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Continuous isomorphisms onto separable groups

Luis Felipe Morales López

Abstract

A condensation *is a one-to-one continuous function onto. We give sufficient conditions for a Tychonoff space to admit a condensation onto a separable dense subspace of the Tychonoff cube* $\mathbb{I}^{\mathfrak{c}}$ *and discuss the differences that arise when we deal with topological groups, where condensation is understood as a* continuous isomorphism. *We also show that every Abelian group* G *with* $|G| \leq 2^{\mathfrak{c}}$ *admits a separable, precompact, Hausdorff group topology, where* $\mathfrak{c} = 2^{\omega}$.

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1. Introduction

A *condensation* is a bijective continuous function. If X and Y are spaces and $f : X \to Y$ is a condensation, we can assume that X and Y have the same underlying set and the topology of X is finer than the topology of Y. In this case we say that the topology of Y is a *subtopology of* X or that X *condenses onto* Y.

The problem of finding conditions under which a space X admits a subtopology with a given property $\mathcal{Q}$ has been extensively studied by many authors. It is known that every Hausdorff space X with $nw(X) \leq \kappa$ can be condensed onto a Hausdorff space Y with $w(Y) \leq \kappa$ (see [7, Lemma 3.1.18]). Similar results remain valid in the classes of regular or Tychonoff spaces. In [16], the authors found several necessary and sufficient conditions for a topological space to admit a connected Hausdorff or regular subtopology. It is shown in [11] that every non-compact metrizable space has a connected Hausdorff subtopology. Druzhinina showed in [9] that every metrizable space X with $w(X) \geq 2^{\omega}$ and *achievable extent* admits a weaker connected metrizable topology. Recently, Yengulalp [17] generalized this result by removing the *achievable extent* condition.

In topological groups (and other algebraic structures with topologies), the concept of condensation has a natural counterpart: *Continuous Isomorphism*, a homomorphism and a condensation at the same time.

At the end of the 70's, Arhangel'skii proved in [2] that every topological group G with $nw(G) \leq \kappa$ admits a continuous isomorphism onto a topological group H with $w(H) \leq \kappa$. In [15] Shakhmatov gave a construction that implies similar statements for topological rings, modules, and fields. C. Hernández modified Shakhmatov's construction and extended that result to many algebraic structures with regular and Tychonoff topologies (see [8]).

As a corollary to Katz's theorem about isomorphic embeddings into products of metrizable groups (see [1, Corollary 3.4.24]) one can easily deduce that if G is an ω-balanced topological group and the neutral element of G is a G_δ-set, then there exists a continuous isomorphism of G onto a metrizable topological group. Pestov showed that the condition on G being ω-balanced can not be removed (see [14]).

In Theorem 3.2 of this paper we present conditions that a Tychonoff space must satisfy in order to admit a condensation onto a separable dense subspace of the Tychonoff cube of weight 2^ω. In Corollary 4.2 we show that those conditions are not sufficient if we want to have a continuous isomorphism from a topological group to a separable group and in Theorem 4.3 we give sufficient and necessary conditions in order for a topological isomorphism from a subgroup of the product of compact metrizable Abelian groups onto a separable group to exist.

As Arhangel'skii showed in [4], every continuous homomorphism of a countably compact group X onto a compact group Y of Ulam nonmeasurable cardinality is open. In Example 4.4 we construct a condensation of a Tychonoff countably compact space with cellularity 2^ω onto a separable compact space with cardinality $2^\mathfrak{c}$ thus showing that Arhangel'skii result cannot be generalized to arbitrary spaces. Finally, we show in Theorem 5.11 that every Abelian group of cardinality less than or equal to $2^\mathfrak{c}$ admits a precompact separable Hausdorff group topology.

2. Notation and terminology

We use $\mathbb{I}$ for the unit interval $[0,1]$, $\mathbb{T}$ for the unit circle, $\mathbb{N}$ for the set of positive integers, $\mathbb{Z}$ for the integers, $\mathbb{Q}$ for the rational numbers, and $\mathbb{R}$ for the set of real numbers.

Let X be a space. As usual, we denote by $w(X)$, $nw(X)$, $\chi(X)$, $\psi(X)$, $d(X)$ the weight, network weight, character, pseudocharacter, and density of X, respectively.

We say that $Z \subset X$ is a zero-set if there exists a real-valued continuous function $f : X \to \mathbb{R}$ such that $Z = f^{-1}(0)$.

Let $\{f_\alpha : \alpha \in A\}$ be a family of functions, where $f_\alpha : X \to Y_\alpha$ for each $\alpha \in A$. We denote by $\triangle\{f_\alpha : \alpha \in A\}$ the diagonal product of the family $\{f_\alpha : \alpha \in A\}$.

Suppose that $\eta = \{G_\alpha : \alpha \in A\}$ is a family of topological groups and $\Pi\eta = \prod_{\alpha\in A} G_\alpha$ is the topological product of the family η. Then the Σ-product of η, denoted by $\Sigma\Pi\eta$, is the subgroup of $\Pi\eta$ consisting of all points $g \in \Pi\eta$ such that $|\{\alpha \in A : \pi_\alpha(g) \neq e_\alpha\}| \leq \omega$ and the σ-product of η, denoted by $\sigma\Pi\eta$ is the subgroup of $\Pi\eta$ consisting of all points $g \in \Pi\eta$ such that $|\{\alpha \in A : \pi_\alpha(g) \neq e_\alpha\}| < \omega$, where $\pi_\alpha : \Pi\eta \to G_\alpha$ is the natural projection of $\Pi\eta$ onto G_α and $e_\alpha \in G_\alpha$ is the neutral element of G_α, for every $\alpha \in A$. It is easy to see that both $\Sigma\Pi\eta$ and $\sigma\Pi\eta$ are dense subgroups of $\Pi\eta$. A description of properties of these subgroups can be found in [1, Section 1.6].

If X is a Tychonoff space and G is a topological group, we denote by βX the Čech-Stone compactification of X (see [7, Section 3.6]), and by ρG the Raĭkov completion of G (see [1, Section 3.6]).

The next definitions are standard in group theory (see [10, Section 1.1]). Let G be a group, e the neutral element of G, and $g \in G$ an element of G distinct from e. We denote by $\langle g\rangle$ the *cyclic subgroup of G generated by g*. The order of g is $o(g) = |\langle g\rangle|$. If $o(g) = \infty$ then $\langle g\rangle$ is isomorphic to $\mathbb{Z}$. The set $tor(G)$ of the elements $g \in G$ with $o(g) < \infty$ is called *the torsion part of G*. If G is Abelian, $tor(G)$ is a subgroup of G.

We say that the group G is:

- *torsion-free* if for every element $g \in G \setminus \{e\}$, $o(g) = \infty$;
- *a torsion group* if for every element $g \in G$, $o(g) < \infty$;
- *bounded torsion* if there exists $n \in \mathbb{N}$ such that $g^n = e$ for every $g \in G$;
- *unbounded torsion* if G is torsion and for each $n \in \mathbb{N}$ there exists $g \in G$ such that $o(g) > n$;
- *a divisible group* if for every $g \in G$ and $n \in \mathbb{N}$, there is $h \in G$ such that $h^n = g$;
- *a p-group*, for a prime p, if the order of any element of G is a power of p.

If G is an Abelian torsion group, then G is the direct sum of p-groups G_p (see [10, Theorem 8.4]). The subgroups G_p are called the *p-components* of G.

Let p be a prime number. The set of p^nth complex roots of the unity, with $n \in \mathbb{N}$ forms the multiplicative subgroup $\mathbb{Z}_{p^\infty}$ of $\mathbb{T}$. For every prime p, the group $\mathbb{Z}_{p^\infty}$ is divisible.

3. Condensations and subtopologies

Not every space has a separable subtopology. For example, a compact Hausdorff space X has a separable Hausdorff subtopology only if X is separable. Let us extend this fact to a wider class of spaces.

We recall that a Hausdorff space X is *ω-bounded* if the closure of any countable subset of X is compact.

Proposition 3.1. *Let X be an ω-bounded non-separable space. Then X does not admit a condensation onto a separable Hausdorff space.*

Proof. By our assumptions, for every countable subset S of X we have that $X \setminus \bar{S} \neq \varnothing$. Let $f : X \to Y$ be a condensation onto a Hausdorff space Y and D a countable subset of Y. Then $S = f^{-1}(D)$ is a countable subset of X, and $\bar{S}$ is compact. Take an element $x \in X \setminus \bar{S}$. Observe that $f(\bar{S})$ is compact, $D \subset f(\bar{S})$ and $f(x) \notin f(\bar{S})$, so D cannot be dense in Y. □

The next theorem gives sufficient conditions on a Tychonoff space to admit a condensation onto a separable dense subspace of $\mathbb{I}^{\mathfrak{c}}$, where $\mathfrak{c} = 2^{\omega}$.

Theorem 3.2. *Let X be a Tychonoff space with $nw(X) \leq 2^{\omega}$. Suppose that X contains an infinite, closed, discrete, and C^*-embedded subset A. Then X can be condensed onto a separable dense subspace of $\mathbb{I}^{\mathfrak{c}}$.*

Proof. We can assume that $|A| = \omega$. By the Hewitt-Marczewski-Pondiczery theorem, we know that $d(\mathbb{I}^{\mathfrak{c}}) = \aleph_0$. Let $D = \{d_n : n \in \omega\}$ be a countable dense subset of $\mathbb{I}^{\mathfrak{c}}$, $\mathcal{N}$ a network for X, $|\mathcal{N}| \leq 2^{\omega}$, and $A = \{x_n : n \in \omega\}$ an enumeration of A. Let $g : A \to D$ be a bijection, where $g(x_n) = d_n$ for each $n \in \omega$. For every $\alpha < \mathfrak{c}$, let $f_\alpha = p_\alpha \circ g$, where $p_\alpha : \mathbb{I}^{\mathfrak{c}} \to \mathbb{I}_{(\alpha)}$ denotes the natural projection of $\mathbb{I}^{\mathfrak{c}}$ to the α-th factor.

Our goal is to construct a family of continuous functions $\{g_\alpha : X \to \mathbb{I}\}_{\alpha<\mathfrak{c}}$ such that g_α extends f_α for every $\alpha < \mathfrak{c}$ in a way that, given two different points $x, y \in X$, there exists $\alpha < \mathfrak{c}$ such that $g_\alpha(x) \neq g_\alpha(y)$.

If $n, m \in \omega$ are distinct, there exists $\alpha < \mathfrak{c}$ such that $f_\alpha(x_n) = p_\alpha(d_n) \neq p_\alpha(d_m) = f_\alpha(x_m)$. Therefore any family of extensions of f_α's separates the points of A. So, given two distinct points $x, y \in X$, it is only necessary to consider the cases when one point is in A and the other is not, and when neither is in A.

Since A is closed and X is Tychonoff, for each $y \in X \setminus A$ there exists $V_y \in \mathcal{N}$ such that $y \in V_y$ and A and $\overline{V_y}$ can be separated by zero-sets. Let $G_1 = \{V_y : y \in X \setminus A\}$ and note that $G_1 \subset \mathcal{N}$. In particular, $|G_1| \leq \mathfrak{c}$.

For each $V \in G_1$, choose disjoint zero-sets Z_V, Z'_V in X such that $V \subset Z_V$ and $A \subset Z'_V$ and let $\mathcal{Z} = \{(Z_V, Z'_V) : V \in G_1\}$ and $\mathcal{C}_1 = A \times \mathcal{Z}$. It is clear that $|\mathcal{C}_1| \leq \mathfrak{c}$.

Let $F = \{(x, y) : x, y \in X \setminus A,\ x \neq y\}$. For each pair $(x, y) \in F$, we can find subsets $U = U_{(x,y)} \in \mathcal{N}, V = V_{(x,y)} \in \mathcal{N}$ with $x \in U$ and $y \in V$ such that there exist pairwise disjoint zero-sets $Z_{(U,V)}$, Z_U, Z_V with $A \subset Z_{(U,V)}$, $U \subset Z_U$ and $V \subset Z_V$. Let $G_2 = \{(U_{(x,y)}, V_{(x,y)}) : (x, y) \in F\}$. It is clear that $G_2 \subset \mathcal{N} \times \mathcal{N}$, therefore $|G_2| \leq \mathfrak{c}$. For each $(U, V) \in G_2$, choose pairwise disjoint zero-sets $Z_{(U,V)}, Z_U, Z_V$ such that $A \subset Z_{(U,V)}$, $U \subset Z_U$ and $V \subset Z_V$ and let $\mathcal{C}_2 = \{(Z_{(U,V)}, Z_U, Z_V) : (U, V) \in G_2\}$. Then $|\mathcal{C}_2| \leq \mathfrak{c}$.

Put $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Clearly $|\mathcal{C}| \leq \mathfrak{c}$. Let $\mathcal{C} = \{C_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{C}$.

Case 1: $C_\alpha \in \mathcal{C}_1$. Then C_α has the form (x_n, Z_V, Z'_V), for some $n \in \mathbb{N}$ and $V \in G_1$. Since Z_V and Z'_V are disjoint zero-sets, there exists a continuous mapping $r_\alpha : X \to \mathbb{I}$ such that $Z_V = r_\alpha^{-1}(0)$ and $Z'_V = r_\alpha^{-1}(1)$. By the definition of Z'_V we have that $A \subset Z'_V$. Since A is C^*-embedded in X, there exists a continuous mapping $\tilde{f}_\alpha : X \to \mathbb{I}$ that extends f_α. We have two subcases, $f_\alpha(x_n) \neq 0$ or $f_\alpha(x_n) = 0$.

If $f_\alpha(x_n) \neq 0$, we define $g_\alpha : X \to \mathbb{I}$, by $g_\alpha = \tilde{f}_\alpha \cdot r_\alpha$. Then $g_\alpha(Z_V) \subset \{0\}$ and for each $x \in A$, $g_\alpha(x) = \tilde{f}_\alpha(x) \cdot r_\alpha(x) = f_\alpha(x)$, therefore g_α is an extension of f_α. In particular, $g_\alpha(x_n) = f_\alpha(x_n) \notin g_\alpha(Z_V)$.

If $f_\alpha(x_n) = 0$, we define $g_\alpha : X \to \mathbb{I}$, by $g_\alpha = 1 - r_\alpha + \tilde{f}_\alpha \cdot r_\alpha$. For any $x \in A$, since $A \subset Z'_V$ we have that $r_\alpha(x) = 1$, therefore $g_\alpha(x) = 1 - r_\alpha(x) + \tilde{f}_\alpha(x) \cdot r_\alpha(x) = \tilde{f}_\alpha(x) = f_\alpha(x)$, so g_α is a continuous extension of f_α. If $y \in Z_V$, then $g_\alpha(y) = 1$, so $g_\alpha(Z_V) \subset \{1\}$.

In both subcases, we have extended f_α to a continuous function g_α such that $g_\alpha(x_n) \notin g_\alpha(Z_V)$.

Case 2: $C_\alpha \in \mathcal{C}_2$. Then C_α has the form $(Z_{(U,V)}, Z_U, Z_V)$ for some $(U, V) \in G_2$, where $Z_{(U,V)}$, Z_U, and Z_V are disjoint zero-sets and $A \subset Z_{(U,V)}$. As in Case 1, there exists a continuous function $\tilde{f}_\alpha : X \to \mathbb{I}$ that extends f_α such that $\tilde{f}_\alpha(Z_U) \subset \{1\}$. As $Z = Z_{(U,V)} \cup Z_U$ is a zero-set disjoint from Z_V, there exists a continuous function $r_\alpha : X \to \mathbb{I}$ such that $Z_V = r_\alpha^{-1}\{0\}$ and $Z = r_\alpha^{-1}\{1\}$. Let $g_\alpha = \tilde{f}_\alpha \cdot r_\alpha$. For each $x \in A$, $g_\alpha(x) = \tilde{f}_\alpha(x) \cdot r_\alpha(x) = f_\alpha(x)$. If $x \in Z_U$, then $g_\alpha(x) = \tilde{f}_\alpha(x) \cdot r_\alpha(x) = 1$. If $x \in Z_V$, then $g_\alpha(x) = \tilde{f}_\alpha(x) \cdot r_\alpha(x) = 0$. Therefore $g_\alpha(Z_U) \cap g_\alpha(Z_V) = \varnothing$.

Thus, we have constructed a family $\{g_\alpha : alpha < \mathfrak{c}\}$ of continuous functions. Given two different elements $x, y \in X$, we have three possibilities: $x, y \in A$, or $x \in A$ and $y \notin A$, or $x, y \in X \setminus A$. The functions g_α are extensions of the mappings f_α, therefore they separate the elements of A. If $x \in A$ and $y \notin A$, then we are in Case 1 and there exists $\alpha < \mathfrak{c}$ such that $C_\alpha = (x_n, Z_V, Z'_V)$ with $x = x_n$ and $y \in Z_V$. Since $g_\alpha(x_n) \notin g_\alpha(Z_V)$, we have in particular that $g_\alpha(x) \neq g_\alpha(y)$. If both points x, y are in $X \setminus A$, we are in Case 2 and there exists $\alpha < \mathfrak{c}$ such that $C_\alpha = (Z_{(U,V)}, Z_U, Z_V)$ with $x \in Z_U$ and $y \in Z_V$. Since $g_\alpha(Z_U) \cap g_\alpha(Z_V) = \varnothing$ we have that $g_\alpha(x) \neq g_\alpha(y)$. Hence the family $\{g_\alpha : \alpha < \mathfrak{c}\}$ separates the elements of X.

Let $\tilde{g} : X \to \mathbb{I}^\mathfrak{c}$, $\tilde{g} = \Delta\{g_\alpha : \alpha < \mathfrak{c}\}$ and $Y = \tilde{g}(X)$. Since g_α's separate the elements of X, $\tilde{g}$ is a continuous injective mapping. Besides, for each $n \in \omega$ and every $\alpha < \mathfrak{c}$, $\tilde{g}(x_n)(\alpha) = g_\alpha(x_n) = f_\alpha(x_n)$, so $D \subset \tilde{g}(X)$. Thus Y is a dense separable subspace of $\mathbb{I}^\mathfrak{c}$.

□

Since the space $Y = \tilde{g}(X)$ in Theorem 3.2 is a dense subspace of $\mathbb{I}^\mathfrak{c}$, it is *κ-metrizable* and *perfectly κ-normal* (see [3], [13]). In particular, every regular closed subset of Y is a zero-set.

Let us show that one cannot remove any assumption in Theorem 3.2.

If we put $X = \beta\mathbb{N}$ and $A = \mathbb{N}$, then we have an example showing that the condition on A being closed can not be dropped in Theorem 3.2.

As to the condition on A being discrete, take a non-separable Hausdorff compact space X with $w(X) \leq 2^{\omega}$, for example, the Alexandroff double circle [7, Example 3.1.26]. Let A be any infinite closed subset of X. By the Urysohn's Lemma, A is C^*-embedded, but there is no condensation of X onto a separable space.

Let $X = (W \times W_0) \setminus \{(\omega_1, \omega)\}$, where $W = \{\alpha : \alpha \leq \omega_1\}$ and $W_0 = \{\alpha : \alpha \leq \omega\}$ carry the order topology. This space, known as *the Tychonoff plank*, is a Tychonoff space that has the property that the closure of any countable subset A is also countable, and if $A \subset (W \setminus \{\omega_1\}) \times W_0$, then $\bar{A}$ is compact. It is not difficult to see that every continuous real-valued function on X can be extended over $W \times W_0$, that is, $\beta X = W \times W_0$. Let $A = \{\omega_1\} \times (W_0 \setminus \{\omega\})$. Clearly A is an infinite closed discrete subset of X. Suppose that we have a condensation $f : X \to Y$ onto a Tychonoff space Y. Let $D \subset Y$ be a countable subset of Y and $S = f^{-1}(D)$. Since f is a bijection we have that S is countable. The closure $\bar{S}$ of S in βX is a compact countable subset of βX, so $X \setminus \bar{S} \neq \varnothing$. Take an element $x \in X \setminus \bar{S}$. Let $F : \beta X \to \beta Y$ be a continuous extension of the mapping f. Observe that $F(\bar{S})$ is countable and compact (hence closed in Y), $F(x) \notin F(\bar{S})$, and $F(\bar{S})$ contains D. Therefore D can not be dense in Y. This example shows that the condition "A is C^*-embedded in X" cannot be removed from Theorem 3.2.

4. Topological Groups: Continuous Isomorphisms

We recall that a topological group G is ω-narrow if it can be covered by countably many translates of any neighborhood of the identity of G.

The following results show that the conditions on X in Theorem 3.2 are not sufficient to ensure the existence of a continuous isomorphism of X onto a separable topological group in the case when X is a topological group.

Theorem 4.1. *Let κ be an infinite cardinal with $\kappa \leq 2^{\omega}$, $\mathbb{T}$ the circle group, and G a subgroup of $\Sigma\mathbb{T}^{\kappa}$. Then there exists a continuous isomorphism $\varphi : G \to H$ onto a separable topological group H if and only if $\psi(G) \leq \omega$.*

Proof. Let $\varphi : G \to H$ be a continuous isomorphism onto a separable topological group H and $D \subset H$ a countable dense subset. There exists a continuous homomorphism $\overline{\varphi} : \varrho G \to \varrho H$ that extends φ. Note that the Raĭkov completion of G is the closure of G in $\mathbb{T}^{\kappa}$, $H \subset \overline{\varphi}(\varrho G) \subset \varrho H$, and $\overline{\varphi}(\varrho G)$ is a compact group containing H as a dense subgroup. Therefore $\overline{\varphi}(\varrho G) = \varrho H$.

Clearly $\varphi^{-1}(D) \subset G \subset \Sigma(\mathbb{T}^{\kappa})$ and $|\varphi^{-1}(D)| = \omega$, therefore $B = \overline{\varphi^{-1}(D)}$ (the closure in $\mathbb{T}^{\kappa}$) is a compact metrizable subspace of $\Sigma\mathbb{T}^{\kappa}$ (see [1, Propositions 1.6.29 and 1.6.30]). So $\overline{\varphi}(B)$ is compact and contains D as a dense subspace, which implies that $\overline{\varphi}(B) = \varrho H$. Also ϱH is compact and has a countable network as a continuous image of B, so $w(\varrho H)$ and $w(H)$ are less or equal than ω. Thus $\varphi : G \to H$ is a continuous isomorphism of G to a metrizable space, therefore $\psi(G) \leq \omega$.

Let us verify the other implication. Observe that the identity element e of G is a G_δ-set, because $\psi(G) \leq \omega$. Since G is a subgroup of a compact group, it is ω-narrow. By Corollary 3.4.25 of [1], there exists a continuous isomorphism of G onto a second countable (hence separable) group. □

The next corollary follows from Theorem 4.1.

Corollary 4.2. *Let $G = \sigma(\mathbb{T}^\kappa)$ be the σ-product of κ-many copies of the circle group, where κ is an infinite cardinal. If there exists a continuous isomorphism $\varphi : G \to H$ onto a separable topological group H, then $\kappa = \omega$.*

Proof. It is easy to verify that $\psi(\sigma(\mathbb{T}^\kappa)) = \kappa$, so the conclusion follows from Theorem 4.1. □

The topological group $G = \sigma(\mathbb{T}^\kappa)$ contains infinite, closed, discrete, C^*-embedded subspaces. By Theorem 3.2, if $\kappa \leq 2^\omega$ we can find a condensation of G onto a separable Tychonoff space, but if $\kappa > \omega$, there is no continuous isomorphism of G onto a separable topological group.

Theorem 4.1 can be generalized if we replace $\mathbb{T}^\kappa$ by the product of any family of compact metrizable Abelian groups:

Theorem 4.3. *Let $\eta = \{G_\alpha : \alpha \in \kappa\}$ be a family of compact metrizable groups with $\kappa \leq 2^\omega$, and G be a subgroup of $\Sigma = \Sigma\Pi\eta$. Then there exists a continuous isomorphism $\varphi : G \to H$ of G onto a separable topological group H if and only if $\psi(G) \leq \omega$.*

The proof of this fact is almost the same as in the Theorem 4.1, and we omitted.

Arhangel'skii showed in [4, Corollary 12] that every continuous homomorphism of a countably compact topological group onto a compact group of Ulam nonmeasurable cardinality is open. In particular, if there exists a continuous isomorphism of a countably compact topological group G onto a compact group of Ulam nonmeasurable cardinality, then G is compact.

The next example shows that one cannot extent this result to topological spaces.

Example 4.4. There exists a condensation of a countably compact non-separable Tychonoff space onto a separable compact space of Ulam nonmeasurable cardinality, $2^{\mathfrak{c}}$.

Let $Y = \beta\mathbb{N}$ be the Čech-Stone compactification of the natural numbers and $Z = Y \setminus \mathbb{N}$. By [7, Example 3.6.18], Z contains a family $\mathcal{A}$ of cardinality $\mathfrak{c}$ consisting of pairwise disjoint non-empty open sets. Let $\pi_1 : Y \times Z \to Y$ and $\pi_2 : Y \times Z \to Z$ be the natural projections to the first and the second factor respectively. Since Y is compact, π_2 is a closed mapping. By [7, Theorem 3.5.8], Z is a compact space because it is the remainder of a locally compact space and so, π_1 is a closed mapping too.

By [7, Theorem 3.6.14], every infinite closed subset S of both Y and Z has cardinality equal to $2^{\mathfrak{c}}$. Let M be an infinite subset of $Y \times Z$. It is clear that at least one of the set, $\pi_1(M)$ or $\pi_2(M)$, is infinite. Suppose that $\pi_1(M)$ is infinite. Since the projection π_1 is closed, $\pi_1(\overline{M})$ is a closed subset of Y, so $\pi_1(\overline{M})$ and $\overline{M}$ have cardinality equal to $2^{\mathfrak{c}}$.

Our goal is to construct a countably compact non-separable subspace $X \subset Y \times Z$ such that $\pi_1(X) = Y$, $\pi_1|_X$ is a one-to-one mapping, and $\pi_2(X) \cap A \neq \varnothing$ for every $A \in \mathcal{A}$.

Recall that $[Y]^\omega$ is the family of subsets of Y with cardinality ω. Let $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ be a faithful enumeration of $\mathcal{A}$ and choose $z_\alpha \in A_\alpha$ for each $\alpha < \mathfrak{c}$. Let also $Y = \{y_\beta : \beta < 2^{\mathfrak{c}}\}$ and $[Y]^\omega = \{F_\gamma : \mathfrak{c} \leq \gamma < 2^{\mathfrak{c}}\}$ be faithful enumerations of Y and $[Y]^\omega$ respectively such that $F_{\mathfrak{c}} \subset \{y_\beta : \beta < \mathfrak{c}\}$.

We shall define a transfinite sequence $\{X_\gamma : \mathfrak{c} \leq \gamma < 2^{\mathfrak{c}}\}$ of subsets of $Y \times Z$ satisfying the following conditions for each γ with $\mathfrak{c} \leq \gamma < 2^{\mathfrak{c}}$:

($\mathbf{i}_\gamma$): $X_\beta \subset X_\gamma$ if $\mathfrak{c} \leq \beta < \gamma$;
($\mathbf{ii}_\gamma$): the restriction of π_1 to X_γ is a one-to-one mapping;
($\mathbf{iii}_\gamma$): $F_\gamma \subset \pi_1(X_\gamma)$;
($\mathbf{iv}_\gamma$): $\pi_1^{-1}(F_\gamma) \cap X_\gamma$ has an accumulation point in X_γ;
($\mathbf{v}_\gamma$): $|X_\gamma| \leq |\gamma|$.

For every $\alpha < \mathfrak{c}$ put $\overline{x}_\alpha = (y_\alpha, z_\alpha)$ and let $X'_{\mathfrak{c}} = \{\overline{x}_\alpha : \alpha < \mathfrak{c}\}$. By our enumeration of $[Y]^\omega$, $F_{\mathfrak{c}} \subset \pi_1(X'_{\mathfrak{c}})$. Put $B_{\mathfrak{c}} = \overline{\pi_1^{-1}(F_{\mathfrak{c}}) \cap X'_{\mathfrak{c}}}$. Since π_1 is closed and $F_{\mathfrak{c}} \subset \pi_1(B_{\mathfrak{c}})$, the cardinality of $\pi_1(\overline{B_{\mathfrak{c}}})$ is equal to $2^{\mathfrak{c}}$, so we can choose $x_{\mathfrak{c}} \in \overline{B_{\mathfrak{c}}}$ such that $\pi_1(x_{\mathfrak{c}}) \notin \pi_1(X'_{\mathfrak{c}})$.

Put $X_{\mathfrak{c}} = X'_{\mathfrak{c}} \cup \{x_{\mathfrak{c}}\}$. Conditions ($\mathrm{ii}_{\mathfrak{c}}$), ($\mathrm{iii}_{\mathfrak{c}}$), ($\mathrm{iv}_{\mathfrak{c}}$), and ($\mathrm{v}_{\mathfrak{c}}$) are clearly satisfied, condition ($\mathrm{i}_{\mathfrak{c}}$) is vacuous.

Suppose that for some γ with $\mathfrak{c} \leq \gamma < 2^{\mathfrak{c}}$, X_ξ are defined for all ξ, $\mathfrak{c} \leq \xi < \gamma$. Let $\tilde{X}_\gamma = \bigcup_{\mathfrak{c} \leq \xi < \gamma} X_\xi$. We have two possibilities. If $F_\gamma \subset \pi_1(\tilde{X}_\gamma)$, then put $X'_\gamma = \tilde{X}_\gamma$. If $F_\gamma \setminus \pi_1(\tilde{X}_\gamma) \neq \varnothing$, then choose an arbitrary point $x_y \in \pi_1^{-1}(y)$ for each $y \in F_\gamma \setminus \pi_1(\tilde{X}_\gamma)$ and put $X'_\gamma = \tilde{X}_\gamma \cup \{x_y : y \in F_\gamma \setminus \pi_1(\tilde{X}_\gamma)\}$. In both cases, $F_\gamma \subset \pi_1(X'_\gamma)$.

Since conditions (i_ξ) and (v_ξ) are satisfied for all $\mathfrak{c} \leq \xi < \gamma$, $|X'_\gamma| \leq |\gamma| < 2^{\mathfrak{c}}$.

Let $B_\gamma = \overline{\pi_1^{-1}(F_\gamma) \cap X'_\gamma}$. Since π_1 is a closed mapping, $|\pi_1(B_\gamma)| = 2^{\mathfrak{c}}$, so there exists $x_\gamma \in B_\gamma$ such that $\pi_1(x_\gamma) \notin \pi_1(X'_\gamma)$.

Let $X_\gamma = X'_\gamma \cup \{x_\gamma\}$. Clearly condition ($\mathrm{i}_\gamma$) is satisfied.

Since conditions (i_ξ) and (ii_ξ) are satisfied for every ξ with $\mathfrak{c} \leq \xi < \gamma$, $\pi_1|_{\tilde{X}_\gamma}$ is a one-to-one mapping. By our definition of X'_γ, $\pi_1|_{X'_\gamma}$ is a one-to-one mapping too. Finally, by our choose of x_γ, $\pi_1(x_\gamma) \notin \pi_1(X'_\gamma)$, so $\pi_1|_{X_\gamma}$ is a one-to-one mapping by (ii_γ).

As $F_\gamma \subset \pi_1(X'_\gamma)$ and $x_\gamma \in X_\gamma$, (iii_γ) and (iv_γ) are satisfied.

Since (v_ξ) and (i_ξ) are satisfied for every ξ with $\mathfrak{c} \leq \xi < \gamma$, $|\tilde{X}_\gamma| \leq |\gamma|$. As $|X_\gamma \setminus \tilde{X}_\gamma| \leq \omega$, we conclude that $|X_\gamma| \leq |\gamma|$

Put $X = \bigcup_{\mathfrak{c} \leq \gamma < 2^{\mathfrak{c}}} X_\gamma$ and let $f : X \to Y$ be the restriction of π_1 to X. Since conditions (i_γ) and (ii_γ) are satisfied for all γ with $\mathfrak{c} \leq \gamma < 2^{\mathfrak{c}}$, f is a continuous one-to-one function. Let $y \in Y$ be an arbitrary element of Y and $F \in [Y]^\omega$ be a subset of Y with $y \in F$. Then there exists γ, $\mathfrak{c} \leq \gamma < 2^{\mathfrak{c}}$ such that $F = F_\gamma$. By (iii$_\gamma$),

$$y \in F = F_\gamma \subset \pi_1(X_\gamma) \subset \pi_1(X) = f(X),$$

so $f(X) = Y$. Therefore f is a condensation of X onto Y.

Let B be an arbitrary infinite countable subset of X. Then $F = f(B)$ is an infinite countable subset of Y and there exists $\gamma < 2^{\mathfrak{c}}$ such that $F = F_\gamma$. By (iv$_\gamma$), $B = f^{-1}(F) = \pi_1^{-1}(F_\gamma) \cap X_\gamma$ has an accumulation point in X_γ and in X. This means that X is countably compact.

Since $A \cap \pi_2(X) \supset A \cap \pi_2(X_{\mathfrak{c}}) \neq \varnothing$ for every $A \in \mathcal{A}$, X cannot be separable.

5. Separable Group Topologies for Abelian Groups

In this section we prove that every Abelian group G with $|G| \leq 2^{\mathfrak{c}}$ admits a separable precompact Hausdorff group topology. To do this, we divide the job in three parts:

Case 1.: There is $x \in G$ with $o(x) = \infty$.
Case 2.: G is a bounded torsion group.
Case 3.: G is an unbounded torsion group.

We say that a topological group is *monothetic* if it has a dense cyclic subgroup. The next result is proved in [12, Corollary 25.15]:

Lemma 5.1. *The group $\mathbb{T}^\kappa$ is monothetic if and only if $\kappa \leq \mathfrak{c}$.*

Let us begin with the case when G is a non-torsion group (Case 1).

Theorem 5.2. *Let G be an Abelian group. Suppose that $|G| \leq 2^{\mathfrak{c}}$ and there is an element $x \in G$ of infinite order. Then there exists a separable precompact Hausdorff group topology on G.*

Proof. The main idea of the proof is to define a monomorphism $\overline{\varphi} : G \to \mathbb{T}^{\mathfrak{c}}$ such that $\overline{\varphi}(G)$ will be separable. First we do this in the case when G is divisible.

Let H be a minimal divisible subgroup of G with $x \in H$. Since $o(x)$ is infinite, H is isomorphic to $\mathbb{Q}$. By Lemma 5.1, there exists $a \in \mathbb{T}^{\mathfrak{c}}$ such that $\overline{\langle a \rangle} = \mathbb{T}^{\mathfrak{c}}$. Let $\varphi : H \to \mathbb{T}^{\mathfrak{c}}$ be a monomorphism such that $\varphi(x) = a$. For every $\beta < \mathfrak{c}$, put $\varphi_\beta = p_\beta \circ \varphi$, where $p_\beta = \mathbb{T}^{\mathfrak{c}} \to \mathbb{T}_{(\beta)}$ is the projection of $\mathbb{T}^{\mathfrak{c}}$ to the β's factor.

Let $\kappa = |G| > \omega$. Since G is divisible, it is isomorphic to the direct sum $H \oplus \bigoplus_{\alpha \in A} G_\alpha$, where each G_α is a subgroup of G isomorphic either to $\mathbb{Q}$ or $\mathbb{Z}_{p^\infty}$ for some prime number p, and A is an index set of cardinality κ (see [10, Theorem 23.1]). For each $\alpha \in A$, let $\varrho_\alpha : G_\alpha \to F$ be the isomorphism of G_α onto F, where F is either $\mathbb{Q}$ or $\mathbb{Z}_{p^\infty}$ for some prime number p.

Consider A as a subspace of the space $2^{\mathfrak{c}}$ with the product topology. Let $\mathcal{B}$ be the canonical base of $2^{\mathfrak{c}}$, we know that $|\mathcal{B}| = \mathfrak{c}$.

For each $g \in G$, let $h_g \in H$ and $k \in \bigoplus_{\alpha \in A} G_\alpha$ be such that $g = h_g + k$. If $g \in G \setminus H$, then $k \neq e$ and there exists a non-empty finite subset $c(g) \subset A$ such that $k \in \bigoplus_{\alpha \in c(g)} G_\alpha$. For every $\alpha \in c(g)$, take $k_\alpha \in G_\alpha$ such that $k =$

$\sum_{\alpha\in c(g)} k_\alpha$. Choose an arbitrary $\alpha(g)\in c(g)$ such that $k_{\alpha(g)}$ is not the identity of the group $G_{\alpha(g)}$. Let $U_g\in\mathcal{B}$ be an open set satisfying $U_g\cap c(g)=\{\alpha(g)\}$. Thus for each $g\in G\setminus H$ we have defined a pair $(h_g,U_g)\in H\times\mathcal{B}$.

The cardinality of the set $P=\{(h_g,U_g): g\in G\setminus H\}$ is less than or equal to $|H\times\mathcal{B}|=\omega\cdot\mathfrak{c}=\mathfrak{c}$. Let $P=\{P_\beta:\beta<\mathfrak{c}\}$ be an enumeration of P, where P_β is a pair (h_β,U_β) with $h_\beta\in H$ and $U_\beta\in\mathcal{B}$. For each $\beta<\mathfrak{c}$, we define a homomorphism $\psi_\beta:\bigoplus_{\alpha\in A} G_\alpha\to\mathbb{T}$ as follows:

If $\varphi_\beta(h_\beta)=1$, we can define ψ_β such that $\psi_\beta|_{G_\alpha}=\varrho_\alpha$ if $\alpha\in U_\beta$, and $\psi_\beta|_{G_\alpha}=\mathbf{1}$, otherwise. If $\varphi_\beta(h_\beta)\neq 1$, we define $\psi_\beta\equiv\mathbf{1}$.

Let $\overline{\varphi}_\beta$ be the homomorphism defined by $\overline{\varphi}_\beta=\varphi_\beta\oplus\psi_\beta$. It is clear that, for each $\beta<\mathfrak{c}$, $\overline{\varphi}_\beta$ is an extension of φ_β, therefore $\overline{\varphi}=\triangle_{\beta<\mathfrak{c}}\overline{\varphi}_\beta$ is an extension of φ and $\ker(\overline{\varphi})\cap H=\ker(\varphi)=\{e\}$.

Choose $g\in G\setminus H$. Then $g=h_g+\sum_{\alpha\in c(g)}k_\alpha$, where $k_\alpha\in G_\alpha$ for each $\alpha\in c(g)$. There exists $\beta<\mathfrak{c}$ such that $h_g=h_\beta$ and $U_g=U_\beta$, so

$$\overline{\varphi}_\beta(g)=\varphi_\beta(h_\beta)\cdot\psi_\beta(\sum_{\alpha\in c(g)}k_\alpha)=\varphi_\beta(h_\beta)\cdot\prod_{\alpha\in c(g)}\psi_\beta(k_\alpha)=\varphi_\beta(h_\beta)\cdot\psi_\beta(k_{\alpha(g)}).$$

We have two cases. If $\varphi_\beta(h_\beta)=1$ then $\psi_\beta(k_{\alpha(g)})=\varrho_{\alpha(g)}(g(\alpha(g)))\neq 1$ and, therefore $\overline{\varphi}_\beta(g)\neq 1$. If $\varphi_\beta(h_\beta)\neq 1$, then $\psi_\beta\equiv\mathbf{1}$. It follows that $\overline{\varphi}_\beta(g)\neq 1$.

Since $\overline{\varphi}$ is a monomorphism of G to the group $\mathbb{T}^{\mathfrak{c}}$, with $\overline{\varphi}(\langle x\rangle)=\langle a\rangle$, and $\langle a\rangle$ is a dense subset of $\mathbb{T}^{\mathfrak{c}}$, it follows that $\overline{\varphi}(G)$ is a precompact, separable, Hausdorff topological group.

In general, every infinite Abelian group G can be seen as a subgroup of a divisible group $\tilde{G}$ with $|\tilde{G}|=|G|$ (see [10, Theorem 24.1]). As shown above, there exists a monomorphism $\phi:\tilde{G}\to\mathbb{T}^{\mathfrak{c}}$ such that $\phi(x)=a$, where $x\in G$ is an element of infinite order and $a\in\mathbb{T}^{\mathfrak{c}}$ with $\langle a\rangle$ dense in $\mathbb{T}^{\mathfrak{c}}$. Therefore $\phi(G)$ contains $\langle a\rangle$ as a dense subset. So, the restriction $\overline{\varphi}=\phi|_G:G\to\phi(G)$ is the isomorphism we are looking for.

□

The next step is to consider a bounded torsion group G (Case 2). In this case we are going to use the next lemmas.

Lemma 5.3. *Let p be a prime number, $m\in\mathbb{N}$, and P the subgroup of $\mathbb{T}$ consisting of all p^m-th complex roots of unity. For every $s\in P$ and $k\leq m$ with $o(s)\leq p^k$, there exists $s_k\in P$ such that $o(s_k)=p^k$ and $s=s_k^{p^k/o(s)}$.*

Proof. Since $s\in P$, the $o(s)=p^n$ for some $n\leq m$ and there exists a, a non-negative integer number $a<p^n$, such that $s=e^{2a\pi i/p^n}$. Observe that a is not divisible by p. Let $s_k=e^{2a\pi i/p^k}$. It is clear that $o(s_k)=p^k$ and

$$s_k^{p^k/o(s)}=(e^{2a\pi i/p^k})^{p^k/o(s)}=e^{2a\pi i/p^n}=s.$$

□

Lemma 5.4. *Let p be a prime number, $m\in\mathbb{N}$, P the subgroup of $\mathbb{T}$ consisting of all p^m-th complex roots of unity and $H=P^{\mathfrak{c}}$. For every $h\in H$ there exists $g\in H$ with $o(g)=p^m$ and $h\in\langle g\rangle$.*

Proof. Let h be an element of H. For every $\alpha < \mathfrak{c}$, let $n_\alpha \in \mathbb{N}$ such that $o(h(\alpha)) = p^{n_\alpha}$. Since $n_\alpha \leq m$ for every $\alpha < \mathfrak{c}$, there exists

$$n_h = \max\{k_\alpha : \alpha < \mathfrak{c}\}.$$

Denote by $d = m - n_h$ and, for every $\alpha < \mathfrak{c}$, let $k_\alpha = n_\alpha + d$. It is clear that $k_\alpha \leq m$. By Lemma 5.3 for each $\alpha < \mathfrak{c}$ we can find $h^*_\alpha \in P$ such that $o(h^*_\alpha) = p^{k_\alpha}$ and $h(\alpha) = {h^*_\alpha}^{p^{k_\alpha}/o(h(\alpha))}$. Note that $\frac{p^{k_\alpha}}{o(h(\alpha))} = \frac{p^{k_\alpha}}{p^{n_\alpha}} = p^d$.

Let g be the element of H such that $g(\alpha) = h^*_\alpha$. Observe that for every $\alpha < \mathfrak{c}$, $p^d \cdot g(\alpha) = h(\alpha)$, then $h = p^d \cdot g$.

By our definition of n_h, there exists $\beta < c$ such that $o(h(\beta)) = p^{n_h}$. Since the order of $h(\beta)$ is equal than p^{n_h}, $o(h^*_\beta) = p^d \cdot o(h(\beta)) = p^m$ and then $o(g) = p^m$. □

Theorem 5.5. *Let G be an Abelian bounded torsion group such that $|G| \leq 2^{\mathfrak{c}}$. Then there exists a separable precompact Hausdorff group topology for G.*

Proof. We can assume that $|G| > \omega$. Suppose first that for every $g \in G$, the order of g is a power of a fixed prime number p. Since G is a bounded torsion group, we can find $k \in \mathbb{N}$ with $o(g) \leq p^k$ for each $g \in G$. Hence there exists a set $\{g_\alpha : \alpha \in A\} \subset G$ such that $G = \bigoplus_{\alpha \in A} \langle g_\alpha \rangle$ (see [10, Theorem 17.2]). For each $\alpha \in A$, let $\varrho_\alpha : \langle g_\alpha \rangle \to \mathbb{T}$ be the monomorphism defined by $\varrho_\alpha(g_\alpha) = e^{2\pi i/n_\alpha}$, where $n_\alpha = o(g_\alpha)$.

For every $n \leq k$, put $A_n = \{\alpha \in A : o(g_\alpha) = p^n\}$ and $m = \max\{n : |A_n| \geq \omega\}$. Let $J_0 = \{\alpha_j : j \in \omega\}$ be an infinite countable subset of A_m, $G_0 = \bigoplus_{\alpha \in J_0} \langle g_\alpha \rangle$, $J = (\bigcup_{n \leq m} A_n) \setminus J_0$ and $F = \bigcup_{n > m} A_n$. Observe that $G' = \bigoplus_{\alpha \in F} \langle g_\alpha \rangle$ is finite and $|G_0| = \omega$. So $G = G' \oplus G_0 \oplus \bigoplus_{\alpha \in J} \langle g_\alpha \rangle$.

Let $H = P^{\mathfrak{c}}$, where P is the subgroup of $\mathbb{T}$ consisting of all p^m-th complex roots of unity. By the Hewitt-Marczewski-Pondiczery theorem, H is separable. Let D be a countable dense subgroup of H. Since D is a bounded torsion group, it is direct sum of cyclic groups, i.e., $D = \bigoplus_{j \in \omega} \langle d_j \rangle$. By Lemma 5.4 we can assume that $o(d_j) = p^m$ for every $n \in \omega$.

Let φ be a monomorphism $\varphi : G_0 \to H$ such that $\varphi(g_{\alpha_j}) = d_j$ for every $n \in \omega$. We will extend this monomorphism to $\bar{G} = G_0 \oplus \bigoplus_{\alpha \in J} \langle g_\alpha \rangle$. For every $\beta < \mathfrak{c}$, let $\varphi_\beta = p_\beta \circ \varphi$, where $p_\beta : H \to P_{(\beta)}$ is the natural projection of H onto the β-th factor.

Consider J as a subspace of the space $2^{\mathfrak{c}}$ endowed with the product topology. Let $\mathcal{B}$ be the base of canonical open sets in $2^{\mathfrak{c}}$, $|\mathcal{B}| = \mathfrak{c}$.

For every $\bar{g} \in \bar{G}$, there exists $\bar{g}_0 \in G_0$ and a finite set $c(\bar{g}) \subset J$ such that $\bar{g} = \bar{g}_0 + l_{\bar{g}}$, where $l_{\bar{g}} \in \bigoplus_{\alpha \in c(\bar{g})} \langle g_\alpha \rangle$. If $\bar{g} \in \bar{G} \setminus G_0$, then $c(\bar{g}) \neq \varnothing$. Let $\alpha_{\bar{g}} \in c(\bar{g})$ be an arbitrary element of $c(\bar{g})$ and choose $U_{\bar{g}} \in \mathcal{B}$ such that $U_{\bar{g}} \cap c(\bar{g}) = \{\alpha_{\bar{g}}\}$.

The set $S = \{(\bar{g}_0, U_{\bar{g}}) : \bar{g} \in \bar{G} \setminus G_0\}$ has cardinality less than or equal to $|G_0 \times \mathcal{B}| = \omega \cdot \mathfrak{c} = \mathfrak{c}$. Let $S = \{S_\beta : \beta < \mathfrak{c}\}$ be an enumeration of S, where S_β is a pair (a_β, U_β) with $a_\beta \in G_0$ and $U_\beta \in \mathcal{B}$.

If $\varphi_\beta(a_\beta) = 1$, then let $\psi_\beta : \bigoplus_{\alpha \in J} \langle g_\alpha \rangle \to P$ be a homomorphism such that $\psi_\beta|_{\langle g_\alpha \rangle} = \varrho_\alpha$ for each $\alpha \in U_\beta$ and $\psi_\beta(g_\alpha) = 1$ if $\alpha \in J \setminus U_\beta$. If $\varphi_\beta(a_\beta) \neq 1$, put

$\psi_\beta \equiv \mathbf{1}$. Let $\overline{\varphi}_\beta = \varphi_\beta \oplus \psi_\beta$. It is clear that, for each $\beta < \mathfrak{c}$, $\overline{\varphi}_\beta$ is an extension of φ_β. Therefore $\overline{\varphi} = \triangle_{\beta<\mathfrak{c}}\overline{\varphi}_\beta$ is an extension of φ and $\ker(\overline{\varphi}) \cap G_0 = \ker(\varphi) = \{e\}$.

Choose $\bar{g} \in \bar{G} \setminus G_0$. Then $\bar{g} = \bar{g}_0 + l_{\bar{g}}$ where $l_{\bar{g}} = \sum_{\alpha\in c(\bar{g})} l_\alpha \in \bigoplus_{\alpha\in c(\bar{g})}\langle g_\alpha\rangle$. There exists $\beta < \mathfrak{c}$ such that $\bar{g}_0 = a_\beta$ and $U_{\bar{g}} = U_\beta$. By our definition of U_β, $U_\beta \cap c(\bar{g}) = \{\alpha_{\bar{g}}\}$ and $l_{\alpha_{\bar{g}}}$ is different from the identity element. It follows that

$$\overline{\varphi}_\beta(\bar{g}) = \overline{\varphi}_\beta(\bar{g}_0+l_{\bar{g}}) = \varphi_\beta(a_\beta)\cdot\psi_\beta(l_{\bar{g}}) = \varphi_\beta(a_\beta)\cdot \prod_{\alpha\in c(\bar{g})} \psi_\beta(l_\alpha) = \varphi_\beta(a_\beta)\cdot\psi_\beta(l_{\alpha_{\bar{g}}}).$$

If $\varphi_\beta(a_\beta) = 1$ then $\psi_\beta(l_{\alpha_{\bar{g}}}) = \varrho_\alpha(l_{\alpha_{\bar{g}}}) \neq 1$ because ϱ_α is an isomorphism and $l_{\alpha_{\bar{g}}}$ is different from the neutral element. Therefore

$$\overline{\varphi}_\beta(\bar{g}) = \varphi_\beta(a_\beta)\cdot\psi_\beta(l_{\alpha_{\bar{g}}}) = 1\cdot\psi_\beta(l_{\alpha_{\bar{g}}}) \neq 1.$$

If $\varphi_\beta(a_\beta) \neq 1$, then $\psi_\beta(l_{\alpha_{\bar{g}}}) = 1$. It follows that

$$\overline{\varphi}_\beta(\bar{g}) = \varphi_\beta(a_\beta)\cdot\psi_\beta(l_{\alpha_{\bar{g}}}) = \varphi_\beta(a_\beta)\cdot 1 \neq 1.$$

So $\overline{\varphi}$ is a monomorphism of $\bar{G}$ to $P^{\mathfrak{c}}$ such that $D \subset \overline{\varphi}(G_0)$. Hence $\overline{\varphi}(\bar{G})$ is a precompact, separable topological group.

Let us consider G' as a finite subgroup of $\mathbb{T}^F$. If $g \in G$, then there exist $g' \in G'$ and $\bar{g} \in \bar{G}$ such that $g = g' + \bar{g}$. Let $\tilde{\varphi} : G \to G' \times P^{\mathfrak{c}}$, $\tilde{\varphi}(g) = (g', \overline{\varphi}(\bar{g}))$. Then $\tilde{\varphi}$ is a monomorphism of G to $G' \times P^{\mathfrak{c}}$ and therefore G has a precompact, separable, Hausdorff group topology.

Now, suppose that G is an arbitrary bounded torsion Abelian group and let $G = \bigoplus_{i\leq n} G_{p_i}$ be the decomposition of G into the direct sum of p-primary components (see [10, Theorem 8.4]). As shown above, each G_{p_i} admits has a precompact, separable, Hausdorff group topology. Since the number of factors is finite, $\bigoplus_{i\leq n} G_{p_i}$ is algebraically isomorphic to $\prod_{i\leq n} G_{p_i}$, so G admits a precompact, separable, Hausdorff group topology as well.

□

5.1. **The Case of Unbounded Torsion Groups.** The case when G is an unbounded torsion Abelian group requires special attention. In this case, we will adapt some ideas from the proof of [5, Theorem 2.3].

We call a connected open subset V of $\mathbb{T}$ an *open arc*, and we denote by $l(V)$ the length of V.

Lemma 5.6. *Suppose that V is an open arc in $\mathbb{T}$, z_1, $z_2 \in \mathbb{T}$, n, $m \in \mathbb{N}$, $1 \leq n < m$, and $4\pi/m < l(V)$. Then there exists $y \in V$ such that $my = z_1$ and $ny \neq z_2$.*

Proof. Let $z \in \mathbb{T}$ and $k \in \mathbb{N}$. The distance between any two different k-th roots of z is a multiple of $2\pi/k$. Since $l(V) > 4\pi/m$, there are two distinct m-th roots of z_1 in V. Let y_1, y_2 be two elements of V such that $my_1 = my_2 = z_1$ and the distance between y_1 and y_2 is $2\pi/m$. Note that y_1 and y_2 can not be both n-th roots of z_2, otherwise the distance between them would be greater than or equal to $2\pi/n$, and it would follow that $m \leq n$ contradicting the assumptions of the lemma. □

Lemma 5.7. *Let K be a countable subgroup of $\mathbb{T}^{\mathfrak{c}}$ and $f' \in K$, $m \geq 2$. Suppose that $\{V_\alpha : \alpha < \mathfrak{c}\}$ is a family of open arcs of $\mathbb{T}$ such that $4\pi/m < l(V_\alpha)$, for every $\alpha < \mathfrak{c}$. Then there exists $f \in \prod\{V_\alpha : \alpha < \mathfrak{c}\}$ such that $mf = f'$ and $nf \notin K$ for each n with $1 \leq n < m$.*

Proof. Let $K \times \{1, ..., m-1\} = \{(h_k, n_k) : k \in \omega\}$ be an enumeration of $K \times \{1, ..., m-1\}$. For each $k < \omega$ we will define $\alpha_k < \mathfrak{c}$ and $x_{\alpha_k} \in \mathbb{T}$ satisfying the following conditions:

(i_k): $\alpha_k \neq \alpha_j$ if $j < k$;
(ii_k): $x_{\alpha_k} \in V_{\alpha_k}$;
(iii_k): $mx_{\alpha_k} = f'(\alpha_k)$;
(iv_k): $n_k x_{\alpha_k} \neq h_k(\alpha_k)$.

Let $\alpha_0 < \mathfrak{c}$ be an arbitrary ordinal. By Lemma 5.6 (with $V = V_{\alpha_0}$, $z_1 = f'(\alpha_0)$, $z_2 = h_0(\alpha_0)$, $n = n_0$) we can choose an element $x_{\alpha_0} \in V_{\alpha_0}$ that satisfies (ii_0), (iii_0) and (iv_0). Condition (i_0) is vacuous.

Suppose that for every $j < k$ we have chosen α_j and x_{α_j} such that conditions (i_j) - (iv_j) are satisfied. We can pick $\alpha_k < \mathfrak{c}$ that satisfies (i_k). By Lemma 5.6 with $V = V_{\alpha_k}$, $z_1 = f'(\alpha_k)$, $z_2 = h_k(\alpha_k)$, and $n = n_k$, we can choose x_{α_k} that satisfies (ii_k) - (iv_k).

Finally, for each $\alpha \in \mathfrak{c} \setminus \{\alpha_k : k \in \omega\}$ we use Lemma 5.6 again with $V = V_\alpha$, $z_1 = f'(\alpha)$, $z_2 = 1$, $n = 1$ to select $x_\alpha \in V_\alpha$ such that $mx_\alpha = f'(\alpha)$.

We define $f \in \mathbb{T}^{\mathfrak{c}}$ by $f(\alpha) = x_\alpha$ for each $\alpha < \mathfrak{c}$. Then:

- $f(\alpha) \in V_\alpha$ for each $\alpha < \mathfrak{c}$, therefore $f \in \prod\{V_\alpha : \alpha < \mathfrak{c}\}$.
- $mf(\alpha) = mx_\alpha = f'(\alpha)$ for each $\alpha < \mathfrak{c}$, so $mf = f'$.
- Given $n \in \{1, ..., m-1\}$ and $h \in K$, there exists $k \in \omega$ such that $(h, n) = (h_k, n_k)$. By conditions (iii_k) and (iv_k), we have that

$$nf(\alpha_k) = n_k x_{\alpha_k} \neq h_k(\alpha_k) = h(\alpha_k).$$

Since $h \in K$ is arbitrary, $nf \notin K$ for every $n < m$.

□

The proof of the following lemma can be found in [1, Lemma 1.1.5]:

Lemma 5.8. *Let G and G^* be Abelian topological groups, K and K^* subgroups of G and G^*, respectively. Suppose that there exist $x \in G$, $x^* \in G^*$, $m \in \mathbb{N}$, $m \geq 2$ and an isomorphism $\psi : K \to K^*$ that satisfy the following conditions:*

- *$mx \in K$ and $mx^* \in K^*$;*
- *$nx \notin K$ and $nx^* \notin K^*$ for every $n \in \mathbb{N}$, $1 \leq n < m$;*
- *$\psi(mx) = mx^*$.*

Then there exists a unique isomorphism $\varphi : K + \langle x \rangle \to K^* + \langle x^* \rangle$ extending ψ such that $\varphi(x) = x^*$.

Now we are going to give some definitions from group theory. A system $\{a_1, ..., a_k\}$ of a group G is called *independent* if

$$n_1 a_1 + ... + n_k a_k = 0 \quad (n_i \in \mathbb{Z})$$

implies

$$n_1 a_1 = ... = n_k a_k = 0.$$

We say that an infinite system L of the group G is independent if any finite subset of L is independent. By the *rank* $r(G)$ of an Abelian group G is meant the cardinal number of a maximal independent system in G. The *torsion-free rank* $r_0(G)$ is the cardinal of the maximal independent system which contains only elements of infinite order. For each prime number p, the *p-rank* $r_p(G)$ of G is the cardinal of a maximal independent system which contains only elements whose orders are powers of p.

The next lemma can be found in [6, Lemma 3.17].

Lemma 5.9. *Let G and G^* be Abelian groups such that $|G| \leq r(G^*)$ and $|G| \leq r_p(G^*)$ for every prime number p. Suppose that H is a subgroup of G satisfying $r(H) < r(G^*)$ and $r_p(H) < r_p(G^*)$ for every prime p. If G^* is a divisible group, then every monomorphism $\varphi : H \to G^*$ can be extended to a monomorphism $\psi : G \to G^*$.*

Now we are in position to prove the following theorem.

Theorem 5.10. *Let G be an unbounded torsion Abelian group with $|G| \leq 2^{\mathfrak{c}}$. Then G admits a separable, precompact, Hausdorff group topology.*

Proof. Let $\mathcal{V}$ be a countable base for the topology of $\mathbb{T}$ consisting of open arcs such that $\mathbb{T} \in \mathcal{V}$. Since G is an unbounded torsion group, we can choose a subset $S \subset G \setminus \{e\}$ such that $|nS| = \omega$ for every $n \in \mathbb{N}$, where e is the unity of G.

Consider $\mathfrak{c}$ as the topological space 2^ω and let $\mathfrak{B}$ be the canonical base for 2^ω consisting of non-empty clopen subsets of 2^ω. Then $|\mathfrak{B}| = \omega$.

Let $\mathbb{U}$ be the set of all finite open covers of 2^ω formed by pairwise disjoint sets. For $\mathcal{U} \in \mathbb{U}$ and $\alpha < \mathfrak{c}$ let $U_{\alpha,\mathcal{U}}$ denote the unique $U \in \mathcal{U}$ such that $\alpha \in U$. Put $\mathbb{E} = \{(\mathcal{U}, \upsilon) : \mathcal{U} \in \mathbb{U}$ and $\upsilon : \mathcal{U} \to \mathcal{V}$ is a function$\}$. For $(\mathcal{U}, \upsilon) \in \mathbb{E}$, let $F(\mathcal{U}, \upsilon) = \prod\{\upsilon(U_{\alpha,\mathcal{U}}) : \alpha < \mathfrak{c}\}$.

Clearly $\mathbb{E}$ is countable. Let $\mathbb{E} = \{(\mathcal{U}_k, \upsilon_k) : k \in \omega\}$ be an enumeration of $\mathbb{E}$ such that $\mathcal{U}_0 = \{2^\omega\}$ and $\upsilon_0(2^\omega) = \mathbb{T}$.

For each $k < \omega$, choose $n_k \in \mathbb{N}$ such that

$$4\pi/n_k < \min\{l(\upsilon(U)) : U \in \mathcal{U}_k\}.$$

By recursion on $k \in \omega$ we will choose an element $x_k \in S$ and define a map $\varphi_k : H_k = \langle\{x_j : j \leq k\}\rangle \to \mathbb{T}^{\mathfrak{c}}$ satisfying the following conditions:

(i$_k$): $\varphi_k(x_k) \in F(\mathcal{U}_k, \upsilon_k)$;
(ii$_k$): φ_k is a monomorphism;
(iii$_k$): $\varphi_k|_{H_j} = \varphi_j$ for all $j < k$.

Pick any element x_0 in S and let $\varphi_0 : \langle x_0\rangle \to \mathbb{T}^{\mathfrak{c}}$ be an arbitrary monomorphism. Then conditions (i$_0$) and (ii$_0$) are satisfied, while condition (iii$_0$) is vacuous. Now let $k \in \mathbb{N}$, and suppose that $x_j \in S$ and a map φ_j satisfying (i$_j$), (ii$_j$) and (iii$_j$) have already been constructed for every $j < k$.

Put $H'_k = \bigcup_{j<k} H_j$. Since (ii$_j$), (iii$_j$) hold for every $j < k$, the function $\varphi'_k = \bigcup_{j<k} \varphi_j : H'_k \to \mathbb{T}^{\mathfrak{c}}$ is a monomorphism. Since $\{x_j : j < k\} \subset S$ is finite, $n_k!S \setminus H'_k \neq \varnothing$. Therefore there exists $x_k \in S$ such that $n_k!x_k \notin H'_k$. In particular, $nx_k \notin H'_k$ for all $n \leq n_k$. Let $K = \varphi'_k(H'_k)$. For $\alpha < \mathfrak{c}$, put $V_\alpha = \upsilon_k(U_{\alpha,\mathcal{U}})$. By the choice of n_k we have that $4\pi/n_k \leq l(\upsilon_k(U_{\alpha,\mathcal{U}})) = l(V_\alpha)$ for every $\alpha < \mathfrak{c}$.

Let $m = \min\{n \in \mathbb{N} : nx_k \in H'_k\}$ and $f' = \varphi'_k(mx_k) \in K$. Then $m > n_k$ and $4\pi/m < 4\pi/n_k < l(V_\alpha)$ for every $\alpha < \mathfrak{c}$. Note that $m \geq 2$. By Lemma 5.7 we can find $f \in F(\mathcal{U}, \upsilon) = \prod\{\upsilon(U_{\alpha,\mathcal{U}}) : \alpha < \mathfrak{c}\}$ such that $mf = f'$ and $nf \notin K$ for $n < m$. Put $H_k = \langle\{x_j : j \leq k\}\rangle$. By Lemma 5.8 we can extend φ'_k to a monomorphism $\varphi_k : H_k \to \mathbb{T}^{\mathfrak{c}}$ with $\varphi_k(x_k) = f$.

We are going to verify that φ_k satisfies (i$_k$), (ii$_k$) and (iii$_k$). As $\varphi_k(x_k) = f \in F(\mathcal{U}, \upsilon)$, the condition (i$_k$) is satisfied. By Lemma 5.8, φ_k is a monomorphism that extends φ'_k, so (ii$_k$) and (iii$_k$) are satisfied.

Let $H = \bigcup_{k\in\omega} H_k$ and $\varphi = \bigcup_{k\in\omega} \varphi_k$. Since (ii$_k$) and (iii$_k$) are fulfilled for every $k \in \omega$, we have that $\varphi : H \to \mathbb{T}^{\mathfrak{c}}$ is a monomorphism.

We claim that $\varphi(H \cap S)$ is a dense subset of $\mathbb{T}^{\mathfrak{c}}$. Let W be a non-empty open set of $\mathbb{T}^{\mathfrak{c}}$. Then there exist a finite subset $I = \{\alpha_1, ..., \alpha_n\}$ of $\mathfrak{c}$ and non-empty open arcs $V_{\alpha_1}, ..., V_{\alpha_n} \in \mathcal{V}$ such that $\prod_{\alpha<\mathfrak{c}} W_\alpha \subset W$, where $W_\alpha = V_\alpha$ if $\alpha \in I$ and $W_\alpha = \mathbb{T}$ otherwise. Let $\mathcal{U} = \{U_1, ..., U_n\} \in \mathbb{U}$ be such that $\alpha_i \in U_i$ for every $i \leq n$ and take $\upsilon : \mathcal{U} \to \mathcal{V}$, $\upsilon(U_i) = V_{\alpha_i}$. Then $(\mathcal{U}, \upsilon) \in \mathbb{E}$ and therefore there exists $k \in \omega$ such that $(\mathcal{U}, \upsilon) = (\mathcal{U}_k, \upsilon_k)$. Clearly $F(\mathcal{U}_k, \upsilon_k) = F(\mathcal{U}, \upsilon) \subset \prod_{\alpha<\mathfrak{c}} W_\alpha \subset W$. Since $x_k \in S \cap H_k \subset H \cap S$ and $\varphi(x_k) = \varphi_k(x_k) \in F(\mathcal{U}_k, \upsilon_k)$, it follows that $\varphi(H \cap S) \cap W \neq \varnothing$. This implies the density of $\varphi(H \cap S)$ in $\mathbb{T}^{\mathfrak{c}}$.

By Lemma 5.9, the monomorphism φ can be extended to a monomorphism $\psi : G \to \mathbb{T}^{\mathfrak{c}}$. Therefore $\psi(G)$ is a dense separable subgroup of $\mathbb{T}^{\mathfrak{c}}$.

□

By Theorems 5.2, 5.5 and 5.10 we conclude:

Theorem 5.11. *Let G be an Abelian group with $|G| \leq 2^{\mathfrak{c}}$. Then G admits a separable, precompact, Hausdorff group topology.*

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Luis Felipe Morales López (luisfelipemorales@live.com.mx)
Posgrado en Matemáticas, Departamento de Matemáticas, Universidad Autónoma Metropolitana - Iztapalapa, Av. San Rafael Atlixco 186, Col. Vicentina, Iztapalapa, C.P. 09340, D.F., México.