}_p(A).$$

(10) Straightforward. □

Let $(X, \mathscr{T})$ be a topological space which is given in Example 4.4. Take $A = \{d, e\}$. Then $\mathrm{Ext}_\alpha(A) = \{a\}$ and $\mathrm{Ext}_p(A) = \{a, b, c\}$. Thus the reverse inclusion of Theorem 4.45(1) is not valid. Let $A = \{b, e\}$ and $B = \{c, d, e\}$. Then $\mathrm{Ext}_p(B) = \{a\} \subseteq \{a, c, d\} = \mathrm{Ext}_p(A)$. This shows that the converse of (5) in Theorem 4.45 is not valid. Now let $A = \{d, e\}$ and $B = \{c\}$. Then $\mathrm{Ext}_p(A \cup B) = \{a\} \neq \{a, b\} = \{a, b, c\} \cap \{a, b, d, e\} = \mathrm{Ext}_p(A) \cap \mathrm{Ext}_p(B)$ which shows that the equality in Theorem 4.45(6) is not valid. Finally let $A = \{a, b\}$ and $B = \{c, d, e\}$. Then $\mathrm{Ext}_p(A \cap B) = \{a, b, c, d, e\}$ and $\mathrm{Ext}_p(A) \cup \mathrm{Ext}_p(B) = \{a, c, d, e\}$. This shows that the equality in Theorem 4.45(7) is not valid.

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