Applied General Topology
© Universidad Politécnica de Valencia
Volume 12, no. 2, 2011
pp. 163-173

On a type of generalized open sets

Bishwambhar Roy*

Abstract

In this paper, a new class of sets called μ-generalized closed (briefly μg-closed) sets in generalized topological spaces are introduced and studied. The class of all μg-closed sets is strictly larger than the class of all μ-closed sets (in the sense of Á. Császár). Furthermore, g-closed sets (in the sense of N. Levine) is a special type of μg-closed sets in a topological space. Some of their properties are investigated here. Finally, some characterizations of μ-regular and μ-normal spaces have been given.

2010 MSC: *54D10, 54D15, 54C08, 54C10.*

Keywords: *μ-open set, μg-closed set, μ-regular space, μ-normal space.*

1. Introduction

In the past few years, different forms of open sets have been studied. Recently, a significant contribution to the theory of generalized open sets, was extended by A. Császár. Especially, the author defined some basic operators on generalized topological spaces.

It is observed that a large number of papers is devoted to the study of generalized open like sets of a topological space, containing the class of open sets and possessing properties more or less similar to those of open sets. For example, [22] has introduced g-open sets, [4, 30, 2] sg-open sets, [25] pg-open sets, [27, 28] $g\alpha$-open sets, [13] δg^*-open sets, [21, 17] bg-open sets.

Owing to the fact that corresponding definitions have many features in common, it is quite natural to conjecture that they can be obtained and a considerable part of the properties of generalized open sets can be deduced from suitable more general definitions. The purpose of this paper is to point

*The author acknowledges the financial support from UGC, New Delhi.

out extremely elementary character of the proofs and to get many unknown results by special choice of the generalized topology.

We recall some notions defined in [9]. Let X be a non-empty set, $expX$ denotes the power set of X. We call a class $\mu \subseteq expX$ a generalized topology [9], (briefly, GT) if $\varnothing \in \mu$ and union of elements of μ belongs to μ. A set X, with a GT μ on it is said to be a generalized topological space (briefly, GTS) and is denoted by (X, μ). The θ-closure [35] (resp. δ-closure [35]) of a subset A of a topological space (X, τ) is defined by $\{x \in X : clU \cap A \neq \varnothing$ for all $U \in \tau$ with $x \in U\}$ (resp. $\{x \in X : A \cap U \neq \varnothing$ for all regular open sets U containing $x\}$, where a subset A is called regular open if $A = int(cl(A))$). A is called δ-closed [35] (resp. θ-closed [35]) if $A = cl_\delta A$ (resp. $A = cl_\theta A$) and the complement of a δ-closed set (resp. θ-closed) set is known as a δ-open (resp. θ-open) set. A subset A of a topological space (X, τ) is called preopen [29] (resp. semiopen [23], δ-preopen [33], δ-semiopen [32], α-open [27], β-open [1], b-open [21]) if $A \subseteq int(cl(A))$ (resp. $A \subseteq cl(int(A))$,$A \subseteq int(cl_\delta A)$, $A \subseteq cl(int_\delta A)$, $A \subseteq int(cl(int(A)))$, $A \subseteq cl(int(clA))$, $A \subseteq cl(int(A)) \cup int(cl(A))$). We note that for any topological space (X, τ), the collection of all open sets denoted by τ (preopen sets denoted by $PO(X)$, semi-open sets denoted by $SO(X)$, δ-open sets denoted by $\delta O(X)$, δ-preopen sets denoted by δ-$PO(X)$, δ-semiopen sets denoted by δ-$SO(X)$, α-open sets denoted by $\alpha O(X)$, β-open sets denoted by $\beta O(X)$, θ-open sets denoted by $\theta O(X)$, b-open sets denoted by $BO(X)$ or $\gamma O(X)$) forms a GT.

For a GTS (X, μ), the elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [9, 10]). Obviously in a topological space (X, τ), if one takes τ as the GT, then c_μ becomes equivalent to the usual closure operator. Similarly, c_μ becomes pcl, scl, cl_δ, pcl_δ, scl_δ, cl_α, cl_β, bcl if μ stands for $PO(X)$ (resp. $SO(X)$, $\delta O(X)$, δ-$PO(X)$, δ-$SO(X)$, $\alpha O(X)$, $\beta O(X)$, $BO(X)$ or $\gamma O(X)$).

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma : expX \to expX$ is said to be idempotent iff $A \subseteq B \subseteq X$ implies $\gamma(\gamma(A))$ $= \gamma(A)$ and monotonic iff $\gamma(A) \subseteq \gamma(B)$. It is also well known from [10, 11] that if g is a GT on X and $A \subseteq X$, $x \in X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \varnothing$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

In this paper we introduce the concepts of μg-closed sets and μg-open sets. It is shown that many results in previous papers can be considered as special cases of our results.

2. Properties of μg-closed sets

Definition 2.1. Let (X, μ) be a GTS. Then a subset A of X is called a μ-generalized closed set (or in short, μg-closed set) iff $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ where U is μ-open in X. The complement of a μg-closed set is called a μg-open set.

Remark 2.2.

(i) If (X, τ) is a topological space, the definition of g-open set [22] (resp. sg-open set [4, 2], pg-open set [25], $g\alpha$-open set [27], δg^*-open set [13], bg-open set [21] or γg-open set [17]) can be obtained by taking $\mu = \tau$ (resp. $SO(X)$, $PO(X)$, $\alpha O(X)$, $\delta O(X)$, $\gamma O(X)$).

(ii) Every μ-open set in a GTS (X, μ) is μg-open. In fact, if A is a μ-open set in (X, μ), then $X \setminus A$ is a μ-closed set. Let $X \setminus A \subseteq U \in \mu$. Then $c_\mu(X \setminus A) = X \setminus A \subseteq U$. Thus $X \setminus A$ is a μg-closed set and hence A is a μg-open set.

The converse of Remark 2.2(ii) is not true as seen from the next example :

Example 2.3. *Let $X = \{a, b, c\}$ and $\mu = \{\varnothing, X, \{a\}, \{b, c\}, \{a, c\}\}$. Then (X, μ) is a GTS. It is easy to verify that $\{c\}$ is μg-open in (X, μ) but not μ-open.*

The next two examples show that the union (intersection) of two μg-open sets is not in general μg-open.

Example 2.4.

(a) *Let $X = \{a, b, c\}$ and $\mu = \{\varnothing, X, \{a\}\}$. Then (X, μ) is a GTS. It can be shown that if $A = \{b\}$ and $B = \{c\}$, then A and B are two μg-open sets but $A \cup B = \{b, c\}$ is not a μg-open set.*

(b) *Let $X = \{a, b, c, d\}$ and $\mu = \{\varnothing, X, \{a, b\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\}$. Then (X, μ) is a GTS. It follows from Remark 2.2(ii) that $\{a, b\}$ and $\{a, c, d\}$ are two μg-open sets but it is easy to check that their intersection $\{a\}$ is not μg-open.*

Theorem 2.5. *A subset A of a GTS (X, μ) is μg-closed iff $c_\mu(A) \setminus A$ contains no non-empty μ-closed set.*

Proof. Let F be a μ-closed subset of $c_\mu(A) \setminus A$. Then $A \subseteq F^c$ (where F^c denotes as usual the complement of F). Hence by μg-closedness of A, we have $c_\mu(A) \subseteq F^c$ or $F \subseteq (c_\mu(A))^c$. Thus $F \subseteq c_\mu(A) \cap (c_\mu(A))^c = \varnothing$, i.e., $F = \varnothing$.

Conversely, suppose that $A \subseteq U$ where U is μ-open. If $c_\mu(A) \nsubseteq U$, then $c_\mu(A) \cap U^c$ $(\neq \varnothing)$ is a μ-closed subset of $c_\mu(A) \setminus A$ - a contradiction. Hence $c_\mu(A) \subseteq U$. ☐

Theorem 2.6. *If a μg-closed subset A of a GTS (X, μ) be such that $c_\mu(A) \setminus A$ is μ-closed, then A is μ-closed.*

Proof. Let A be a μg-closed subset such that $c_\mu(A) \setminus A$ is μ-closed. Then $c_\mu(A) \setminus A$ is a μ-closed subset of itself. Then by Theorem 2.5, $c_\mu(A) \setminus A = \varnothing$ and hence $c_\mu(A) = A$, showing A to be a μ-closed set. ☐

That the converse is false follows from the following example.

Example 2.7. *Let $X = \{a, b, c\}$ and $\mu = \{\varnothing, \{a\}, \{a, b\}\}$. Then (X, μ) is a GTS. It is easy to observe that $\{b, c\}$ is μ-closed and hence a μg-closed set (by Remark 2.2), but $c_\mu(A) \setminus A = \varnothing$, which is not μ-closed.*

Theorem 2.8. *Let A be a μg-closed set in a GTS (X, μ) and $A \subseteq B \subseteq c_\mu(A)$. Then B is μg-closed.*

Proof. Let $B \subseteq U$, where U is μ-open in (X, μ). Since A is μg-closed and $A \subseteq U$, $c_\mu(A) \subseteq U$. Now, $B \subseteq c_\mu(A) \Rightarrow c_\mu(B) \subseteq c_\mu(A)$. So $c_\mu(B) \subseteq U$. □

Theorem 2.9. *In a GTS (X, μ), $\mu = \Omega$ (the collection of all μ-closed sets) iff every subset of X is μg-closed.*

Proof. Suppose $\mu = \Omega$ and A $(\subseteq X)$ be such that $A \subseteq U \in \mu$. Then $c_\mu(A) \subseteq c_\mu(U) = U$ and hence A is μg-closed.

Conversely, suppose that every subset of X is μg-closed. Let $U \in \mu$. Then $U \subseteq U$ and by μg-closedness of U, we have $c_\mu(U) \subseteq U$, i.e., $U \in \Omega$. Thus $\mu \subseteq \Omega$.

Now, if $F \in \Omega$ then $F^c \in \mu$, so $F^c \in \Omega$ (as $\mu \subseteq \Omega$), i.e., $F \in \mu$. □

Theorem 2.10. *A subset A of a GTS (X, μ) is μg-open iff $F \subseteq i_\mu(A)$, whenever F is μ-closed and $F \subseteq A$.*

Proof. Obvious and hence omitted. □

Theorem 2.11. *A set A is μg-open in a GTS (X, μ) iff $U = X$ whenever U is μ-open and $i_\mu(A) \cup A^c \subseteq U$.*

Proof. Suppose U is μ-open and $i_\mu(A) \cup A^c \subseteq U$. Now, $U^c \subseteq (i_\mu(A))^c \cap A = c_\mu(X \setminus A) \setminus (X \setminus A)$. Since U^c is μ-closed and $X \setminus A$ is μg-closed, by Theorem 2.5, $U^c = \varnothing$, i.e., $U = X$.

Conversely, let F be a μ-closed set and $F \subseteq A$. Then by Theorem 2.10, it is enough to show that $F \subseteq i_\mu(A)$. Now, $i_\mu(A) \cup A^c \subseteq i_\mu(A) \cup F^c$, where $i_\mu(A) \cup F^c$ is μ-open. Hence by the given condition, $i_\mu(A) \cup F^c = X$, i.e., $F \subseteq i_\mu(A)$. □

Theorem 2.12. *A subset A of a GTS (X, μ) is μg-closed iff $c_\mu(A) \setminus A$ is μg-open.*

Proof. Suppose A is μg-closed and $F \subseteq c_\mu(A) \setminus A$, where F is a μ-closed subset of X. Then by Theorem 2.5, $F = \varnothing$ and hence $F \subseteq i_\mu[c_\mu(A) \setminus A]$. Then by Theorem 2.10, $c_\mu(A) \setminus A$ is μg-open.

Conversely, suppose that $A \subseteq U$ where U is μ-open. Now, $c_\mu(A) \cap U^c \subseteq c_\mu(A) \cap A^c = c_\mu(A) \setminus A$. Since $c_\mu(A) \cap U^c$ is μ-closed and $c_\mu(A) \setminus A$ is μg-open, $c_\mu(A) \cap U^c = \varnothing$ (by Theorem 2.5). Thus $c_\mu(A) \subseteq U$, i.e., A is μg-closed. □

Definition 2.13. A GTS (X, μ) is said to be

(i) μ-T_0 [34] iff $x, y \in X$, $x \neq y$ implies the existence of $K \in \mu$ containing precisely one of x and y.

(ii) μ-T_1 [34] iff $x, y \in X$, $x \neq y$ implies the existence of $K, K^1 \in \mu$ such that $x \in K$, $y \notin K$ and $x \notin K^1$, $y \in K^1$.

(iii) μ-$T_{1/2}$ iff every μg-closed set is μ-closed.

Remark 2.14. A topological space (X,τ) is T_i [16] (resp. *semi*-T_i [4], *pre*-T_i [25], α-T_i [28], δ-T_i [13], b-T_i [21]) for $i = 0, 1/2, 1$ by taking $\mu = \tau$ (resp. $SO(X)$, $PO(X)$, $\alpha O(X)$, $\delta O(X)$, $BO(X)$ or $\gamma O(X)$).

Theorem 2.15. *If a GTS (X,μ) is μ-$T_{1/2}$ then it is μ-T_0.*

Proof. Suppose that (X,μ) is not a μ-T_0 space. Then there exist distinct points x and y in X such that $c_\mu(\{x\}) = c_\mu(\{y\})$. Let $A = c_\mu(\{x\}) \cap \{x\}^c$. We shall show that A is μg-closed but not μ-closed. Suppose that $A \subseteq V \in \mu$. We have to show that $c_\mu(A) \subseteq V$. Thus it is enough to show that $c_\mu(\{x\}) \subseteq V$ (as $A \subseteq c_\mu(\{x\})$). Again, since $c_\mu(\{x\}) \cap \{x\}^c = A \subseteq V$, we need only to show that $x \in V$. In fact, if $x \notin V$, then $y \in c_\mu(\{x\}) \subseteq V^c$ (as V^c is μ-closed). So $y \in A \subseteq V^c$ and hence $y \in V \cap V^c$ - a contradiction.
If $x \in U \in \mu$, then $U \cap A \supseteq \{y\} \neq \varnothing$, and hence $x \in c_\mu(A)$. Clearly, $x \notin A$ and thus A is not μ-closed. □

Example 2.16. *Let $X = \{a,b,c,d\}$ and $\mu = \{\varnothing, X, \{a,b\}, \{a,c,d\}, \{a,b,d\}, \{b,c,d\}\}$. Then (X,μ) is a GTS. Clearly, this GTS is μ-T_0 and it can be shown that the collection of all μg-open sets are $\{\varnothing, X, \{d\}, \{a,b\}, \{a,c,d\}, \{a,b,d\}, \{b,c,d\}\}$. Thus this space is not μ-$T_{1/2}$.*

Theorem 2.17. *If a GTS (X,μ) is μ-T_1 then it is μ-$T_{1/2}$.*

Proof. Suppose that A is a subset of X which is not μ-closed. Take $x \in c_\mu(A) \setminus A$. Then $\{x\} \subseteq c_\mu(A) \setminus A$ and $\{x\}$ is μ-closed (as (X,μ) is μ-T_1). Thus by Theorem 2.5, A is not μg-closed. □

Example 2.18. *Let $X = \{a,b,c,d\}$ and $\mu = \{\varnothing, X, \{d\}, \{a,b\}, \{b,c,d\}, \{a,c,d\}, \{a,b,d\}\}$. Then (X,μ) is a GTS. It is easy to verify that (X,μ) is μ-$T_{1/2}$ but not μ-T_1.*

Definition 2.19. A GTS (X,μ) is said to be μ-symmetric iff for each $x, y \in X$, $x \in c_\mu(\{y\}) \Rightarrow y \in c_\mu(\{x\})$.

Remark 2.20. It is easy to check that the above definition of a μ-symmetric space GT unifies the existing definitions of δ-symmetric space [8], (δ,p)-symmetric space [5], α-symmetric [6], δ-semi symmetric space [7] if (X,τ) is a topological space and $\mu = \delta O(X)$, δ-$PO(X)$, $\alpha O(X)$, δ-$SO(X)$ respectively.

Theorem 2.21. *A GTS (X,μ) is μ-symmetric iff $\{x\}$ is μg-closed for each $x \in X$.*

Proof. Let $\{x\} \subseteq U \in \mu$ and (X,μ) be μ-symmetric but $c_\mu(\{x\}) \nsubseteq U$. Then $c_\mu(\{x\}) \cap U^c \neq \varnothing$. Let $y \in c_\mu(\{x\}) \cap U^c$. Then $x \in c_\mu(\{y\}) \subseteq U^c \Rightarrow x \notin U$ - a contradiction.

Conversely, let for each $x \in X$, $\{x\}$ is μg-closed and $x \in c_\mu(\{y\}) \subseteq (c_\mu(\{x\}))^c$ (as $\{y\}$ is μg-closed). Thus $x \in (c_\mu(\{x\}))^c$ - a contradiction. □

Corollary 2.22. *If a GTS (X,μ) is μ-T_1 then it is μ-symmetric.*

Example 2.23. *Let $X = \{a,b\}$ and $\mu = \{\varnothing, X\}$. Then (X,μ) is a μ-symmetric space which is not μ-T_1.*

Theorem 2.24. *A GTS (X,μ) is μ-symmetric and μ-T_0 iff (X,μ) is μ-T_1.*

Proof. If (X,μ) is μ-T_1 then it is μ-symmetric (by Corollary 2.22) and μ-T_0 (by Definition 2.13).

Conversely, let (X,μ) be μ-symmetric and μ-T_0. We shall show that (X,μ) is μ-T_1. Let $x,y \in X$ and $x \neq y$. Then by μ-T_0-ness of (X,μ), there exists $U \in \mu$ such that $x \in U \subseteq \{y\}^c$. Then $x \notin c_\mu(\{y\})$ and hence $y \notin c_\mu(\{x\})$. Thus there exists $V \in \mu$ such that $y \in V$ and $x \notin V$. Thus (X,μ) is μ-T_1. $\square$

Theorem 2.25. *If (X,μ) is μ-symmetric, then (X,μ) is μ-T_0 iff (X,μ) is μ-$T_{1/2}$ iff (X,μ) is μ-T_1.*

Proof. Follows from Theorem 2.24 and the fact that μ-$T_1 \Rightarrow \mu$-$T_{1/2} \Rightarrow \mu$-T_0. $\square$

3. Preservation of μg-closed sets

Definition 3.1. Let (X,μ_1) and (Y,μ_2) be two GTS's. A mapping $f : (X,\mu_1) \to (Y,\mu_2)$ is said to be

(i) (μ_1,μ_2) continuous [9] iff $f^{-1}(G_2) \in \mu_1$ for each $G_2 \in \mu_2$;
(ii) (μ_1,μ_2)-closed iff for any μ_1-closed subset A of X, $f(A)$ is μ_2-closed in Y.

Theorem 3.2. *Let (X,μ_1) and (Y,μ_2) be two GTS's and $f : (X,\mu_1) \to (Y,\mu_2)$ be (μ_1,μ_2)-continuous and (μ_1,μ_2)-closed mapping. If A is $\mu_1 g$-closed in X then $f(A)$ is $\mu_2 g$-closed in Y.*

Proof. Let $f(A) \subseteq G_2$, where G_2 is a μ_2-open set in Y. Then $A \subseteq f^{-1}(G_2)$, where $f^{-1}(G_2)$ is a μ_1-open set in X. Thus by $\mu_1 g$-closedness of A, $c_{\mu_1}(A) \subseteq f^{-1}(G_2)$. Thus $f(c_{\mu_1}(A)) \subseteq G_2$ and $f(c_{\mu_1}(A))$ is μ_2-closed in Y. It thus follows that $c_{\mu_2}(f(A)) \subseteq c_{\mu_2}(f(c_{\mu_1}(A))) = f(c_{\mu_1}(A)) \subseteq G_2$. Thus $f(A)$ is $\mu_2 g$-closed in Y. $\square$

Theorem 3.3. *Let (X,μ_1) and (Y,μ_2) be two GTS's and $f : (X,\mu_1) \to (Y,\mu_2)$ be a (μ_1,μ_2)-continuous and (μ_1,μ_2)-closed mapping. If B is a $\mu_2 g$-closed set in Y, then $f^{-1}(B)$ is $\mu_1 g$-closed in X.*

Proof. Suppose that B is a $\mu_2 g$-closed set in Y and $f^{-1}(B) \subseteq G_1$, where G_1 is μ_1-open in X. We shall show that $c_{\mu_1}(f^{-1}(B)) \subseteq G_1$. Now $f[c_{\mu_1}(f^{-1}(B)) \cap G_1^c] \subseteq c_{\mu_2}(B) \setminus B$ and by Theorem 2.5, $f[c_{\mu_1}(f^{-1}(B)) \cap G_1^c] = \varnothing$. Thus $c_{\mu_1}(f^{-1}(B)) \cap G_1^c = \varnothing$. Thus $c_{\mu_1}(f^{-1}(B)) \subseteq G_1$ and hence $f^{-1}(B)$ is $\mu_1 g$-closed in X. $\square$

Next two examples show that (μ_1,μ_2)-continuity and (μ_1,μ_2)-closedness in both of the above theorems are essential.

Example 3.4. *Let $X = \{a,b,c,d\}$, $\mu_1 = \{\varnothing, X, \{a,b\}, \{c,d\}, \{a,c,d\}, \{a,b,d\}\}$ and $\mu_2 = \{\varnothing, X, \{a,b\}, \{c,d\}, \{a,c,d\}\}$. Then (X,μ_1) and (X,μ_2) are two GTS's. Consider the identity mapping $f : (X,\mu_1) \to (X,\mu_2)$. It is easy to see* *that f is a (μ_1, μ_2)-continuous mapping which is not (μ_1, μ_2)-closed. The families of $\mu_1 g$-open and $\mu_2 g$-open sets are respectively $\{\varnothing, X, \{a\}, \{d\}, \{c,d\}, \{a,d\}, \{a,b\}, \{a,c,d\}, \{a,b,d\}\}$ and $\{\varnothing, X, \{a\}, \{c\}, \{d\}, \{c,d\}, \{a,d\}, \{a,b\}, \{a,c\}, \{a,c,d\}, \{a,b,d\}, \{a,b,c\}\}$. We note that $\{d\}$ is $g\mu_2$-closed but $f^{-1}(\{d\})$ is not $g\mu_1$-closed.*

Again, the identity map h defined by $h : (X, \mu_2) \to (X, \mu_1)$ is not a (μ_2, μ_1)-continuous mapping but it is (μ_2, μ_1)-closed. Clearly, $\{d\}$ is a $\mu_2 g$-closed set but $h(\{d\})$ is not a $\mu_1 g$-closed set.

Example 3.5. *Let $X = \{a,b,c,d\}$, $\mu_1 = \{\varnothing, X, \{a,b\}, \{c,d\}, \{a,c,d\}, \{a,b,d\}\}$ and $\mu_2 = \{\varnothing, X, \{a,b\}, \{a,b,d\}, \{a,c,d\}\}$. Then (X, μ_1) and (X, μ_2) are GTS's. Now, consider the identity map $f : (X, \mu_1) \to (X, \mu_2)$. It is easy to verify that f is a (μ_1, μ_2)-continuous mapping which is not (μ_1, μ_2)-closed. The family of $g\mu_1$-open and $g\mu_2$-open sets are respectively $\{\varnothing, X, \{a\}, \{d\}, \{c,d\}, \{a,d\}, \{a,b\}, \{a,c,d\}, \{a,b,d\}\}$ and $\{\varnothing, X, \{a\}, \{d\}, \{a,b\}, \{a,d\}, \{a,b,d\}, \{a,c,d\}\}$. We note that $\{a,b\}$ is $\mu_1 g$-closed but $f(\{a,b\})$ is not $\mu_2 g$-closed.*

Again, consider the identity map $h : (X, \mu_2) \to (X, \mu_1)$. Then, clearly h is a (μ_2, μ_1)-closed map which is not (μ_2, μ_1)-continuous. Clearly, $\{a,b\}$ is $\mu_1 g$-closed but $h^{-1}(\{a,b\})$ is not a $\mu_2 g$-closed set.

4. Properties of μ-regular and μ-normal spaces

Definition 4.1. A GTS (X, μ) is said to be μ-regular if for each μ-closed set F of X not containing x, there exist disjoint μ-open set U and V such that $x \in U$ and $F \subseteq V$.

Remark 4.2. Regular space, pre-regular space, semi-regular space, β-regular space, α-regular space are defined and studied in [16, 31, 15, 19, 20] respectively. The above definition gives a unified version of all these definitions if μ takes the role of τ, $PO(X)$, $SO(X)$, $\beta O(X)$, $\alpha O(X)$ respectively.

Theorem 4.3. *For a GTS (X, μ) the followings are equivalent:*

(a) *X is μ-regular.*
(b) *For each $x \in X$ and each $U \in \mu$ containing x, there exists $V \in \mu$ such that $x \in V \subseteq c_\mu(V) \subseteq U$.*
(c) *For each μ-closed set F of X, $\cap\{c_\mu(V) : F \subseteq V \in \mu\} = F$.*
(d) *For each subset A of X and each $U \in \mu$ with $A \cap U \neq \varnothing$, there exists a $V \in \mu$ such that $A \cap V \neq \varnothing$ and $c_\mu(V) \subseteq U$.*
(e) *For each non-empty subset A of X and each μ-closed subset F of X with $A \cap F = \varnothing$, there exist $U, V \in \mu$ such that $A \cap V \neq \varnothing$, $F \subseteq W$ and $W \cap V = \varnothing$.*
(f) *For each μ-closed set F with $x \notin F$ there exist $U \in \mu$ and a μg-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$.*
(g) *For each $A \subseteq X$ and each μ-closed set F with $A \cap F = \varnothing$ there exist a $U \in \mu$ and a μg-open set V such that $A \cap U \neq \varnothing$, $F \subseteq V$ and $U \cap V = \varnothing$.*
(h) *For each μ-closed set F of X, $F = \cap\{c_\mu(V) : F \subseteq V, V$ is μg-open$\}$.*

Proof. **(a) ⇒ (b) :** Let U be a μ-open set containing x. Then $x \notin X \setminus U$, where $X \setminus U$ is μ-closed. Then by (a) there exist $G, V \in \mu$ such that $X \setminus U \subseteq G$ and $x \in V$ and $G \cap V = \varnothing$. Thus $V \subseteq X \setminus G$ and so $x \in V \subseteq c_\mu(V) \subseteq X \setminus G \subseteq U$.

(b) ⇒ (c) : Let $X \setminus F \in \mu$ be such that $x \notin F$. Then by (b) there exists $U \in \mu$ such that $x \in U \subseteq c_\mu(U) \subseteq X \setminus F$. So, $F \subseteq X \setminus c_\mu(U) = V$ (say)$\in \mu$ and $U \cap V = \varnothing$. Thus $x \notin c_\mu(V)$. Hence $F \supseteq \cap\{c_\mu(V) : F \subseteq V \in \mu\}$.

(c) ⇒ (d) : Let $U \in \mu$ with $x \in U \cap A$. Then $x \notin X \setminus U$ and hence by (c) there exists a μ-open set W such that $X \setminus U \subseteq W$ and $x \notin c_\mu(W)$. We put $V = X \setminus c_\mu(W)$, which is a μ-open set containing x and hence $A \cap V \neq \varnothing$ (as $x \in A \cap V$). Now $V \subseteq X \setminus W$ and so $c_\mu(V) \subseteq X \setminus W \subseteq U$.

(d) ⇒ (e) : Let F be a μ-closed set as in the hypothesis of (e). Then $X \setminus F$ is a μ-open set and $(X \setminus F) \cap A \neq \varnothing$. Then there exists $V \in \mu$ such that $A \cap V \neq \varnothing$ and $c_\mu(V) \subseteq X \setminus F$. If we put $W = X \setminus c_\mu(V)$, then $F \subseteq W$ and $W \cap V = \varnothing$.

(e) ⇒ (a) : Let F be a μ-closed set not containing x. Then by (e), there exist $W, V \in \mu$ such that $F \subseteq W$ and $x \in V$ and $W \cap V = \varnothing$.

(a) ⇒ (f) : Obvious as every μ-open set is μg-open (by Remark 2.2).

(f) ⇒ (g) : Let F be a μ-closed set such that $A \cap F = \varnothing$ for any subset A of X. Thus for $a \in A$, $a \notin F$ and hence by (f), there exist a $U \in \mu$ and a μg-open set V such that $a \in U$, $F \subseteq V$ and $U \cap V = \varnothing$. So $A \cap U \neq \varnothing$.

(g) ⇒ (a) : Let $x \notin F$, where F is μ-closed. Since $\{x\} \cap F = \varnothing$, by (g) there exist a $U \in \mu$ and a μg-open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \varnothing$. Now put $V = i_\mu(W)$. Then $F \subseteq V$ (by Theorem 2.10) and $U \cap V = \varnothing$.

(c) ⇒ (h) : We have $F \subseteq \cap\{c_\mu(V) : F \subseteq V$ and V is μg-open$\} \subseteq \cap\{c_\mu(V) : F \subseteq V$ and V is μ-open$\} = F$.

(h) ⇒ (a) : Let F be a μ-closed set in X not containing x. Then by (h) there exists a μg-open set W such that $F \subseteq W$ and $x \in X \setminus c_\mu(W)$. Since F is μ-closed and W is μg-open, $F \subseteq i_\mu(W)$ (by Theorem 2.10). Take $V = i_\mu(W)$. Then $F \subseteq V$, $x \in X \setminus c_\mu(V) = U$ (say) (as $(X \setminus F) \cap V = \varnothing$) and $U \cap V = \varnothing$. □

Definition 4.4. A GTS (X, μ) is μ-normal [12] if for any pair of disjoint μ-closed subsets A and B of X, there exist disjoint μ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Remark 4.5. Normal space, pre-normal space, semi-normal space, α-normal space, β-normal space, γ-normal space are defined and studied in [16, 31, 2, 20, 19, 17] respectively. The above definition gives a unified version of all these definitions if μ takes the role of τ, $PO(X)$, $SO(X)$, $\alpha O(X)$, $\beta O(X)$ respectively.

Theorem 4.6. *For a GTS (X, μ) the followings are equivalent:*

(a) *X is μ-normal;*
(b) *For any pair of disjoint μ-closed sets A and B, there exist disjoint μg-open sets U and V such that $A \subseteq U$ and $B \subseteq V$;*
(c) *For every μ-closed set A and μ-open set B containing A, there exists a μg-open set U such that $A \subseteq U \subseteq c_\mu(U) \subseteq B$;*
(d) *For every μ-closed set A and every μg-open set B containing A, there exists a μ-open set U such that $A \subseteq U \subseteq c_\mu(U) \subseteq i_\mu(B)$;*
(e) *For every μg-closed set A and every μ-open set B containing A, there exists a μ-open set U such that $A \subseteq c_\mu(A) \subseteq U \subseteq c_\mu(U) \subseteq B$.*

Proof. **(a) $\Rightarrow$ (b) :** Let A and B be two disjoint μ-closed subsets of X. Then by μ-normality of X, there exist disjoint μ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Then U and V are μg-open by Remark 2.2.

(b) $\Rightarrow$ (c) : Let A be a μ-closed set and B be a μ-open set containing A. Then A and B^c are two disjoint μ-closed sets in X. Then by (b), there exist disjoint μg-open sets U and V such that $A \subseteq U$ and $B^c \subseteq V$. Thus $A \subseteq U \subseteq X \setminus V \subseteq B$. Again, since B is μ-open and $X \setminus V$ is μg-closed, $c_\mu(X \setminus V) \subseteq B$. Hence $A \subseteq U \subseteq c_\mu(U) \subseteq B$.

(c) $\Rightarrow$ (d) : Let A be a μ-closed subset of X and B be a μg-open set containing A. Since B is a μg-open set containing A and A is μ-closed, by Theorem 2.10, $A \subseteq i_\mu(B)$. Thus by (c) there exists a μg-open set U such that $A \subseteq U \subseteq c_\mu(U) \subseteq i_\mu(B)$.

(d) $\Rightarrow$ (e) : Let A be a μg-closed set and B be a μ-open set in X containing A. $A \subseteq B$ implies $c_\mu(A) \subseteq B$, where $c_\mu(A)$ is μ-closed and B is μg-open (as B is μ-open). Then by (d), there exists a μ-open set U such that $A \subseteq c_\mu(A) \subseteq U \subseteq c_\mu(U) \subseteq i_\mu(B)$. Thus $A \subseteq c_\mu(A) \subseteq U \subseteq c_\mu(U) \subseteq B$.

(e) $\Rightarrow$ (a) : Let A and B be two disjoint μ-closed subsets of X. Then A is μg-closed and $A \subseteq X \setminus B$, where $X \setminus B$ is μ-open. Thus by (e), there exists a μ-open set U such that $A \subseteq c_\mu(A) \subseteq U \subseteq c_\mu(U) \subseteq X \setminus B$. Thus $A \subseteq U$, $B \subseteq X \setminus c_\mu(U)$ and $U \cap (X \setminus c_\mu(U)) = \varnothing$. Hence X is μ-normal. □

Remark 4.7. (a) By using $\mu = \tau$ [22] (resp. $PO(X)$ [25], $SO(X)$ [4], $\alpha O(X)$ [27], $\delta O(X)$ [13], $BO(X)$ [17, 21]) on a topological space (X, τ) several modifications of g-closed sets (resp. sg-closed sets, $g\alpha$-closed sets, δg^*-closed sets, bg-closed sets) are introduced and investigated. Since each of τ, $PO(X)$, $SO(X)$, $\alpha O(X)$, $\delta O(X)$, $BO(X)$ forms a GT on X, the characterizations of each of the families are obtained from μg-open set.

(b) The definition of many other similar types of generalized closed sets can be defined on a topological space (X, τ) from the definition of μg-closed set by replacing μ by the corresponding GT on X.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ. **12** (1983), 77–90.

[2] S. P. Arya and T. M. Nour, *Characterizations of s-normal spaces*, Indian J. Pure Appl. Math. **21**, no. 8 (1990), 717–719.

[3] Y. Beceren and T. Noiri, *Some functions defined by semi-open sets and β-open sets*, Chaos Solitons Fractals **36** (2008), 1225–1231.

[4] P. Bhattacharyya and B. K. Lahiri, *Semi-generalized closed sets in topology*, Indian J. Math. **29** (1987), 376–382.

[5] M. Caldas, T. Fukutake, S. Jafari and T. Noiri, *Some applications of δ-preopen sets in topological spaces*, Bull. Inst. Math. Acad. Sinica **33**, no. 3 (2005), 261–275.

[6] M. Caldas, D. N. Georgiou and S. Jafari, *Characterizations of low separation axioms via α-open sets and α-closure operator*, Bull. Soc. Paran. Mat. **(3)21** (2003), 97–111.

[7] M. Caldas, D. N. Georgiou and T. Noiri, *More on δ-semiopen sets*, Note di Mathematica **22**, no. 2 (2003), 113–126.

[8] M. C. Caldas and S. Jafari, *On δD-sets and associated weak separation axioms*, Bull. Malaysian Math. Soc. **25** (2002), 173–185.

[9] Á. Császár, *Generalized topology, generalized continuity*, Acta Math. Hungar. **96** (2002), 351–357.

[10] Á. Császár, *Generalized open sets in generalized topologies*, Acta Math. Hungar. **106** (2005), 53–66.

[11] Á. Császár, *δ- and θ-modifications of generalized topologies*, Acta Math. Hungar. **120** (2008), 275–279.

[12] Á. Császár, *Normal generalized topologies*, Acta Math. Hungar. **115**, no. 4 (2007), 309–313.

[13] J. Dontchev, I. Arokiarani and K. Balachandran, *On generalized δ-closed sets and almost weakly Hausdroff spaces*, Questions Answers Gen. Topology **18**, no. 1 (2000), 17–30.

[14] C. Dorsett, *Semi normal spaces*, Kyungpook Math. J. **25**(1985), 173–180.

[15] C. Dorsett, *Semi regular spaces*, Soochow J. Math. **8** (1982), 45–53.

[16] J. Dugunji, *Topology*, Allyn and Bacon, Boston, 1966.

[17] E. Ekici, *On γ-normal spaces*, Bull. Math. Soc. Sci. Math. Roumanie **50 (98)** (2007), 259–272.

[18] E. Ekici, *On almost πgp-continuous functions*, Chaos Solitons Fractals **32** (2007), 1935–1944.

[19] S. Jafari and T. Noiri, *On β-quasi irresolute functions*, Mem. Fac. Sci. Kochi Univ.(Math.) **21** (2000), 53–62.

[20] A. Kar and P. Bhattacharyya, *Bitopological α-compact spaces*, Riv. Mat. Parma **7**, no. 1 (2002), 159–176.

[21] A. Keskin and T. Noiri, *On bD-sets and associated separation axioms*, Bull. Iran. Math. Soc. **35**, no. 1 (2009), 179–198.

[22] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo **19** (1970), 89–96.

[23] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.

[24] S. N. Maheshwari and R. Prasad, *On s-regular spaces*, Glasnik Mat. **10 (30)** (1975), 347–350.

[25] H. Maki, J. Umehara and T. Noiri, *Every topological space is pre-$T_{1/2}$*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math. **17** (1996), 33–42.

[26] H. Maki, J. Umehara and K. Yamamura, *Characterizations of $T_{1/2}$-spaces using generalized ν-sets*, Indian J. Pure Appl. Math. **19**, no. 7 (1998), 634–640.
[27] H. Maki, R. Devi and K. Balachandran, *Associated topologies of generalizrd α-closed sets and α-generalized closed sets*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math. **15** (1994), 51–63.
[28] H. Maki, R. Devi and K. Balachandran, *Generalized α-closed sets in topology*, Bull. Fukuoka Univ. Ed., Part-III **42** (1993), 13–21.
[29] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.
[30] T. Noiri, *Semi-normal spaces and some functions*, Acta Math. Hungar. **65**, no. 3 (1994), 305-311.
[31] M. C. Pal and P. Bhattacharyya, *Feeble and strong forms of preirresolute functions*, Bull. Malaysian Math. Soc. **19** (1996), 63–75.
[32] J. H. Park, D. S. Song and R. Saadati, *On generalized δ-semiclosed sets in topological spaces*, Chaos Solitons Fractals **33** (2007), 1329–1338.
[33] S. Raychaudhuri and M. N. Mukherjee, *On δ-almost continuity and δ-preopen sets*, Bull. Inst. Math. Acad. Sinica **21** (1993), 357–366.
[34] P. Sivagami and D. Sivaraj, $\bigvee$ *and* $\bigwedge$*- sets of generalized topologies*, Scientia Magna **5**, no. 1 (2009), 83–93.
[35] N. V. Veličko, *H-closed topological spaces*, Mat. Sb. **70** (1966), 98–112.

(Received December 2010 – Accepted July 2011)

Bishwambhar Roy (bishwambhar_roy@yahoo.co.in)
Department of Mathematics, Women's Christian College, 6 Greek Church Row, Kolkata-700026, India.