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Aspects of RG-spaces

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Abstract

A Tychonoff space X which satisfies the property that $G(X) = C(X_\delta)$ is called an RG-space, where $G(X)$ is the minimal regular ring extension of C(X) inside $F(X)$, the ring of all functions from X to R, and X_δ is the topology on X generated by its G_δ-sets. We correct an error that we found in the proof of [19, Theorem 3.4] *and show that RG-spaces must satisfy a finite dimensional condition.*
We also introduce a new class of topological spaces which we call almost k-Baire spaces. The class of almost Baire spaces is a particular instance. We show that every RG-space is an almost Baire space but not necessarily a Baire space. However RG-spaces of countable pseudocharacter must be Baire and, furthermore, their dense sets have dense interiors.

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1. Introduction

Let X be a Tychonoff space, let $C(X)$ be the ring of real-valued continuous functions defined on X, and let $F(X)$ be the ring of all real-valued functions defined on X. It is clear that both of these are commutative semiprime rings sharing the same identity. Moreover the ring $F(X)$ is regular (in the sense of Von Neumann). The (unique) smallest regular ring lying between $C(X)$ and $F(X)$, denoted $G(X)$, was studied intensively in [19].

For any function $f \in F(X)$, the quasi-inverse of f is given by:

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$$f^*(x) = \begin{cases} 0 & \text{if } x \in Z(f) \\ \frac{1}{f(x)} & \text{if } x \in coz(f) \end{cases}$$

where $Z(f) = \{x : f(x) = 0\}$ and $coz(f) = X - Z(f)$. A subset of X is called a zero-set (cozero-set) if it has the form $Z(f)$ ($coz(f)$) for some function $f \in C(X)$. The set of all zero-sets in X is denoted $Z(X)$. For a topological space X, a point p is called a P-point if p is in the interior of each zero-set containing it. A topological space X is called a P-space if every point in X is a P-point [10, 4L]. A space X is a P-space if and only if $C(X)$ is a regular ring, or equivalently if every G_δ-set is open [10, 4J]. A space X is called an almost P-space if every non-empty zero-set has a non-empty interior. Details on almost P-spaces appear in [15]. Algebraic background, for example on regular rings and quasi-inverses, can be found in [13].

2. RG-spaces

If (X, τ) is a topological space then the family of its G_δ-sets forms an open base for a (potentially) stronger topology on X known in the literature as the δ-topology. It is denoted τ_δ and the new space is written as X_δ. In [12] it was shown that $G(X) = \{\ \sum_{i=1}^n f_i g_i^* :\ \ f_i, g_i \in C(X),\ n \geq 1\}$, so each function in $G(X)$ is continuous in the δ-topology. Thus $G(X)$ contains $C(X)$ and is a subring of $C(X_\delta)$.

Definition 2.1 ([12])**.** Let X be a topological space. Then X is called a **regular good** space, denoted an **RG**-space, if $G(X) = C(X_\delta)$.

It is clear from the definition of RG-spaces that every P-space is an RG-space because if X is a P-space, then $G(X) = C(X) = C(X_\delta)$. There are many examples and non-examples of RG-spaces in the literature, for example in [12] and [19]. Interestingly, for any space X, whether RG or not, any function in $G(X)$ is continuous on a dense open subset of X [12].

3. A Theorem revisited and repaired

We recall from [10] that a prime ideal P is called a z-ideal if $a \in P$ whenever $b \in P$ and $Z(a) = Z(b)$.

Definition 3.1. By the (Krull) z-dimension of a maximal ideal we mean the supremum of the lengths of chains of prime z-ideals lying in it. The Krull z-dimension of $C(X)$ is the supremum of the dimensions of the maximal ideals of $C(X)$.

Our first goal in this note is to revisit [19, Theorem 3.4]. For completeness let us recall the result that was claimed.

Theorem 3.2. *If the Krull z-dimension of $C(X)$ is infinite then X is not an* RG *space and* $\mathrm{rg}(X) = \infty$.

Regrettably, the proof given in [19] is mistaken. The assertion in the last paragraph of the proof that $clB_{k,t}$ contains $Q_{k,t}$ and no other prime from the array is not justified because a countably infinite operation is used in defining $B_{k,t}$. Below we give a correct proof of the result.

(The method) We use the following idea. A space X can be shown not to be RG if one can apply the following technique to its set of prime z-ideals. By **"the method"** we mean the selection of a countably infinite array of prime z-ideals belonging to disjoint chains, say D_n, of finite but globally unbounded lengths. One also needs a countably infinite family of disjoint clopen sets in X_δ such that each clopen set contains precisely one chain from the array in its $\beta(X_\delta)$-closure. Then an appropriate function is defined using the method of [5, Theorem 3.1 part (3)]. This is done by assigning fixed values (taken from $C(X)$) on each clopen set, and letting it be zero elsewhere. The constants chosen come from the (proper) containment of a prime in its successor in its chain. By virtue of the topology on X_δ the global function thus defined will be in $C(X_\delta)$ but it will not lie in $G(X)$ because of the unbounded nature of the lengths of the chains D_n.

Lemma 3.3. *Suppose the following three conditions hold for a space X.*

1. *X is an RG-space,*
2. *$C(X)$ has a chain of prime z-ideals of length n, and*
3. *There is a subpace Y_δ that is clopen in X_δ such that all of the primes in the chain lie in the closure of Y_δ in $\beta(X_\delta)$.*

Then there is a function $h \in G(X)$ that vanishes on $X - Y$ and has regularity degree at least n.

Proof. Construct a function k as in [19, Case 1, page 80]. Then get h by multiplying k by the idempotent that is 1 on Y and 0 and $X - Y$. The idempotent is in $G(X) = C(X_\delta)$ so the product is as well. □

Lemma 3.4. *Suppose that X is of infinite prime Krull z-dimension. Then X contains a family of pairwise disjoint chains D_n indexed by n, so that the length of the chain D_n is equal to n.*

Proof. This is done by an easy induction. The case $n = 1$ is clear. Suppose we have chosen disjoint chains $D_1, ...D_n$. Since the lengths of the chains is unbounded, there is one of length at least $n(n + 1)/2 + (n + 1)$. Even if this chain has some overlap with the chains $D_1, ..., D_n$ there must be $n + 1$ primes in it that do not occur in the previous chains. These $n + 1$ primes give us a new chain D_{n+1} disjoint from all of the previous chains. □

Lemma 3.5. *Assume, if possible, that X is an RG-space of infinite Krull z-dimension. Then X contains a countably infinite family of disjoint subspaces Y_n, each clopen in the δ-topology such that each Y_n supports a function $h_n \in C(X_\delta)$ that vanishes outside of Y_n and has regularity degree at least n.*

Proof. Since X is of infinite dimension Lemma 3.4 applies. Let D_n have the same meaning as in Lemma 3.4.

We construct the disjoint clopen sets inductively as follows.

First step. The sets D_1 and D_2 are finite and disjoint in the Boolean space $T = Spec(X_\delta) = \beta(X_\delta)$ so they can be separated by complementary clopen sets U and $T - U$, say $D_1 \subset U$, $D_2 \subset T - U$. The subspaces $A = U \cap X_\delta$ and $B = (T - U) \cap X_\delta$ are clopen subsets of X_δ so the union of their T-closures is T. They are also RG-spaces in the (relative) X-topology by [12, Theorem 2.3(a)].

We know that D_1 is in the T-closure of A, D_2 is in the T-closure of B, and that the other D_n lie wholely or partly in the T-closures of A and B and certainly in the union of the latter. Although we know that A and B are RG-spaces with union X we do not assert that (at least) one of them is of infinite dimension. This would hold if we knew that X were the free union of A and B by [16, Prop 4.7] but we do not know this. Fortunately we only want to be able to apply Lemma 3.3. Observe that one (or both) of A and B is "infinite" with respect to X in the following sense: it is not possible that both A and B have a finite bound on the cardinalities of their intersections with the chains $D_3, D_4, ...$ If it is B that has this property then we let $Y_1 = A$ and continue to the next step in which we will be working inside B. If, on the other hand, there is a global finite bound on the cardinalities of the intersections of the chains $D_3, D_4, ...$ with B then we let $Y_1 = B$ in the knowledge that B has a chain of length 2 and therefore one of length 1, and we continue with A. In both cases Y_1 is clopen in the δ-topology, has a chain of length 1 in its T-closure, and has a complement $X - Y_1$ with an "infinite" property.

Second step. We work with the space $X - Y_1$ which has disjoint chains of all finite lengths in its T-closure, in particular, one called S_2 of length 2 and one of length 3 called S_3.

Working in the T-closure, again, find a clopen subspace W of $T - cl_T(Y_1)$ that separates A_2 and A_3, say $A_2 \subset W, A_3 \subset T - cl_T(Y_1) - W$. If the T-closure of $((T - Y_1) - W) \cap X$ has the "infinite property" we let $Y_2 = W \cap X_\delta$. If not, we let $Y_2 = ((T - Y_1) - W) \cap X_\delta$. In both cases Y_2 is clopen in X_δ, has a chain of length 2 in its T-closure, and $(X - Y_1) - Y_2$ has the "infinite" property.

Now one simply continues the process to get the sequence Y_n. □

We can now present our main result of this section.

Theorem 3.6. *If X is an RG-space then X is of finite Krull z-dimension.*

Proof. Let Y_n denote the disjoint subsets given by Lemma 3.5. Each is clopen in X_δ. Let $Y = \cup Y_n$. By standard properties of P-spaces Y is also clopen. For each n let h_n be the function provided by Lemma 3.3 on Y. Let h be defined on X by, $h = h_n$ on Y_n for each n, and h is zero on $X - Y$. Then $h \in C(X_\delta)$ and $h \notin G(X)$ by the argument of [19, Theorem 3.1] (see also [19, Theorem 3.4)].

Remark 3.7. As was shown in [1, p82] the spectrum of the epimorphic hull is the set of prime d-ideals of $C(X)$ under the patch topology. Exactly the same arguments show that if dimension is measured using prime d-ideals, then an infinite dimension is inconsistent with having $H(X)$ be a ring of continuous functions when X is realcompact. Note as well that an infinite dimension when measured using prime d-ideals will also prevent X from being an RG space.

Remark 3.8. Notice that the proof of Lemma 3.5 contains the following fact of interest. If X is RG, and A is a subset of X which is clopen in X in the δ-topology, then A is also an RG-space. This is simply an application of [12, Theorem 2.3(a)]. There is no immediate converse however. The space Ψ is known not to be RG. It is discrete in the δ-topology, both N and $\Psi - N$ are discrete, and therefore RG, and the idempotent function on Ψ which is 1 on N and 0 on $\Psi - N$ is certainly in $G(\Psi)$. A converse can be achieved with lifting properties, like the C^*-embeddedness of A and its complement. We can get by with a bit less, as follows.

Proposition 3.9. *Let X be the union of two disjoint subspaces A and B, suppose A and B are RG-spaces which are G-embedded in X and suppose that the function e that is 1 on A and 0 on B lies in $G(X)$. Then X is an RG-space.*

Proof. The proof is straightforward. Take $f \in C(X_\delta)$. It suffices to see that both ef and $(1-e)f$ lie in $G(X)$. Let us check the result for ef. The function $f|A$ lies in $C(A_\delta)$ and therefore in $G(A)$. That means it lifts to a function $h \in G(X)$. Now $eh \in G(X)$ and coincides with ef. Similarly $(1-e)f$ coincides with $(1-e)k$ for some $k \in G(X)$ □

Corollary 3.10. *Suppose that a space X is the union of two disjoint Lindelof subspaces A and B which are RG one of which is open. Suppose further, that the characteristic function of A lies in $G(X)$. Then X is an RG-space.*

Proof. Suppose that A is the open subspace. Since it is Lindelof, it is a cozero set of X and is therefore G-embedded in X. The space B is closed in the normal space X and is therefore C-embedded, hence G-embedded in X as well. □

4. Nowhere separable spaces

The following theorem can be compared with [12, 2.2].

Theorem 4.1. *Let X be an RG-space and $(Z_n)_{n=1}^{\infty}$ be a sequence of nowhere dense zero-sets in X. Then $\bigcup_{n=1}^{\infty} Z_n$ is a nowhere dense subset.*

Proof. Let $S = \bigcup_{n=1}^{\infty} Z_n$, $A_1 = Z_1$ and $A_m = Z_m - (\bigcup_{i=1}^{m-1} Z_i)$ for each $m \geq 2$. Then by well-known properties of P-spaces, $\{A_n : n \in N\}$ is a collection of clopen subsets in X_δ, and therefore $\{A_n : n \in N\} \cup \{X - S\}$ is a clopen partition of X_δ. Let $f : X_\delta \longrightarrow R$ be defined by $f(A_n) = \{n+1\}$ for each $n \in N$ and $f(X - S) = \{1\}$. Then $f \in G(X)$, and since X is an RG-space there is a dense open subset D of X on which f is continuous. Now suppose that $cl(\bigcup_{n=1}^{\infty} Z_n)$ has an interior point p. Then there is an open subset U_p containing p such that $U_p \subseteq cl(\bigcup_{n=1}^{\infty} Z_n)$, which means that for each y in U_p and each neighborhood W_y of y we have $W_y \cap (\bigcup_{n=1}^{\infty} Z_n) \neq \phi$. Since D is a dense subset, then $D \cap U_p \neq \phi$. Let $y \in D \cap U_p$. There are two cases:

(1) If $f(y) = 1$, then there is an open neighborhood W_y of y such that $f(W_y) \subseteq (0, \frac{3}{2})$. So $W_y \cap (\bigcup_{n=1}^{\infty} Z_n) = \phi$, which is a contradiction.
(2) If $f(y) = k+1$, then $y \in A_k$, and therefore there is an open neighborhood W_y of y such that $f(W_y) \subseteq (k + \frac{2}{3}, k + \frac{4}{3})$. Then $W_y \subseteq A_k \subseteq Z_k$, which is also a contradiction. Thus $\bigcup_{n=1}^{\infty} Z_n$ is a nowhere dense subset of X. □

Recall that a topological space X is called **separable at a point** p if there exists an open set O containing p such that O is separable. A topological space X is called **nowhere separable** if X is not separable at any of its points. Details appear in [6].

It is an open question whether an RG-space must have almost P-points. There certainly are separable RG-spaces, even countable ones and these have isolated points. One does have the following implication.

Theorem 4.2. *If X is an RG-space with no almost P-points then X is nowhere separable.*

Proof. Let X be an RG-space with no almost P-points that is separable at some point. Then there is a countable subset $\{a_n : n \in N\}$ such that $int(cl(\bigcup_{n=1}^{\infty}\{a_n\})) \neq \phi$. For each a_n pick a nowhere dense zero-set Z_n such that $a_n \in Z_n$. Then $int(cl(\bigcup_{n=1}^{\infty} Z_n)) \neq \phi$ which contradicts theorem 4.1. Thus X is a nowhere separable. □

5. Spaces of countable pseudocharacter and Blumberg spaces

Definition 5.1. A topological space X is said to be of **countable pseudocharacter** if every point in X is a G_δ-set. (This is equivalent to saying that X_δ is discrete.)

Recall that a topological space X is called **Blumberg** if every real-valued function defined on X has a continuous restriction to a dense subset [22]. In [4] J. C. Bradford and C. Goffman proved that every Blumberg space is Baire and in [14] R. Levy showed that there is consistently a compact Hausdorff, and therefore Baire, space which is not Blumberg.

As noted below an RG-space need not be Baire and hence need not be Blumberg. Hoever, we can obtain the Blumberg property for one particular class of RG-spaces as follows.

Theorem 5.2. *Let X be an RG-space of countable pseudocharacter. Then X is Blumberg and hence also Baire.*

Proof. Suppose X is an RG-space of countable pseudocharacter. Since X_δ is a discrete space, every function on X is in $G(X)$, and thus by [12, Prop. 2.1] every real-valued function defined on X can be restricted continuously to a dense open subset. So X satisfies the Blumberg property and hence X is a Baire space. □

Definition 5.3 (cf [3])**.** Let X be a topological space. Then X is called an **almost resolvable** space if it is a countable union of sets with void interiors.

Theorem 5.4. *If X is an RG-space of countable pseudocharacter then X is not an almost resolvable space.*

Proof. Let X be an RG-space of countable pseudocharacter and suppose X is almost resolvable. Then there is a countable collection $\{F_n : n \in N\}$ of sets each with void interior such that $X = \bigcup_{n=1}^{\infty} F_n$. Let $A_1 = F_1$ and $A_n = F_n - (\bigcup_{m=1}^{n-1} F_m)$ for each $n \geq 2$. Then $\{A_n : n \in N\}$ is a a countable collection of disjoint sets with void interior and $X = \bigcup_{n=1}^{\infty} A_n$. Define $f : X \longrightarrow R$ by $f(A_n) = n$ for each $n \in N$. Since X_δ is a discrete space, then $f \in C(X_\delta) = G(X)$, which implies that f is continuous on a dense open subset D, which is a contradiction because f is not continuous at any point. Thus X is not almost resolvable. $\square$

We know from Theorem 5.2 that RG-spaces of countable pseudocharacter are Baire. In fact, one can do a bit better as follows.

Lemma 5.5. *If X is an RG-space of countable pseudocharacter then every countable union of nowhere dense subsets is nowhere dense.*

Proof. Let X be an RG-space and $(A_n)_{n=1}^{\infty}$ be a sequence of nowhere dense subsets of X. Let $S = \bigcup_{n=1}^{\infty} A_n$, $F_1 = A_1$ and $F_m = A_m - (\bigcup_{i=1}^{m-1} A_i)$ for each $m \geq 2$. Then $\{F_n : n \in N\}$ is a collection of disjoint nowhere dense subsets of X, and therefore $\{F_n : n \in N\} \cup \{X - S\}$ is a partition of X. Now define $f : X_\delta \longrightarrow R$ by $f(F_n) = \{n+1\}$ for each $n \in N$ and $f(X - S) = \{1\}$. Then $f \in C(X_\delta) = G(X)$, which implies that there is a dense open subset D of X such that $f|D$ is a continuous function. Suppose $cl(\bigcup_{n=1}^{\infty} F_n)$ has an interior point p. Then there is an open subset U_p containing p such that $U_p \subseteq cl(\bigcup_{n=1}^{\infty} F_n)$, that is $\forall\ y \in U_p$ and for each neighborhood W_y of y we have $W_y \cap (\bigcup_{n=1}^{\infty} F_n) \neq \phi$. Since $D \cap U_p \neq \phi$, let y be any point in $\in D \cap U_p$. Again we have two cases:
(1) If $f(y) = 1$, then there is an open neighborhood W_y of y such that $f(W_y) \subseteq (0, \frac{3}{2})$. Hence $W_y \cap (\bigcup_{n=1}^{\infty} F_n) = \phi$, which is a contradiction.
(2) If $f(y) = k+1$, then $y \in F_k$. So there is an open neighborhood W_y of y such that $f(W_y) \subseteq (k + \frac{2}{3}, k + \frac{4}{3})$, and therefore $W_y \subseteq F_k \subseteq A_k$, which is a contradiction too.
Thus $\bigcup_{n=1}^{\infty} A_n$ is a nowhere dense subset of X. $\square$

A topological space X can have a dense subset K such that K^c is somewhere dense or even a dense subset. This is will be a relevant point for RG-spaces.

Lemma 5.6. *Let X be a topological space. Then X is either has the property that every dense subset has a nowhere dense complement or X has a resolvable cozero subspace.*

Proof. Suppose D is a dense subset such that D^c is somewhere dense. Then there is a non-empty cozero subset U such that $U \subseteq cl(D^c)$. Let $A = D \cap U$ and $B = D^c \cap U$. Then A and B are disjoint dense subsets of U. Hence U is a resolvable cozero subspace. $\square$

Theorem 5.7. *Let X be an RG-space of countable pseudocharacter. Then*

(1) *Every dense subset of X has a nowhere dense complement.*
(2) *Every countable intersection of dense sets has a dense interior.*
(3) *In particular, every dense set has a dense interior.*

Proof. (1) Since every cozero subset of X is an RG-space of countable pseudocharacter then it cannot be resolvable. Thus the result follows directly by lemma 5.6.
(2) Let $(D_n)_{n=1}^{\infty}$ be a sequence of dense subsets of X. Let $A_n = X - D_n$ for each $n \in N$. Then $(A_n)_{n=1}^{\infty}$ is a sequence of nowhere subsets of X, which implies that $\bigcup_{n=1}^{\infty} A_n$ is a nowhere dense subset of X. Hence the result follows directly from the fact that $cl(T)^c = int(T^c)$ for any subset T of X.
(3) This follows directly from (2). □

Recall that a topological space X is called **open hereditarily irresolvable** (or simply o.h.i) if each open subspace of X is irresolvable [7, def 1.2]. In [9] Ganster proved that a topological space X is open hereditarily irresolvable if and only if every dense set of X has a dense interior. Since every RG-space of countable pseudocharacter is open hereditarily irresolvable, one also deduce part (3) of theorem 5.7 from an understanding of Ganster's work.

Before finishing this section, we use work of Tamariz and Villegas to prove that under the assumption $V = L$, every RG-space of countable pseudocharacter has a dense set of isolated points. This matters because the presence of even almost P-points in RG-spaces is an open question. First we recall proposition 4.10 of [21] which says that, assuming $V = L$, every space without isolated points is almost resolvable.

Theorem 5.8. *Assume $V = L$. Then every RG-space of countable pseudocharacter is scattered.*

Proof. Since every cozero subspace of X is also an RG-space of countable pseudocharacter, then by proposition 4.10 of [21] and theorem 5.4, every cozero set has an isolated point. It follows immediately that X has a dense set of isolated points. Now if Y is any subspace of X then Y is also of countable pseudocharacter, and since it is C-embedded in X in the δ-topology Y is RG itself by [12, Theorem 2.3(a)]. Thus Y has a dense set of isolated points by the first part of the proof, which means that X is scattered. □

6. Almost Baire spaces

Recall that a topological space X is called k-Baire, where k is a fixed cardinal number, if the intersection of fewer than k dense open sets is dense [20]. Thus the usual Baire spaces are $\aleph_1$-Baire spaces. It is clear that the intersection of all dense open subset of X is a dense subset if and only if X has a dense subset of isolated points, which means that if X has a dense subset of isolated points then X is a k-Baire space for any cardinal number k. Thus every scattered space is a k-Baire space for any cardinal number k.
In a general Tychonoff space an open subset need not be a cozero-set, and the

collection of all dense cozero-sets can have any cardinality. For these reasons we will introduce the following class of spaces.

Definition 6.1. Let X be a topological space and k be a cardinal number. Then we will call X an almost k-Baire space if any collection of fewer than k dense cozero-sets has a dense intersection. We will call X almost-Baire if X is an almost $\aleph_1$-Baire space.

If X is a topological space and k is a fixed cardinal number, then X is almost k-Baire if and only if the union of fewer than k nowhere dense zero-sets has an empty interior. It is clear that every clopen subspace of almost k-Baire space is an almost k-Baire space, and a space X is almost k-Baire if and only if X has a dense subspace which is almost k-Baire. Every k-Baire space is an almost k-Baire space, but the converse is not true in general as we will see next.
If X is an RG-space then it is clear from theorem 4.1 that every countable intersection of dense cozero subsets of X has a dense interior. Recall that a space X is an almost P-space if and only if every non-empty countable intersection of open sets has a non-empty interior. It is clear that every almost P-space is almost k-Baire for each cardinal number k.

Corollary 6.2. *Every RG-space is an almost-Baire space.*

Proof. This follows directly from theorem 4.1. □

RG-spaces need not be Baire. In [8] the authors gave two examples. First they gave a regular P-space without isolated points. Secondly they gave an example of a Tychonoff space X with a dense set of isolated points such that X_δ is not a Baire space. Thus an RG-space does not have to be Baire, and consequently an almost k-Baire space need not be a k-Baire space.

Corollary 6.3. *The cozero-sets and zero-sets of RG-spaces are almost-Baire spaces.*

Definition 6.4 (cf [18])**.** Let X be a topological space. Then the subset gX is defined to be the intersection of all dense cozero subsets of X.

It is clear that gX is the set of almost P-points in X. If X is an RG-space, then it follows from theorem 4.2 that every countable subset of $X - gX$ is a nowhere dense subset of X.

Proposition 6.5. *Let X be a topological space. Then:*

(1) *X is almost Baire implies that every dense open C^*-embedded subset in X is almost Baire.*
(2) *X is almost k-Baire for each cardinal number k if and only if gX is dense.*

Proof. (1) Let X be an almost Baire space, let U be a dense open C^*-embedded subset in X and let $V_n, n = 1, 2, 3, ...$ be a collection of dense cozero-sets in U. Since U is C^*-embedded in X then for each n, there is a dense cozero-set W_n in X such that $W_n \cap U = V_n$. But X is almost Baire. Therefore $\bigcap_{n=1}^{\infty} W_n$

is a dense subset of X which implies that $\bigcap_{n=1}^{\infty} W_n \cap U = \bigcap_{n=1}^{\infty} V_n$ is a dense subset of U. Thus U is almost Baire.
(2) ($\Longrightarrow$) Let X be almost k-Baire for each cardinal number k. Suppose there exists a non-empty open subset U such that $U \cap gX = \phi$. For each $x \in U$ choose a nowhere dense zero-set Z_x such that $x \in Z_x$ and let $V_x = X - Z_x$. Then V_x is a dense cozero-set for each $x \in U$ and $U \cap \bigcap_{x \in U} V_x = \phi$, which contradicts the fact that X is an almost k-Baire space for each cardinal number k. Thus gX is a dense subset of X.
($\Longleftarrow$) This is clear from the fact that gX is contained in every dense cozero-set. □

7. Open questions

Question 7.1 (cf [19]). Are all RG-spaces of finite regularity degree?

Question 7.2 (cf Corollary 3.10). If X is the union of two disjoint Lindelof RG-subspaces, must X be RG?

Question 7.3. Is the intersection of two RG-spaces RG? What about the case where both of the spaces are Lindelof?

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