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Scott-representability of some spaces of Tall and Miškin

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Abstract. In this paper we show that a variation of a technique of Miškin and Tall yields a cocompact completely regular Moore space that is Scott-domain-representable and has a closed G_δ-subspace that is not Scott-domain-representable. This clarifies the general topology of Scott-domain-representable spaces and raises additional questions about Scott-domain representability in Moore spaces.

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1. Introduction

A domain is a continuous poset $(P, \sqsubseteq)$ in which each non-empty directed subset has a supremum. A Scott domain is a domain in which each nonempty bounded set has a supremum. (For more details, see Section 2.) Representing mathematical objects as the set of maximal elements of a domain or of a Scott domain is an idea that originated in theoretical computer science.

Every domain carries a natural topology, called the Scott topology, and a topological space is said to be *domain representable* (respectively, *Scott-domain-representable*) if it is homeomorphic to the set of maximal elements of a domain (respectively, a Scott domain) with the relative Scott topology. In recent years, topologists have come to see domain representability and Scott-domain representability as strong completeness properties associated with the Baire category theorem. For example, every subcompact regular space is domain-representable [4] and every domain-representable space is Choquet complete [8], and therefore a Baire space. (See Section 2 for definitions.)

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The basic general topology of domain-representable spaces is fairly well understood. For example, while domain-representability is an open-hereditary property, it is not closed-hereditary (because if X is any completely regular space that is not domain-representable, then the space obtained from βX by isolating all points of $\beta X - X$ is domain-representable [3] and contains X as a closed subspace). Similarly, Scott-domain-representability is open-hereditary and not closed-hereditary (as can be seen by applying the same βX construction described above). Further, any G_δ-subspace of a domain-representable space is domain-representable, as shown in [3], so it is natural to ask whether G_δ-subspaces of Scott-domain-representable spaces inherit Scott-domain representability. Among metrizable spaces, the answer is "Yes," because if X is a Scott-domain representable metric space, then X is completely metrizable. Let Y be a G_δ-subset of X. Then Y is also completely metrizable so that a recent result of Kopperman, Kunzi, and Waszkiewicz [7] shows that Y is Scott-domain representable. The first goal of this paper is to show that, without metrizability, Scott-domain-representability is *not* inherited by (closed) G_δ-subspaces. Furthermore, our example is a Moore space, a particularly nicely-behaved type of generalized metric space.

It was already known that the equivalence among metric spaces of essentially all strong completeness properties (complete metrizability, Scott-domain-representability, Čech-completeness, cocompactness, subcompactness, and domain-representability) breaks down outside of the metric space category. But there is still a rich theory of completeness in the wider class of Moore spaces, and results due to K. Martin, Tall, Rudin, Bennett, Lutzer, and Reed show that

- among Moore spaces, domain-representability is equivalent to subcompactness [4] and is equivalent to Rudin-completeness [2] which is strictly weaker than Moore-completeness [6];
- for completely regular Moore spaces, Moore completeness is equivalent to Čech-completeness [2];
- there is a completely regular Moore space that is Čech-complete but not cocompact [11] and not Scott-domain-representable [5];
- if a Moore space is Scott-domain-representable, then it is completely regular and Moore-complete, Čech complete [8], and cocompact [7];

Additional equivalents of domain-representability among Moore spaces that involve the strong Choquet game are given in [5]. The second goal of this paper is to explore the role of Scott-domain representability in the class of completely regular Moore spaces and we show that a certain Moore space X_0 (due to Miškin [10]) is Scott-domain-representable and contains a closed G_δ-subspace Z (due to Tall [11]) that is not Scott-domain representable. This example raises a natural question about completeness and representability in Moore spaces, namely:

Question 1.1. *Is Scott-domain-representability equivalent to cocompactness among completely regular Moore spaces?*

Kopperman, Kunzi, and Waszkiewicz [7] have characterized Scott-domain-representability in any completely regular space as being a combination of cocompactness and a bi-topological condition ("pairwise complete regularity"), but is not yet clear how to apply their characterization in the Moore space context. A natural place to look for counterexamples to Question 1.1 is in Miškin's construction of a cocompact Moore space, mentioned above. In Section 3 we show that *some* of Miškin's spaces are Scott-domain-representable, but we do not know the answer to the following:

Question 1.2. *Is it true that each of Miškin's spaces in* [10] *is Scott-domain-representable?*

In this paper we show that a certain Čech-complete Moore space constructed by Tall embeds as a closed subspace of a Scott-domain representable Moore space. To what extent is this a general phenomenon? More precisely, we have:

Question 1.3. *Does each completely regular, Čech-complete Moore space X embed in a Moore space $Y(X)$ that is Scott-domain-representable? What if X is required to be a dense subspace of $Y(X)$? What if X is required to be a closed subspace?*

Basic definitions appear in Section 2. Section 3 gives the basic constructions due to Tall and Miškin, and shows that, with some additional restrictions, one of Miškin's spaces is Scott-domain-representable and has a closed G_δ-subspace that is not. Throughout the paper, we reserve the symbols $\mathbb{R}, \mathbb{Q}$, and $\mathbb{P}$ for the usual sets of real, rational, and irrational numbers.

2. Basic definitions

A space X is *cocompact* if it is T_1 and has a collection $\mathcal{C}$ of closed subsets with the following two properties:

a) if $\mathcal{D}$ is a centered [1] subcollection of $\mathcal{C}$, then $\bigcap \mathcal{D} \neq \varnothing$;
b) if U is an open subset of X and $x \in U$, then some $C \in \mathcal{C}$ has $x \in \mathrm{Int}(C) \subseteq C \subseteq U$.

Note that the members of $\mathcal{C}$ might not be the closures of their interiors, even when the interiors are non-void. If one insists that members of $\mathcal{C}$ *are* the closures of their interiors, i.e., are regularly-closed sets, then one obtains a different notion called *regular cocompactness.* The Sorgenfrey line, for example, is cocompact but not regularly cocompact [2].

Cocompactness was introduced by de Groot and his colleagues [1]. Another strong completeness first studied by the Amsterdam school is *subcompactness*, where we say that a space X is *subcompact* if X has a base $\mathcal{B}$ with the property that $\bigcap \mathcal{F} \neq \varnothing$ whenever $\mathcal{F} \subseteq \mathcal{B}$ has the property that if $B_1, B_2 \in \mathcal{F}$, then some $B_3 \in \mathcal{F}$ has $\mathrm{cl}(B_3) \subseteq B_1 \cap B_2$.

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[1] A collection $\mathcal{D}$ is *centered* if $\bigcap\{D_i : i \leq n\} \neq \varnothing$ whenever $\{D_i : i \leq n\}$ is a finite subcollection of $\mathcal{D}$.

To define domain-representability and Scott-domain-representability, we begin with a poset $(S, \sqsubseteq)$. A subset $E \subseteq S$ is *directed* if for each $e_1, e_2 \in E$ some $e_3 \in E$ has $e_1, e_2 \sqsubseteq e_3$. If $\sup(E) \in S$ whenever E is a nonempty directed subset of S, then S is a *dcpo* ("directed-complete partial order"). Given $a, b \in S$, we write $a \ll b$ to mean that whenever $E \subseteq S$ is a directed set with $b \sqsubseteq \sup(E)$, then some $e \in E$ has $a \sqsubseteq e$. The set $\Downarrow(b)$ is defined to be $\{a \in S : a \ll b\}$. In case $\Downarrow(b)$ is directed and has b as its supremum for each $b \in S$, we say that S is *continuous*. If S is a continuous dcpo, then we say that S is a *domain*. If the domain S has the additional property that every nonempty bounded subset of S has a supremum in S, then we say that S is a *Scott domain*. Among domains, Scott domains are easily characterized:

Lemma 2.1. *A domain $(S, \sqsubseteq)$ is a Scott domain if and only if* $\sup(\{a, b\})$ *exists whenever $a, b \in S$ and $a, b \sqsubseteq c$ for some $c \in S$.*

Proof. To prove the nontrivial half of the lemma, suppose E is a nonempty bounded subset of S. Let $f \in S$ be an upper bound for E. If $e_1, e_2, e_3 \in E$, then $\sup(\{e_1, e_2\}) \in S$ and f is an upper bound for $\{\sup(\{e_1, e_2\}), e_3\}$ in S so that $\sup(\sup(e_1, e_2), e_3) \in S$. It is easy to show that $\sup(\sup(e_1, e_2), e_3) = \sup(\sup(e_i, e_j), e_k)$ for each permutation i, j, k of $1, 2, 3$, so that the supremum of each three-element subset of the bounded set E is well-defined. Similarly, $\sup(F)$ is a well-defined point of S for each non-empty finite subset $F \subseteq E$. Now let $D := \{\sup(F) : \varnothing \neq F \subseteq E \text{ and } |F| < \omega\}$. Then D is a directed subset of S so that, S being a domain, $\sup(D) \in S$. Clearly $\sup(D) = \sup(E)$ as required. □

Every poset $(S, \sqsubseteq)$ can be endowed with a special topology called the *Scott topology* in which a set U is open if and only if it satisfies both (i) if $x \sqsubseteq y$ and $x \in U$, then $y \in U$, and (ii) if $E \subseteq S$ is a nonempty directed set with $\sup(E) \in U$, then $E \cap U \neq \varnothing$. In a domain S, the collection of all sets $\Uparrow(a) := \{b \in S : a \ll b\}$ is a base for the Scott topology on S. The set of maximal elements of a domain S is denoted by $\max(S)$. If a topological space X is homeomorphic to the subspace $\max(S)$ of some domain S with the relative Scott topology, then we say that X is *domain-representable*. If S is a Scott-domain and X is homeomorphic to $\max(S)$, then we say that X is *Scott-domain-representable*.

Kopperman, Kunzi, and Waszkiewicz [7] have characterized Scott-domain-representable spaces as being the cocompact spaces that also satisfy a certain bi-topological condition. A short, direct proof of cocompactness of any Scott-domain-representable space is possible and we give it here. A central tool is the following Interpolation Lemma [9].

Lemma 2.2. *Suppose $a \ll c$ in a domain S. Then some $b \in S$ has $a \ll b \ll c$.*

Lemma 2.3. *Let S be a Scott domain. For each $p \in S$, let $\uparrow(p) = \{q \in S : p \sqsubseteq q\}$. Then each set $\uparrow(p) \cap \max(S)$ is a relatively closed subset of* $\max(S)$.

Proof. Suppose that $x \in \max(S)$ is a limit point of $\uparrow(p) \cap \max(S)$. Then for each $q \ll x$, $\Uparrow(q) \cap \uparrow(p) \neq \varnothing$. Consequently p and q have a common extension, so that $r(p,q) := \sup\{p,q\}$ is in S. Let $E := \{r(p,q) : q \ll x\}$. We claim that E is a directed set. For suppose that $r(p,q_1), r(p,q_2) \in E$. Because $\Downarrow(x)$ is directed, some $q_3 \in \Downarrow(x)$ has $q_1, q_2 \sqsubseteq q_3$. Then $r(p,q_3) \in E$ and $r(p,q_i) \sqsubseteq r(p,q_3)$ for $i = 1,2$. Because S is a dcpo, $\sup(E) \in S$ so that some $z \in \max(S)$ has $\sup(E) \sqsubseteq z$. Recall that as a subspace of S, $\max(S)$ is a T_1-space. Therefore, if $z \neq x$, then some $q_4 \ll x$ has $z \notin \Uparrow(q_4)$. Because $q_4 \ll x$, Lemma 2.2 gives $q_5 \in S$ with $q_4 \ll q_5 \ll x$. But $q_5 \ll x$ forces $r(p,q_5) \in E$ so that $q_4 \ll q_5 \sqsubseteq r(p,q_5) \sqsubseteq \sup(E) \sqsubseteq z$. Therefore $z \in \Uparrow(q_5) \subseteq \Uparrow(q_4)$, contrary to our choice of q_4. Therefore, $p \sqsubseteq \sup(E) = z = x$ showing that $x \in \uparrow(p)$ as required. □

Our next result appears in [7]. We present an easy direct proof.

Corollary 2.4. *Suppose S is a Scott domain. Then the subspace of maximal elements of S is cocompact.*

Proof. First, the subspace $\max(S)$ of S is T_1. Second, let $\mathcal{C} = \{\max(S) \cap \uparrow(p) : p \in S\}$. In the light of Lemma 2.3, each member of $\mathcal{C}$ is a closed subset of $\max(S)$. To verify the first part of the cocompactness definition, suppose that $\mathcal{D} \subseteq \mathcal{C}$ is a centered collection. Write $\mathcal{D} = \{\max(S) \cap \uparrow(a) : a \in A\}$. Then, given any finite set $F := \{a_1, \cdots, a_k\} \subseteq A$ we know that $\uparrow(a_1) \cap \cdots \cap \uparrow(a_k) \neq \varnothing$ because $\mathcal{D}$ is centered, so that $\sup(F) \in S$ by Lemma 2.1. Let $\hat{A} := \{\sup(F) : \varnothing \neq F \subseteq A \text{ and } |F| < \omega\}$. Then $\hat{A}$ is directed, so $\sup(\hat{A}) \in S$, say $\sup(\hat{A}) = b \in S$. Then $\varnothing \neq \uparrow(b) \cap \max(S) \subseteq \bigcap \mathcal{D}$, as required.

To verify the second part of the definition of cocompactness, it is enough to consider a point x in a basic open set $\max(S) \cap \Uparrow(q)$. The Interpolation Lemma 2.2 provides a point $p \in S$ with $q \ll p \ll x$. Then $\Uparrow(p)$ is a neighborhood of x with $\Uparrow(p) \subseteq \uparrow(p) \subseteq \Uparrow(q)$ so that x is in the relative interior of $\uparrow(p) \cap \max(S)$ which is contained in the closed set $\uparrow(p) \cap \max(S) \subseteq \max(S) \cap \Uparrow(q)$, as required to show that $\max(S)$ is cocompact. □

3. A variation of spaces of Tall and Miškin

Tall and Miškin began their constructions with a countable subset of the plane that had uncountably many limit points on the x-axis. We need more control and so we replace that countable set by a binary tree T with ω-many levels and use its branch space Y in place of the limit points on the x-axis. This tree may be embedded in the upper half plane in such a way that its branch space corresponds in a natural way to an uncountable set (the Cantor set) on the x-axis. Therefore, the space we will construct is one of the spaces due to Miškin.

Description of the space X and the subspace X_0:

a) The tree T: Let T be a binary tree with ω-many levels. Denote the unique minimal element of T by $\bar{0}$. The level of any $d \in T$ in our tree is denoted by lv(d) and $T(n) = \{d \in T : \text{lv(d)} = \text{n}\}$ so that $T = \bigcup\{T(n) : 0 \leq n < \omega\}$.

b) The branches of T: Let Y be the set of all branches of T, i.e., each $y \in Y$ is a maximal linearly ordered subset of T. We let $e(y, n)$ denote the unique element of the branch y that lies at level n of the tree T. Thus, for example, $e(y, 0) = \bar{0}$ for each $y \in Y$ and if $y_1, y_2 \in Y$ have $e(y_1, n) = e(y_2, n)$, then $e(y_1, k) = e(y_2, k)$ for each $0 \leq k \leq n$.

c) The space X: Let $T^* := \{(d, S) : d \in T,\ \varnothing \neq S \subseteq Y\}$. The underlying set of our space is $X = T^* \cup Y$ and the set X is topologized by isolating each point of T^* and by using the sets

$$N(y, n) = \{y\} \cup \{(d, S) \in T^* : \mathrm{lv(d) \geq n,\ d \in y,\ and\ y \in S}\}$$

as basic neighborhoods of $y \in Y$. Equivalently, $N(y, n) = \{y\} \cup \{(e(y, k), S) \in T^* : n \leq k < \omega \text{ and } y \in S\}$.

d) The subspace $X_0 = X - \{(\bar{0}, Y)\}$. This is a closed and open subspace of X and will be the space used in our example. However, X_0 is not needed until the very end of Section 3.

As in [10], the space X is a cocompact, Čech-complete, completely regular Moore space and therefore so is its closed and open subspace X_0. The subspace X_0 contains a closed (and hence G_δ) -subspace $Z := \{(d, S) \in T^* : (d, S) \neq (\bar{0}, Y), |S| < \omega\} \cup Y$ that is homeomorphic to one of the spaces constructed by Tall in [11]. Consequently, Z is not cocompact and therefore is not Scott-domain-representable (by Corollary 2.4). What remains is to prove that X_0 *is* Scott-domain representable. Our first step is to define the Scott domain that comes very close to representing X, and then to work around the "almost" part of that statement to show that X_0 is Scott-domain representable. We begin by constructing our poset.

The poset $(\mathcal{S}, \sqsubseteq)$

e) The sets $I(B, k)$: Let $\varnothing \neq B \subseteq Y$ and let $k \geq 0$. If $|B| = 1$, then let $I(B, k) := N(y, k)$ where y is the unique point of B. If $|B| \geq 2$, then let

$$I(B, k) := \{(d, S) \in T^* : lv(d) \geq k,\ B \subseteq S, \text{ and } \forall y \in B,\ d \in y\}.$$

Note that the condition $d \in y$ is equivalent to $e(y, \mathrm{lv(d)}) = \mathrm{d}$, a fact that will be used later.

f) Let $\mathcal{S} = \{\{t\} : t \in X\} \cup \{I(B, k) : I(B, k) \neq \varnothing, \varnothing \neq B \subseteq Y,\ 0 \leq k < \omega\}$ and let $\sqsubseteq$ denote reverse inclusion. Consequently, if $t \in X$, then $I(B, k) \sqsubseteq \{t\}$ means $t \in I(B, k)$.

Remark 3.1. If $|B| \geq 2$, one can prove that $I(B, k) = \bigcap\{N(y, k) : y \in B\}$, and that was the way we initially thought of the sets $I(B, k)$. However, that fact is not really needed in our construction.

The next example illustrates how the sets $I(B, k)$ can behave. It introduces special notations, and parts c), d), and e) will be very important tools in the proofs of later lemmas in this section.

Example 3.2. Let d', d'' be the two points of $T(1)$ and recall that $\bar{0}$ is the unique point of $T(0)$. Let $y', y'' \in Y$ have $e(y', 1) = d'$ and $e(y'', 1) = d''$ (so that y', y'' are two branches of T that disagree at level 1 of the tree). Note that $y' \cap y'' = \{(\bar{0}, Y)\} = T(0)$. Then

a) A set of the form $I(B,k)$ can be empty. For example, $I(\{y', y''\}, 1) = \varnothing = I(Y, 1)$ because if $(d,S) \in I(\{y', y''\}, 1)$ then $lv(d) \geq 1$ and $e(y', lv(d)) = d = e(y'', lv(d))$. Because T is a tree and $lv(d) \geq 1$ we must have $d' = e(y', 1) = e(y'', 1) = d''$ so that $y' \cap y''$ contains some element of T at or above level 1, which is false.

b) $I(\{y', y''\}, 0)$ is the infinite set $\{(\bar{0}, S) : y', y'' \in S \subseteq Y\}$ and $I(Y, 0)$ is the singleton set $\{(\bar{0}, Y)\}$.

c) For any $I(B,k)$, if $(d,S) \in I(B,k)$ then $(\hat{d}, S) \in I(B,k)$ where $\hat{d}$ is the unique predecessor of d in level k of the tree T.

d) For any $I(B,k)$, if $(d,S) \in I(B,k)$ then $(d,B) \in I(B,k)$ and $(d,S') \in I(B,k)$ whenever $S \subseteq S' \subseteq Y$. In particular $(d,Y) \in I(B,k)$.

e) From b), c) and d), the only way that $|I(B,k)| = 1$ is for $k = 0$ and $B = Y$, and then $I(Y,0) = \{(\bar{0}, Y)\}$.

Lemma 3.3. *For any $B \subseteq Y$ and any $k \geq 0$, $|I(B,k) \cap Y| \leq 1$. If $|B| \geq 2$ then $I(B,k) \subseteq T^*$ and $\pi_1[I(B,k)]$ is finite, where $\pi_1 : T^* \to T$ is first coordinate projection.*

Proof. The first two assertions follow directly from the definition of the sets $I(B,k)$, so we prove only the final assertion. Because $|B| \geq 2$ we may choose distinct $y_1, y_2 \in B$. Then there is some integer L such that $e(y_1, L) \neq e(y_2, L)$ so that $e(y_1, j) \neq e(y_2, j)$ for each $j \geq L$. Therefore, if $(d,S) \in I(B,k)$, we know that $d \in T$ and $lv(d) < L$, and there are only finitely many such points. □

Lemma 3.4. *The maximal elements of $\mathcal{S}$ are the singleton sets $\{x\}$ where $x \in X$.*

Lemma 3.5. *If $\varnothing \neq B_1 \subseteq B_2 \subseteq Y$ with and $k_1 \leq k_2$, then $I(B_1, k_1) \sqsubseteq I(B_2, k_2)$. Furthermore if $I(B_1, k_1) \sqsubseteq I(B_2, k_2) \neq \varnothing$, then $B_1 \subseteq B_2$ and $k_1 \leq k_2$.*

Proof. First suppose that $B_1 \subseteq B_2$ and $k_1 \leq k_2$. If $|B_2| = 1$ then $B_1 = B_2$. Let y be the unique point of B_2. Then $k_1 \leq k_2$ gives $I(B_2, k_2) = N(y, k_2) \subseteq N(y, k_1) = I(B_1, k_1)$ and hence $I(B_1, k_1) \sqsubseteq I(B_2, k_2)$. In case B_2 has at least two points, then $I(B_2, k_2) \subseteq T^*$ so that each element of $I(B_2, k_2)$ has the form (d,S) where $\mathrm{lv(d)} \geq \mathrm{k_2}$ and $d \in B_2$. Hence $\mathrm{lv(d)} \geq \mathrm{k_2} \geq \mathrm{k_1}$ and $d \in y$ for each $y \in B_2$. Because $B_1 \subseteq B_2$, we have $(d,S) \in I(B_1, k_1)$, as required.

To prove the second claim, note that $I(B_1, k_1) \sqsubseteq I(B_2, k_2)$ gives $I(B_2, k_2) \subseteq I(B_1, k_1)$ because $\sqsubseteq$ is reverse inclusion. Now fix any $(d,S) \in I(B_2, k_2)$. Then $\mathrm{lv(d)} \geq \mathrm{k_2}$, $\mathrm{B_2} \subseteq \mathrm{S}$, and $d \in y$ for all $y \in B_2$. Let $\hat{d}$ be the unique predecessor of d at level k_2 of the tree T. Then (see Example 3.2), $(\hat{d}, S) \in I(B_2, k_2) \subseteq I(B_1, k_1)$ so that $k_2 = \mathrm{lv}(\hat{\mathrm{d}}) \geq \mathrm{k_1}$. Thus $k_1 \leq k_2$. Next, Example 3.2 shows that since $(d,S) \in I(B_2, k_2)$, $(d, B_2) \in I(B_2, k_2) \subseteq I(B_1, k_1)$ so that $B_1 \subseteq B_2$, as required. □

Lemma 3.6. *Let $\mathcal{E} := \{I(B_\alpha, k_\alpha) : \alpha \in A\}$ be a directed subset of $(\mathcal{S}, \sqsubseteq)$ that contains no maximal element of itself. Let $C = \bigcup\{B_\alpha : \alpha \in A\}$.*

a) *If $|C| = 1$ then the set $\{k_\alpha : \alpha \in A\}$ is unbounded, and* $\sup(\mathcal{E}) = \{y\}$ *where y is the unique point of C. (Note that in this case, $y \in Y$.)*
b) *If $|C| \geq 2$, then $\{k_\alpha : \alpha \in A\}$ is bounded and* $\sup(\mathcal{E}) = I(C, L)$ *where* $L = \max\{k_\alpha : \alpha \in A\}$.

Proof. In case (a), it is clear that $\{y\}$ is an upper bound for $\mathcal{E}$, and that no other $\{z\}$ for $z \in Y$ can be an upper bound for $\mathcal{E}$. In addition, each $B_\alpha = \{y\}$. If the set $\{k_\alpha : \alpha \in A\}$ is bounded, let k_β be its largest member. Then $I(B_\beta, k_\beta)$ is the maximal member of $\mathcal{E}$, contrary to hypothesis. Therefore $\{k_\alpha : \alpha \in A\}$ is unbounded, and now it is clear that $\sup \mathcal{E} = \{y\}$.

To prove (b), fix distinct $y_1, y_2 \in C$ and choose $\alpha_i \in A$ with $y_i \in B_{\alpha_i}$ for $i = 1, 2$. Using directedness of $\mathcal{E}$, find $\beta \in A$ with $I(B_{\alpha_i}, k_{\alpha_i}) \sqsubseteq I(B_\beta, k_\beta)$. Then $I(B_\beta, k_\beta) \neq \varnothing$ so that by Lemma 3.5 $y_i \in B_{\alpha_i} \subseteq B_\beta$. According to Lemma 3.3, the set $F := \pi_1[I(B_\beta, k_\beta)]$ is finite.

Next, we claim that some $d \in F$ has $d \in \pi_1[I(B_\alpha, k_\alpha)]$ for each $\alpha \in A$. For contradiction, suppose that corresponding to each $d \in F$ there is some $\gamma(d) \in A$ with $d \notin \pi_1[I(B_{\gamma(d)}, k_{\gamma(d)})]$. Directedness of $\mathcal{E}$ provides some $\eta \in A$ with $I(B_\beta, k_\beta) \sqsubseteq I(B_\eta, k_\eta)$ and such that $I(B_{\gamma(d)}, k_{\gamma(d)}) \sqsubseteq I(B_\eta, k_\eta)$ for each of the finitely many $d \in F$. Choose any $(\bar{d}, S) \in I(B_\eta, k_\eta)$. Then $I(B_\beta, k_\beta) \sqsubseteq I(B_\eta, k_\eta)$ yields $I(B_\eta, k_\eta) \subseteq I(B_\beta, k_\beta)$ so that $\bar{d} \in \pi_1[I(B_\beta, k_\beta)] = F$. Because $\bar{d} \in F$ we know that $\gamma(\bar{d})$ is defined and $\bar{d} \notin \pi_1[I(B_{\gamma(\bar{d})}, k_{\gamma(\bar{d})})]$. Because $I(B_{\gamma(\bar{d})}, k_{\gamma(\bar{b})}) \sqsubseteq I(B_\eta, k_\eta)$ we have $(\bar{d}, S) \in I(B_\eta, k_\eta) \subseteq I(B_{\gamma_{\bar{d}}}, k_{\gamma_{\bar{b}}})$ and that is impossible because we know that $\bar{d} \notin \pi_1[I(B_{\gamma_{\bar{d}}}, k_{\gamma(\bar{d})})]$.

At this stage of the argument, we know that there is some $d_0 \in F$ with $d_0 \in \pi_1[I(B_\alpha, k_\alpha)]$ for each $\alpha \in A$. Then for some $S_\alpha \subseteq Y$ we have $(d_0, S_\alpha) \in I(B_\alpha, k_\alpha)$. Because $B_\alpha \subseteq C$, part (c) of Example 3.2 shows that $(d_0, C) \in I(B_\alpha, k_\alpha)$. Consequently $\mathrm{lv(d_0)} \geq \mathrm{k_\alpha}$ and we conclude that $\mathrm{lv(d_0)}$ is an upper bound for the set $\{k_\alpha : \alpha \in A\}$. Let L be the largest member of the set $\{k_\alpha : \alpha \in A\}$. Note that $\mathrm{lv(d_0)} \geq \mathrm{L}$.

Next we claim that $(d_0, C) \in I(C, L)$. Consider the membership criteria for $I(C, L)$. We already know that $\mathrm{lv(d_0)} \geq \mathrm{L}$ and obviously $C \subseteq C$, so all we must show is that $d_0 \in y$ for each $y \in C$. Fix any $y \in C$. Then there is some $\alpha \in A$ with $y \in B_\alpha$. From above we know that $(d_0, C) \in I(B_\alpha, k_\alpha)$ so that $y \in B_\alpha$ gives $d_0 \in y$ as required. Now we know that $I(C, L) \neq \varnothing$ so that $I(C, L) \in \mathcal{S}$.

According to Lemma 3.5, $I(C, L)$ is an upper bound for $\mathcal{E}$. To complete the proof that $I(C, L) = \sup(\mathcal{E})$, we consider any upper bound $G \in \mathcal{S}$ for $\mathcal{E}$ and we will show that $I(C, L) \sqsubseteq G$. With $I(B_\beta, k_\beta)$ as defined in the second paragraph of this proof, we have $I(B_\beta, k_\beta) \sqsubseteq G$ so that $G \subseteq I(B_\beta, k_\beta)$. Hence $G \subseteq I(B_\beta, k_\beta) \subseteq T^*$ so that either G has the form $G = I(H, m)$ or else $G = \{(e, S)\} \in \max \mathcal{S}$. In the first case, Lemma 3.5 shows that $I(B_\alpha, k_\alpha) \sqsubseteq I(H, m)$ implies $B_\alpha \subseteq H$ and $k_\alpha \leq m$ for each $\alpha \in A$, so that $C \subseteq H$ and $L = \max\{k_\alpha : \alpha \in A\} \leq m$. Hence $I(C, L) \sqsubseteq I(H, m) = G$, as claimed. In the second case, where $G = \{(e, S)\}$, we will show that $(e, S) \in I(C, L)$. Note that $I(B_\alpha, k_\alpha) \sqsubseteq G = \{(e, S)\}$ gives $(e, S) \in I(B_\alpha, k_\alpha)$ so that $\mathrm{lv(e)} \geq \mathrm{k}_\alpha$ and $B_\alpha \subseteq S$ for each α and therefore $C \subseteq S$ and $\mathrm{lv(e)} \geq \max\{\mathrm{k}_\alpha : \alpha \in \mathrm{A}\} = \mathrm{L}$. Furthermore, if $y \in C$ then $y \in B_\alpha$ for some $\alpha \in A$ so that $(e, S) \in I(B_\alpha, k_\alpha)$ guarantees that $e \in y$. Therefore, $I(C, L) \sqsubseteq G$, as required. to show that $I(C, L) = \sup(\mathcal{E})$. □

Lemma 3.7. *In $\mathcal{S}$, we have $I(B_1, k_1) \ll I(B_2, k_2)$ if and only if B_1 is a finite set, $B_1 \subseteq B_2$, and $k_1 \leq k_2$.*

Proof. First suppose $I(B_1, k_1) \ll I(B_2, k_2)$. Then $I(B_1, k_1) \sqsubseteq I(B_2, k_2)$ so that $B_1 \subseteq B_2$ and $k_1 \leq k_2$. We let $\mathcal{F}$ be the collection of all finite subsets of B_2. Then $\mathcal{E} := \{I(F, k_2) : F \in \mathcal{F}\}$ is a directed subset of $\mathcal{S}$ and $I(B_2, k_2) = \sup \mathcal{E}$ so that $I(B_1, k_1) \ll I(B_2, k_2)$ gives $I(B_1, k_1) \sqsubseteq I(F_1, k_2)$ for some $F_1 \in \mathcal{F}$, showing that $B_1 \subseteq F_1$. Since F_1 is finite, so is B_1.

For the converse, suppose that B_1 is a finite set and $B_1 \subseteq B_2$ and $k_1 \leq k_2$ (so that $I(B_1, k_1) \sqsubseteq I(B_2, k_2)$), and suppose that $\mathcal{E} = \{I(B_\alpha, k_\alpha) : \alpha \in A\}$ is a directed subset of $\mathcal{S}$ with $I(B_2, k_2) \sqsubseteq \sup(\mathcal{E})$. If $\mathcal{E}$ contains a maximal element of itself, there is nothing to prove, so assume that $\mathcal{E}$ contains no maximal element.

Let $C := \bigcup\{B_\alpha : \alpha \in A\}$. There are several cases to consider. In case $|C| \geq 2$, Lemma 3.6 gives

$$I(B_1, k_1) \sqsubseteq I(B_2, k_2) \sqsubseteq \sup \mathcal{E} = I(C, L)$$

where L is the largest member of the bounded set $\{k_\alpha : \alpha \in A\}$, say $L = k_\gamma$ for some $\gamma \in A$. Then $I(B_1, k_1) \sqsubseteq I(B_2, K_2) \sqsubseteq I(C, L)$ gives $B_1 \subseteq B_2 \subseteq C$. Therefore, each y in the finite set B_1 is a point of $C = \bigcup\{B_\alpha : \alpha \in A\}$, so we may find $\alpha(y) \in A$ with $y \in B_{\alpha(y)}$. Directedness of the collection $\mathcal{E}$ allows us to find $\beta \in A$ with $I(B_{\alpha(y)}, k_{\alpha(y)}) \sqsubseteq I(B_\beta, k_\beta)$ for each y in the finite set B_1 and therefore $y \in B_{\alpha(y)} \subseteq B_\beta$. Therefore $B_1 \subseteq B_\beta$. Once again using directedness, find $\delta \in A$ with $I(B_\gamma, k_\gamma), I(B_\beta, k_\beta) \sqsubseteq I(B_\delta, k_\delta)$. Then $B_1 \subseteq B_\beta \subseteq B_\delta$ and

$$k_1 \leq \max\{k_\alpha : \alpha \in A\} = L = k_\gamma \leq k_\delta \leq L.$$

Therefore $I(B_1, k_1) \sqsubseteq I(B_\delta, k_\delta) \in \mathcal{E}$ as required.

The remaining case is where $|C| = 1$, say $C = \{z\}$. Then $B_\alpha = \{z\}$ for each $\alpha \in A$. Because $\mathcal{E}$ contains no maximal element of itself, Lemma 3.6 shows that $\sup \mathcal{E} = \{z\}$ and that $\{k_\alpha : \alpha \in A\}$ is unbounded. Choose $\mu \in A$ with $k_\mu > k_1$. Then $I(B_\mu, k_\mu) = N(z, k_\mu) \subseteq N(z, k_1) = I(B_1, k_1)$ so that $I(B_1, k_1) \sqsubseteq I(B_\mu, k_\mu) \in \mathcal{E}$ as required. □

Lemma 3.8. *Suppose $S \in \mathcal{S}$ and $y \in Y$. Then $S \ll \{y\}$ if and only if $S = I(\{y\}, k)$ for some $k \geq 0$.*

Proof. Suppose $S = I(\{y\}, k)$. By Lemma 3.7, $I(\{y\}, k) \ll I(\{y\}, k) \sqsubseteq \{y\}$, so we know that $S = I(\{y\}, k) \ll \{y\}$. For the converse, suppose $S \in \mathcal{S}$ has $S \ll \{y\}$. Then $S \sqsubseteq \{y\}$ so that $y \in S$. By Lemma 3.3, either $S = I(\{y\}, k)$ or else $S = \{y\}$. If $S = \{y\}$ let $\mathcal{E} := \{I(\{y\}, k) : k \geq 0\}$. This is a directed set in $\mathcal{S}$ with $\sup \mathcal{E} = \{y\}$ and yet no member $I(\{y\}, k) \in \mathcal{E}$ has $S = \{y\} \sqsubseteq I(\{y\}, k)$. Therefore, S must have the form $S = I(\{y\}, k)$ as claimed. □

Lemma 3.9. *For $t \in X - Y$, $\{t\} \ll \{t\}$ provided $t \neq (\bar{0}, Y)$.*

Proof. Write $t = (d, S)$ with $(d, S) \neq (\bar{0}, Y)$. To show that $\{t\} \ll \{t\}$, suppose $\{t\} \sqsubseteq \sup \mathcal{E}$ where $\mathcal{E}$ is a directed subset of $\mathcal{S}$. Maximality of $\{t\}$ in $\mathcal{S}$ (see Lemma 3.4) shows that $\sup(\mathcal{E}) = \{t\}$.

If $\mathcal{E}$ contains a maximal member, there is nothing to prove, so for contradiction, suppose $\mathcal{E}$ contains no maximal member of itself. Then the collection $\mathcal{E}$ must be of the form $\mathcal{E} = \{I(B_\alpha, k_\alpha) : \alpha \in A\}$.

Write $C = \bigcup\{B_\alpha : \alpha \in A\}$. If $|C| = 1$, then $C = \{y\} \subseteq Y$, so that Lemma 3.6 shows $\sup \mathcal{E} = \{y\}$ and hence $\{y\} = \{t\}$. That is impossible because $y \in Y$ and $t \in X - Y$. Therefore $|C| \geq 2$.

Because $|C| \geq 2$, from Lemma 3.6 we know that the set $\{k_\alpha : \alpha \in A\}$ is bounded and $\sup \mathcal{E} = I(C, L)$ where L is the maximal element of the bounded set $\{k_\alpha : \alpha \in A\}$. Then $\{t\} = \sup(\mathcal{E}) = I(C, L)$ so that $I(C, L)$ is a singleton. Part (e) of Example 3.2 shows that the set $I(C, L)$ can be a singleton if and only if $C = Y$ and $L = 0$, and then $I(C, L) = \{(\bar{0}, Y)\}$, forcing us to conclude that $t = (\bar{0}, Y)$, which is false. This contradiction completes the proof of the lemma. □

Corollary 3.10. *The poset $(\mathcal{S}, \sqsubseteq)$ is continuous.*

Proof. Consider any element $S \in \mathcal{S}$. If $S \ll S$, then $S \in \Downarrow(S)$, so that $\Downarrow(S)$ is directed with $\sup(\Downarrow(S)) = S$. So suppose $S \ll S$ is false. Then Lemmas 3.8 and 3.9 show that one of the following three statements must be true:

(i) $S = I(B, k)$ where B is infinite, or
(ii) $S = \{y\}$ for some $y \in Y$, or
(iii) $S = \{(\bar{0}, Y)\}$.

If $S = I(B, k)$ where B is infinite, let $\mathcal{F}$ be the collection of all finite subsets of B. Then, by Lemma 3.7, $\Downarrow(I(B, k)) = \{I(F, j) : j \leq k, F \in \mathcal{F}\}$, which is directed and has $I(B, k)$ as its supremum, as required. In case $S = \{y\}$ for some $y \in Y$, then $\Downarrow(S) = \{I(\{y\}, k) : k \geq 1\}$ which is also directed and has supremum $S = \{y\}$, as required. The case where $S = \{(\bar{0}, Y)\}$ is actually a special case of item (i) because $\{(\bar{0}, Y)\} = I(Y, 0)$ as noted in Example 3.2, above. □

Lemma 3.11. *$(\mathcal{S}, \sqsubseteq)$ is a Scott domain.*

Proof. Suppose $U_1, U_2 \in \mathcal{S}$ have a common extension. We may assume that neither U_i is maximal in $\mathcal{S}$ (so that $U_i = I(B_i, k_i)$ for $i = 1, 2$) and that neither of U_1, U_2 is contained in the other. Then there is some $(d, S) \in I(B_1, k_1) \cap I(B_2, k_2)$. Let $C = B_1 \cup B_2$. Because neither of U_1, U_2 is contained in the other, $|C| \geq 2$ and $(d, S) \in I(C, \max(k_1, k_2))$ yields $I(C, \max(k_1, k_2)) \neq \varnothing$ so that $I(C, \max(k_1, k_2)) \in \mathcal{S}$. Clearly $I(C, \max(k_1, k_2))$ is an upper bound for U_1 and U_2.

To show that $I(C, \max(k_1, k_2))$ is the least upper bound of $U_1 = I(B_1, k_1)$ and $U_2 = I(B_2, k_2)$, consider any upper bound $U_3 \in \mathcal{S}$ for U_1 and U_2. From $U_i \sqsubseteq U_3$ we obtain $U_3 \subseteq U_1 \cap U_2$. Because $|C| \geq 2$ we know that $U_3 \subseteq U_1 \cap U_2 \subseteq$

$X - Y$, so that U_3 cannot have the form $\{y\}$ for some $y \in Y$. Therefore either $U_3 = I(D, j)$ for some D and some j, or else $U_3 = \{(\hat{d}, \hat{S})\} \in \max(\mathcal{S})$. In the first case $B_i \subseteq D$ and $j \geq k_i$ for $i = 1, 2$ so that $C \subseteq D$ and $\max(k_1, k_2) \leq j$ and therefore (see Lemma 3.5) $I(C, \max(k_1, k_2)) \sqsubseteq U_3$. In the second case, where $U_3 = \{(\hat{d}, \hat{S})\} \in \max(\mathcal{S})$, for $i = 1, 2$ we know that $(\hat{d}, \hat{S}) \in I(B_i, k_i)$ so that $lv(\hat{d}) \geq k_i$, $B_i \subseteq \hat{S}$, and that for each $y \in B_i, y \in \hat{d}$. Hence $I(C, \max(k_1, k_2)) \sqsubseteq U_3$. Therefore $I(C, \max(k_1, k_2)) = \sup(U_1, U_2)$ as required. □

There is a natural-looking function that sends each $x \in X$ to the element $\{x\} \in \mathcal{S}$. This mapping is 1-1, onto, and continuous from X to $\max(\mathcal{S})$, and it is tempting to think that the function is an a homeomorphism from X onto $\max(\mathcal{S})$. Unfortunately, it is not. The point $(\bar{0}, Y) \in X$ is isolated in X, but the point $\{(\bar{0}, Y)\}$ is not an isolated point of $\max(\mathcal{S})$. We are lucky that $(\bar{0}, Y)$ is the only "bad" point for the natural mapping. Recall that $X_0 = X - \{(\bar{0}, Y)\}$. Then we have:

Lemma 3.12. *The function $h : X_0 \to \max(S) - \{\{(\bar{0}, Y)\}\}$ given by $h(t) = \{t\}$ is a homeomorphism from X_0 onto the open subspace $\max(S) - \{\{(\bar{0}, Y)\}\}$ of $\max(\mathcal{S})$ with the relative Scott topology.*

Proof. Clearly the function h is 1-1 and $h[X_0] = \max(\mathcal{S}) - \{\{(\bar{0}, Y)\}\}$. To prove that h is continuous, it is enough to consider what happens at non-isolated points of X_0, i.e., at points $y \in Y$. Suppose $h(y) \in \Uparrow(p) \cap \max(\mathcal{S})$ where $p \in \mathcal{S}$. Then Lemma 3.8 guarantees that $p = I(\{y\}, k) = N(y, k)$ for some k. We claim that that $h[N(y, k+1)] \subseteq \Uparrow(p)$. Apply Lemma 3.9 to show that if $(d, S) \in N(y, k+1)$ then $(d, S) \neq (\bar{0}, Y)$ so that $h((d, S)) = \{(d, S)\} \ll \{(d, S)\}$. Then note that $p \sqsubseteq \{(d, S)\} \ll \{(d, S)\}$ so that $h(d, S) \in \Uparrow(p)$ as required.

To prove that h is an open mapping onto $\max(\mathcal{S}) - \{\{(\bar{0}, Y)\}\}$, the first step is to recall Lemma 3.9 which shows that if $t \in X - Y$ with $t \neq (\bar{0}, Y)$, i.e., if t is an isolated point of X_0, then in $\mathcal{S}$, $\{t\} \ll \{t\}$ so that $h(t) = \{t\}$ is an isolated point of $\max(\mathcal{S})$. Second, consider any non-isolated point $y \in X_0$ and note that for $k \geq 1$, $h[N(y, k)] = \max(\mathcal{S}) \cap \Uparrow(I(\{y\}, k)$. Therefore h is an open mapping onto $\max(\mathcal{S}) - \{\{(\bar{0}, Y)\}\}$ as required. □

Our next lemma shows that X_0 is Scott-domain-representable.

Lemma 3.13. *The subspace $X_0 = X - \{(\bar{0}, Y)\}$ is Scott-domain-representable.*

Proof. Because $\mathcal{S}$ is a Scott domain, we know that its subspace $\max(\mathcal{S})$ is Scott-domain-representable. It is easy to check that for any domain $\mathcal{D}$, the subspace $\max(\mathcal{D})$ is T_1. Therefore we see that for our Scott domain $\mathcal{S}$, the set $\max(\mathcal{S}) - \{\{(\bar{0}, Y)\}\}$ is an open subspace of the Scott-domain-representable space $\max(\mathcal{S})$. Now recall that any non-empty, relatively open subset of a Scott-domain representable space is also Scott-domain representable, and that completes the proof. □

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