Applied General Topology
© Universidad Politécnica de Valencia
Volume 13, no. 2, 2012
pp. 167-178

Classification of separately continuous mappings with values in σ-metrizable spaces

Olena Karlova

Abstract

We prove that every vertically nearly separately continuous mapping defined on a product of a strong PP-space and a topological space and with values in a strongly σ-metrizable space with a special stratification, is a pointwise limit of continuous mappings.

2010 MSC: *54C08, 54C05, 54E20*

Keywords: *separately continuous mapping, σ-metrizable space, strong PP-space, Baire classification, Lebesgue classification*

1. Introduction

Let X, Y and Z be topological spaces.

By $C(X,Y)$ we denote the collection of all continuous mappings from X to Y.

For a mapping $f : X \times Y \to Z$ and a point $(x,y) \in X \times Y$ we write

$$f^x(y) = f_y(x) = f(x,y).$$

We say that a mapping $f : X \times Y \to Z$ is *separately continuous*, $f \in CC(X\times Y, Z)$, if $f^x \in C(Y,Z)$ and $f_y \in C(X,Z)$ for every point $(x,y) \in X\times Y$. A mapping $f : X\times Y \to Z$ is said to be *vertically nearly separately continuous*, $f \in C\overline{C}(X \times Y, Z)$, if $f_y \in C(X,Z)$ for every $y \in Y$ and there exists a dense set $D \subseteq X$ such that $f^x \in C(Y,Z)$ for all $x \in D$.

Let $B_0(X,Y) = C(X,Y)$. Assume that the classes $B_\xi(X,Y)$ are already defined for all $\xi < \alpha$, where $\alpha < \omega_1$. Then $f : X \to Y$ is said to be *of the α-th Baire class*, $f \in B_\alpha(X,Y)$, if f is a pointwise limit of a sequence of mappings $f_n \in B_{\xi_n}(X,Y)$, where $\xi_n < \alpha$. In particular, $f \in B_1(X,Y)$ if it is a pointwise limit of a sequence of continuous mappings.

In 1898 H. Lebesgue [12] proved that every real-valued separately continuous function of two real variables is of the first Baire class. Lebesgue's theorem was generalized by many mathematicians (see [4, 15, 17, 19, 18, 1, 2, 5, 6, 16] and the references given there). W. Rudin[17] showed that $C\overline{C}(X \times Y, Z) \subseteq B_1(X \times Y, Z)$ if X is a metrizable space, Y a topological space and Z a locally convex topological vector space. Naturally the following question has been arose, which is still unanswered.

Problem 1.1. *Let X be a metrizable space, Y a topological space and Z a topological vector space. Does every separately continuous mapping $f : X \times Y \to Z$ belong to the first Baire class?*

V. Maslyuchenko and A. Kalancha [5] showed that the answer is positive, when X is a metrizable space with finite Čech-Lebesgue dimension. T. Banakh [1] gave a positive answer in the case that X is a metrically quarter-stratifiable paracompact strongly countably dimensional space and Z is an equiconnected space. In [8] it was shown that the answer to Problem 1.1 is positive for metrizable spaces X and Y and a metrizable arcwise connected and locally arcwise connected space Z. It was pointed out in [9] that $CC(X \times Y, Z) \subseteq B_1(X \times Y, Z)$ if X is a metrizable space, Y is a topological space and Z is an equiconnected strongly σ-metrizable space with a stratification $(Z_n)_{n=1}^{\infty}$ (see the definitions below), where Z_n is a metrizable arcwise connected and locally arcwise connected space for every $n \in \mathbb{N}$.

In this paper we generalize the above-mentioned result from [9] to the case of vertically nearly separately continuous mappings. To do this, we introduce the class of strong PP-spaces which includes the class of all metrizable spaces. In Section 3 we investigate some properties of strong PP-spaces. In Section 4 we establish an auxiliary result which generalizes the famous Kuratowski-Montgomery theorem (see [11] and [14]). Finally, in Section 5 we prove that the inclusion $C\overline{C}(X \times Y, Z) \subseteq B_1(X \times Y, Z)$ holds if X is a strongly PP-space, Y is a topological space and Z is a contractible space with a stratification $(Z_n)_{n=1}^{\infty}$, where Z_n is a metrizable arcwise connected and locally arcwise connected space for every $n \in \mathbb{N}$.

2. Preliminary observations

A subset A of a topological space X is *a zero (co-zero) set* if $A = f^{-1}(0)$ ($A = f^{-1}((0,1])$) for some continuous function $f : X \to [0,1]$.

Let $\mathcal{G}_0^*$ and $\mathcal{F}_0^*$ be collections of all co-zero and zero subsets of X, respectively. Assume that the classes $\mathcal{G}_\xi^*$ and $\mathcal{F}_\xi^*$ are defined for all $\xi < \alpha$, where $0 < \alpha < \omega_1$. Then, if α is odd, the class $\mathcal{G}_\alpha^*$ ($\mathcal{F}_\alpha^*$) is consists of all countable intersections (unions) of sets of lower classes, and, if α is even, the class $\mathcal{G}_\alpha^*$ ($\mathcal{F}_\alpha^*$) is consists of all countable unions (intersections) of sets of lower classes. The classes $\mathcal{F}_\alpha^*$ for odd α and $\mathcal{G}_\alpha^*$ for even α are said to be *functionally additive*, and the classes $\mathcal{F}_\alpha^*$ for even α and $\mathcal{G}_\alpha^*$ for odd α are called *functionally multiplicative*. If a set belongs to the α'th functionally additive and functionally multiplicative class, then it is called *functionally ambiguous of the α'th class.* Note that $A \in \mathcal{F}^*_\alpha$ if and only if $X \setminus A \in \mathcal{G}^*_\alpha$.

If a set A is of the first functionally additive (multiplicative) class, we say that A is an F^*_σ (G^*_δ) set.

Let us observe that if X is a perfectly normal space (i.e. a normal space in which every closed subset is G_δ), then functionally additive and functionally multiplicative classes coincide with ordinary additive and multiplicative classes respectively, since every open set in X is functionally open.

Lemma 2.1. *Let $\alpha \geq 0$, X be a topological space and let $A \subseteq X$ be of the α'th functionally multiplicative class. Then there exists a function $f \in B_\alpha(X, [0,1])$ such that $A = f^{-1}(0)$.*

Proof. The hypothesis of the lemma is obvious if $\alpha = 0$.

Suppose the assertion of the lemma is true for all $\xi < \alpha$ and let A be a set of the α'th functionally multiplicative class. Then $A = \bigcap\limits_{n=1}^{\infty} A_n$, where A_n belong to the α_n'th functionally additive class with $\alpha_n < \alpha$ for all $n \in \mathbb{N}$. By assumption, there exists a sequence of functions $f_n \in B_{\alpha_n}(X, [0,1])$ such that $A_n = f_n^{-1}((0,1])$. Notice that for every n the characteristic function χ_{A_n} of A_n belongs to the α-th Baire class. Indeed, setting $h_{n,m}(x) = \sqrt[m]{f_n(x)}$, we obtain a sequence of functions $h_{n,m} \in B_{\alpha_n}(X, [0,1])$ which is pointwise convergent to χ_{A_n}. Now let

$$f(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{A_n}(x).$$

for all $x \in X$. Then $f \in B_\alpha(X, [0,1])$ as a sum of a uniform convergent series of functions of the α'th class. Moreover, it is easy to see that $A = f^{-1}(0)$. $\square$

A topological space X is called

- *equiconnected* if there exists a continuous function $\lambda : X \times X \times [0,1] \to X$ such that
  (1) $\lambda(x, y, 0) = x$;
  (2) $\lambda(x, y, 1) = y$;
  (3) $\lambda(x, x, t) = x$
  for all $x, y \in X$ and $t \in [0,1]$.
- *contractible* if there exist $x^* \in X$ and a continuous mapping $\gamma : X \times [0,1] \to X$ such that $\gamma(x, 0) = x$ and $\gamma(x, 1) = x^*$. A contractible space X with such a point x^* and such a mapping γ is denoted by (X, x^*, γ).

Remark that every convex subset X of a topological vector space is equiconnected, where $\lambda : X \times X \times [0,1] \to X$ is defined by the formula $\lambda(x, y, t) = (1-t)x + ty$, $x, y \in X$, $t \in [0,1]$.

It is easily seen that a topological space X is contractible if and only if there exists a continuous mapping $\lambda : X \times X \times [0,1] \to X$ such that $\lambda(x, y, 0) = x$ and $\lambda(x, y, 1) = y$ for all $x, y \in X$. Indeed, if (X, x^*, γ) is a contractible space, then the formula

$$\lambda(x,y,t)=\begin{cases}\gamma(x,2t), & 0\le t\le \frac12,\\ \gamma(y,-2t+2), & \frac12<t\le 1.\end{cases}$$

defines a continuous mapping $\lambda : X\times X\times[0,1]\to X$ with the required properties. Conversely, if X is equiconnected, then fixing a point $x^*\in X$ and setting $\gamma(x,t)=\lambda(x,x^*,t)$, we obtain that the space (X,x^*,γ) is contractible.

Lemma 2.2. *Let $0\le\alpha<\omega_1$, X a topological space, Y a contractible space, $A_1,\dots,A_n$ be disjoint sets of the α'th functionally multiplicative class in X and $f_i\in B_\alpha(X,Y)$ for each $1\le i\le n$. Then there exists a mapping $f\in B_\alpha(X,Y)$ such that $f|_{A_i}=f_i$ for each $1\le i\le n$.*

Proof. Let $n=2$. In view of Lemma 2.1 there exist functions $h_i\in B_\alpha(X,[0,1])$ such that $A_i=h_i^{-1}(0)$ for $i=1,2$. We set $h(x)=\dfrac{h_1(x)}{h_1(x)+h_2(x)}$ for all $x\in X$. It is easy to verify that $h\in B_\alpha(X,[0,1])$ and $A_i=h^{-1}(i-1)$, $i=1,2$.

Consider a continuous mapping $\lambda : Y\times Y\times[0,1]\to Y$ such that $\lambda(y,z,0)=y$ and $\lambda(y,z,1)=z$ for all $y,z\in Y$. Let

$$f(x)=\lambda(f_1(x),f_2(x),h(x))$$

for every $x\in X$. Clearly, $f\in B_\alpha(X,Y)$. If $x\in A_1$, then $f(x)=\lambda(f_1(x),f_2(x),0)$ $=f_1(x)$. If $x\in A_2$, then $f(x)=\lambda(f_1(x),f_2(x),1)=f_2(x)$.

Assume that the lemma is true for all $2\le k<n$ and let $k=n$. According to our assumption, there exists a mapping $g\in B_\alpha(X,Y)$ such that $g|_{A_i}=f_i$ for all $1\le i<n$. Since $A=\bigcup\limits_{i=1}^{n-1}A_i$ and A_n are disjoint sets which belong to the α'th functionally multiplicative class in X, by the assumption, there is a mapping $f\in B_\alpha(X,Y)$ with $f|_A=g$ and $f|_{F_n}=f_n$. Then $f|_{F_i}=f_i$ for every $1\le i\le n$. ☐

Let $0\le\alpha<\omega_1$. We say that a mapping $f : X\to Y$ is *of the (functional) α-th Lebesgue class*, $f\in H_\alpha(X,Y)$ ($f\in H^*_\alpha(X,Y)$), if the preimage $f^{-1}(V)$ belongs to the α'th (functionally) additive class in X for any open set $V\subseteq Y$.

Clearly, $H_\alpha(X,Y)=H^*_\alpha(X,Y)$ for any perfectly normal space X.

The following statement is well-known, but we present a proof here for convenience of the reader.

Lemma 2.3. *Let X and Y be topological spaces, $(f_k)_{k=1}^\infty$ a sequence of mappings $f_k : X\to Y$ which is pointwise convergent to a mapping $f : X\to Y$, $F\subseteq Y$ be a closed set such that $F=\bigcap\limits_{n=1}^\infty\overline{V}_n$, where $(V_n)_{n=1}^\infty$ is a sequence of open sets in Y such that $\overline{V_{n+1}}\subseteq V_n$ for all $n\in\mathbb{N}$. Then*

$$f^{-1}(F)=\bigcap_{n=1}^\infty\bigcup_{k=n}^\infty f_k^{-1}(V_n). \tag{2.1}$$

Proof. Let $x \in f^{-1}(F)$ and $n \in \mathbb{N}$. Taking into account that V_n is an open neighborhood of $f(x)$ and $\lim\limits_{k\to\infty} f_k(x) = f(x)$, we obtain that there is $k \geq n$ such that $f_k(x) \in V_n$.

Now let x belong to the right-hand side of (2.1), i.e. for every $n \in \mathbb{N}$ there exists a number $k \geq n$ such that $f_k(x) \in V_n$. Suppose $f(x) \notin F$. Then there exists $n \in \mathbb{N}$ such that $f(x) \notin \overline{V_n}$. Since $U = X \setminus \overline{V_n}$ is a neighborhood of $f(x)$, there exists k_0 such that $f_k(x) \in U$ for all $k \geq k_0$. In particular, $f_k(x) \in U$ for $k = \max\{k_0, n\}$. But then $f_k(x) \notin V_n$, a contradiction. Hence, $x \in f^{-1}(F)$. □

Lemma 2.4. *Let X be a topological space, Y a perfectly normal space and $0 \leq \alpha < \omega_1$. Then $B_\alpha(X,Y) \subseteq H^*_\alpha(X,Y)$ if α is finite, and $B_\alpha(X,Y) \subseteq H^*_{\alpha+1}(X,Y)$ if α is infinite.*

Proof. Let $f \in B_\alpha(X,Y)$. Fix an arbitrary closed set $F \subseteq Y$. Since Y is perfectly normal, there exists a sequence of open sets $V_n \subseteq Y$ such that $\overline{V_{n+1}} \subseteq V_n$ and $F = \bigcap\limits_{n=1}^{\infty} \overline{V}_n$. Moreover, there exists a sequence of mappings $f_k : X \to Y$ of Baire classes $< \alpha$ which is pointwise convergent to f on X. By Lemma 2.3, equality (2.1) holds. Now put $A_n = \bigcup\limits_{k=n}^{\infty} f_k^{-1}(V_n)$.

If $\alpha = 0$, then f is continuous and $f^{-1}(F)$ is a zero set in X, since F is a zero set in Y.

Suppose the assertion of the lemma is true for all finite ordinals $1 \leq \xi < \alpha$. We show that it is true for α. Remark that $f_k \in B_{\alpha-1}(X,Y)$ for every $k \geq 1$. By assumption, $f_k \in H^*_{\alpha-1}(X,Y)$ for every $k \in \mathbb{N}$. Then A_n is of the functionally additive class $\alpha - 1$. Therefore, $f^{-1}(F)$ belongs to the α'th functionally multiplicative class.

Assume the assertion of the lemma is true for all ordinals $\omega_0 \leq \xi < \alpha$. For all $k \in \mathbb{N}$ we choose $\alpha_k < \alpha$ such that $f_k \in B_{\alpha_k}(X,Y)$ for every $k \geq 1$. The preimage $f_k^{-1}(V_n)$, being of the (α_k+1)'th functionally additive class, belongs to the α'th functionally additive class for all $k, n \in \mathbb{N}$, provided $\alpha_k + 1 \leq \alpha$. Then A_n is of the α'th functionally additive class, hence, $f^{-1}(F)$ belongs to the $(\alpha+1)$'th functionally multiplicative class. □

Recall that a family $\mathcal{A} = (A_i : i \in I)$ of sets A_i *refines* a family $\mathcal{B} = (B_j : j \in J)$ of sets B_j if for every $i \in I$ there exists $j \in J$ such that $A_i \subseteq B_j$. We write in this case $\mathcal{A} \preceq \mathcal{B}$.

3. PP-spaces and their properties

Definition 3.1. A topological space X is said to be a *(strong) PP-space* if (for every dense set D in X) there exist a sequence $((\varphi_{i,n} : i \in I_n))_{n=1}^{\infty}$ of locally finite partitions of unity on X and a sequence $((x_{i,n} : i \in I_n))_{n=1}^{\infty}$ of families of points of X (of D) such that

$$(\forall x \in X)((\forall n \in \mathbb{N}\ \ x \in \mathrm{supp}\varphi_{i_n,n}) \Longrightarrow (x_{i_n,n} \to x)) \tag{3.1}$$

Remark that Definition 3.1 is equivalent to the following one.

Definition 3.2. A topological space X is a *(strong) PP-space* if (for every dense set D in X) there exist a sequence $((U_{i,n} : i \in I_n))_{n=1}^{\infty}$ of locally finite covers of X by co-zero sets $U_{i,n}$ and a sequence $((x_{i,n} : i \in I_n))_{n=1}^{\infty}$ of families of points of X (of D) such that

$$(\forall x \in X)((\forall n \in \mathbb{N}\ \ x \in U_{i_n,n}) \Longrightarrow (x_{i_n,n} \to x)) \tag{3.2}$$

Clearly, every strong PP-space is a PP-space.

Proposition 3.3. *Every metrizable space is a strong PP-space.*

Proof. Let X be a metrizable space and d a metric on X which generates its topology. Fix an arbitrary dense set D in X. For every $n \in \mathbb{N}$ let $\mathcal{B}_n$ be a cover of X by open balls of diameter $\frac{1}{n}$. Since X is paracompact, for every n there exists a locally finite cover $\mathcal{U}_n = (U_{i,n} : i \in I_n)$ of X by open sets $U_{i,n}$ such that $\mathcal{U}_n \preceq \mathcal{B}_n$. Notice that each $U_{i,n}$ is a co-zero set. Choose a point $x_{i,n} \in D \cap U_{i,n}$ for all $n \in \mathbb{N}$ and $i \in I_n$. Let $x \in X$ and let U be an arbitrary neighborhood of x. Then there is $n_0 \in \mathbb{N}$ such that $B(x, \frac{1}{n}) \subseteq U$ for all $n \geq n_0$. Fix $n \geq n_0$ and take $i \in I_n$ such that $x \in U_{i,n}$. Since $\operatorname{diam} U_{i,n} \leq \frac{1}{n}$, $d(x, x_{i,n}) \leq \frac{1}{n}$, consequently, $x_{i,n} \in U$. □

Example 3.4. The Sorgenfrey line $\mathbb{L}$ is a strong PP-space which is not metrizable.

Proof. Recall that the Sorgenfrey line is the real line $\mathbb{R}$ endowed with the topology generated by the base consisting of all semi-intervals $[a, b)$, where $a < b$ (see [3, Example 1.2.2]).

Let $D \subseteq \mathbb{L}$ be a dense set. For any $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ by $\varphi_{i,n}$ we denote the characteristic function of $[\frac{i-1}{n}, \frac{i}{n})$ and choose a point $x_{i,n} \in [\frac{i}{n}, \frac{i+1}{n}) \cap D$. Then the sequences $\big((\varphi_{i,n} : i \in I_n)\big)_{n=1}^{\infty}$ and $\big((x_{i,n} : i \in I_n)\big)_{n=1}^{\infty}$ satisfy (3.1). □

Proposition 3.5. *Every σ-metrizable paracompact space is a PP-space.*

Proof. Let $X = \bigcup\limits_{n=1}^{\infty} X_n$, where $(X_n)_{n=1}^{\infty}$ is an increasing sequence of closed metrizable subspaces, and let d_1 be a metric on X_1 which generates its topology. According to Hausdorff's theorem [3, p. 297] we can extend the metric d_1 to a metric d_2 on X_2. Further, we extend the metric d_2 to a metric d_3 on X_3. Repeating this process, we obtain a sequence $(d_n)_{n=1}^{\infty}$ of metrics d_n on X_n such that $d_{n+1}|_{X_n} = d_n$ for every $n \in \mathbb{N}$. We define a function $d : X^2 \to \mathbb{R}$ by setting $d(x, y) = d_n(x, y)$ for $(x, y) \in X_n^2$.

Fix $n \in \mathbb{N}$ and $m \geq n$. Let $\mathcal{B}_{n,m}$ be a cover of X_m by d-open balls of diameter $\frac{1}{n}$. For every $B \in \mathcal{B}_{n,m}$ there exists an open set V_B in X such that $V_B \cap X_m = B$. Let $\mathcal{V}_{n,m} = \{V_B : B \in \mathcal{B}_{n,m}\}$ and $\mathcal{U}_n = \bigcup\limits_{m=1}^{\infty} \mathcal{V}_{n,m}$. Then $\mathcal{U}_n$ is an open cover of X for every $n \in \mathbb{N}$. Since X is paracompact, for every $n \in \mathbb{N}$ there exists a locally finite partition of unity $(h_{i,n} : i \in I_n)$ on X subordinated to $\mathcal{U}_n$. For every $n \in \mathbb{N}$ and $i \in I_n$ we choose $x_{i,n} \in X_{k(i,n)} \cap \operatorname{supp} h_{i,n}$, where $k(i, n) = \min\{m \in \mathbb{N} : X_m \cap \operatorname{supp} h_{i,n} \neq \emptyset\}$.

Now fix $x \in X$. Let $(i_n)_{n=1}$ be a sequence of indexes $i_n \in I_n$ such that $x \in \operatorname{supp} h_{i_n,n}$. We choose $m \in \mathbb{N}$ such that $x \in X_m$. It is easy to see that $k(i_n, n) \le m$ for every $n \in \mathbb{N}$. Then $x_{i_n,n} \in X_m$. Since $d_m(x_{i_n,n}, x) \le \operatorname{diam} \operatorname{supp} h_{i_n,n} \le \frac{1}{n}$, $x_{i_n,n} \to x$ in X_m. Therefore, $x_{i_n,n} \to x$ in X. □

Denote by $\mathbb{R}^\infty$ the collection of all sequences with a finite support, i.e. sequences of the form $(\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots)$, where $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{R}$. Clearly, $\mathbb{R}^\infty$ is a linear subspace of the space $\mathbb{R}^\mathbb{N}$ of all sequences. Denote by E the set of all sequences $e = (\varepsilon_n)_{n=1}^\infty$ of positive reals ε_n and let

$$U_e = \{x = (\xi_n)_{n=1}^\infty \in \mathbb{R}^\infty : (\forall n \in \mathbb{N})(|\xi_n| \le \varepsilon_n)\}.$$

We consider on $\mathbb{R}^\infty$ the topology in which the system $\mathcal{U}_0 = \{U_e : e \in E\}$ forms the base of neighborhoods of zero. Then $\mathbb{R}^\infty$ equipped with this topology is a locally convex σ-metrizable paracompact space which is not a first countable space, consequently, non-metrizable.

Example 3.6. The space $\mathbb{R}^\infty$ is a PP-space which is not a strong PP-space.

Proof. Remark that $\mathbb{R}^\infty$ is a PP-space by Proposition 3.5.

We show that $\mathbb{R}^\infty$ is not a strong PP-space. Indeed, let

$$A_n = \{(\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots) : |\xi_k| \le \frac{1}{n} \ (\forall 1 \le k \le n)\},$$

$$D = \bigcup_{m=1}^\infty \bigcap_{n=1}^m (\mathbb{R}^\infty \setminus A_n).$$

Then D is dense in $\mathbb{R}^\infty$, but there is no sequence in D which converges to $x = (0, 0, 0, \dots) \in \mathbb{R}^\infty$. Hence, $\mathbb{R}^\infty$ is not a strong PP-space. □

4. The Lebesgue classification

The following result is an analog of theorems of K. Kuratowski [11] and D. Montgomery [14] who proved that every separately continuous function, defined on a product of a metrizable space and a topological space and with values in a metrizable space, belongs to the first Baire class.

Theorem 4.1. *Let X be a strong PP-space, Y a topological space, Z a perfectly normal space and $0 \le \alpha < \omega_1$. Then*

$$C\overline{H^*_\alpha}(X \times Y, Z) \subseteq H^*_{\alpha+1}(X \times Y, Z).$$

Proof. Let $f \in C\overline{H^*_\alpha}(X \times Y, Z)$. Then for the set $X_{H^*_\alpha}(f)$ there exist a sequence $(\mathcal{U}_n)_{n=1}^\infty$ of locally finite covers $\mathcal{U}_n = (U_{i,n} : i \in I_n)$ of X by co-zero sets $U_{i,n}$ and a sequence $((x_{i,n} : i \in I_n))_{n=1}^\infty$ of families of points of the set $X_{H^*_\alpha}(f)$ satisfying condition (3.2).

We choose an arbitrary closed set $F \subseteq Z$. Since Z is perfectly normal, $F = \bigcap\limits_{m=1}^\infty G_m$, where G_m are open sets in Z such that $\overline{G}_{m+1} \subseteq G_m$ for every

$m \in \mathbb{N}$. Let us verify that the equality

$$f^{-1}(F) = \bigcap_{m=1}^{\infty} \bigcup_{n\geq m}^{\infty} \bigcup_{i\in I_n} U_{i,n} \times (f^{x_{i,n}})^{-1}(G_m). \tag{4.1}$$

holds. Indeed, let $(x_0, y_0) \in f^{-1}(F)$. Then $f(x_0, y_0) \in G_m$ for every $m \in \mathbb{N}$. Fix any $m \in \mathbb{N}$. Since $V_m = f_{y_0}^{-1}(G_m)$ is an open neighborhood of x_0, there exists a number $n_0 \geq m$ such that for all $n \geq n_0$ and $i \in I_n$ the inclusion $x_{i,n} \in V_m$ holds whenever $x_0 \in U_{i,n}$. We choose $i_0 \in I_{n_0}$ such that $x_0 \in U_{i_0,n_0}$. Then $f(x_{i_0,n_0}, y_0) \in G_m$. Hence, (x_0, y_0) belongs to the right-hand side of (4.1).

Conversely, let (x_0, y_0) belong to the right-hand side of (4.1). Fix $m \in \mathbb{N}$. We choose sequences $(n_k)_{k=1}^{\infty}$, $(m_k)_{k=1}^{\infty}$ of numbers $n_k, m_k \in \mathbb{N}$ and a sequence $(i_k)_{k=1}^{\infty}$ of indexes $i_k \in I_{n_k}$ such that

$$m = m_1 \leq n_1 < m_2 \leq n_2 < \cdots < m_k \leq n_k < \ldots,$$

$$x_0 \in U_{i_k,n_k} \quad \text{and} \quad f(x_{i_k,n_k}, y_0) \in G_{m_k} \subseteq G_m \quad \text{for every } k \in \mathbb{N}.$$

Since $\lim\limits_{k\to\infty} x_{i_k,n_k} = x_0$ and the mapping f is continuous with respect to the first variable, $\lim\limits_{k\to\infty} f(x_{i_k,n_k}, y_0) = f(x_0, y_0)$. Therefore, $f(x_0, y_0) \in \overline{G}_m$ for every $m \in \mathbb{N}$. Hence, (x_0, y_0) belongs to the left-hand side of (4.1).

Since $f^{x_{i,n}} \in H_{\alpha}^{*}(Y, Z)$, the sets $(f^{x_{i,n}})^{-1}(G_m)$ are of the functionally additive class α in Y. Moreover, all $U_{i,n}$ are co-zero sets in X, consequently, by [6, Theorem 1.5] the set $E_n = \bigcup\limits_{i\in I_n} U_{i,n} \times (f^{x_{i,n}})^{-1}(G_m)$ belongs to the α'th functionally additive class for every n. Therefore, $\bigcup\limits_{n\geq m} E_n$ is of the α'th functionally additive class. Hence, $f^{-1}(F)$ is of the $(\alpha+1)$'th functionally multiplicative class in $X \times Y$. □

Definition 4.2. We say that a topological space X *has the (strong) L-property* or *is a (strong) L-space*, if for every topological space Y every (nearly vertically) separately continuous function $f : X \times Y \to \mathbb{R}$ is of the first Lebesgue class.

According to Theorem 4.1 every strong PP-space has the strong L-property.

Proposition 4.3. *Let X be a completely regular strong L-space. Then for any dense set $A \subseteq X$ and a point $x_0 \in X$ there exists a countable dense set $A_0 \subseteq A$ such that $x_0 \in \overline{A_0}$.*

Proof. Fix an arbitrary everywhere dense set $A \subseteq X$ and a point $x_0 \in A$. Let Y be the space of all real-valued continuous functions on X, endowed with the topology of pointwise convergence on A. Since the evaluation function $e : X \times Y \to \mathbb{R}$, $e(x, y) = y(x)$, is nearly vertically separately continuous, $e \in H_1(X \times Y, \mathbb{R})$. Then $B = e^{-1}(0)$ is G_δ-set in $X \times Y$. Hence, $B_0 = \{y \in Y : y(x_0) = 0\}$ is a G_δ-set in Y. We set $y_0 \equiv 0$ and choose a sequence $(V_n)_{n=1}^{\infty}$ of basic neighborhoods of y_0 in Y such that $\bigcap\limits_{n=1}^{\infty} V_n \subseteq B_0$. For every n there exist a finite set $\{x_{i,n} : i \in I_n\}$ of X and $\varepsilon_n > 0$ such that $V_n = \{y \in Y : \max\limits_{i\in I_n} |y(x_{i,n})| < \varepsilon_n\}$. Let

$$A_0 = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in I_n} \{x_{i,n}\}.$$

Take an open neighborhood U of x_0 in X and suppose that $U \cap A_0 = \emptyset$. Since X is completely regular and $x_0 \notin X \setminus U$, there exists a continuous function $y : X \to \mathbb{R}$ such that $y(x_0) = 1$ and $y(X \setminus U) \subseteq \{0\}$. Then $y \in \bigcap\limits_{n=1}^{\infty} V_n$, but $y \notin B_0$, a contradiction. Therefore, $U \cap A_0 \neq \emptyset$, and $x_0 \in \overline{A_0}$. □

5. Baire classification and σ-metrizable spaces

We recall that a topological space Y is *B-favorable for a space X*, if $H_1(X, Y) \subseteq B_1(X, Y)$ (see [10]).

Definition 5.1. Let $0 \le \alpha < \omega_1$. A topological space Y is called *weakly B_α-favorable for a space X*, if $H^*_\alpha(X, Y) \subseteq B_\alpha(X, Y)$.

Clearly, every B-favorable space is weakly B_1-favorable.

Proposition 5.2. *Let $0 \le \alpha < \omega_1$, X a topological space, $Y = \bigcup\limits_{n=1}^{\infty} Y_n$ a contractible space, $f : X \to Y$ a mapping, $(X_n)_{n=1}^{\infty}$ a sequence of sets of the α'th functionally additive class such that $X = \bigcup\limits_{n=1}^{\infty} X_n$ and $f(X_n) \subseteq Y_n$ for every $n \in \mathbb{N}$. If one of the following conditions holds*

(i) *Y_n is a nonempty weakly B_α-favorable space for X for all n and $f \in H^*_\alpha(X, Y)$, or*
(ii) *$\alpha > 0$ and for every n there exists a mapping $f_n \in B_\alpha(X, Y_n)$ such that $f_n|_{X_n} = f|_{X_n}$,*

then $f \in B_\alpha(X, Y)$.

Proof. If $\alpha = 0$ then the statement is obvious in case (i).

Let $\alpha > 0$. By [6, Lemma 2.1] there exists a sequence $(E_n)_{n=1}^{\infty}$ of disjoint functionally ambiguous sets of the α'th class such that $E_n \subseteq X_n$ and $X = \bigcup\limits_{n=1}^{\infty} E_n$.

In case (i) for every n we choose a point $y_n \in Y_n$ and let

$$f_n(x) = \begin{cases} f(x), & \text{if } x \in E_n, \\ y_n, & \text{if } x \in X \setminus E_n. \end{cases}$$

Since $f \in H^*_\alpha(X, Y)$ and E_n is functionally ambiguous set of the α'th class, $f_n \in H^*_\alpha(X, Y_n)$. Then $f_n \in B_\alpha(X, Y_n)$ provided Y_n is weakly B_α-favorable for X.

For every n there exists a sequence of mappings $g_{n,m} : X \to Y_n$ of classes $< \alpha$ such that $g_{n,m}(x) \underset{m\to\infty}{\to} f_n(x)$ for every $x \in X$. In particular, $\lim\limits_{m\to\infty} g_{n,m}(x) =$

$f(x)$ on E_n. Since E_n is of the α-th functionally additive class, $E_n = \bigcup\limits_{m=1}^{\infty} B_{n,m}$, where $(B_{n,m})_{m=1}^{\infty}$ is an increasing sequence of sets of functionally additive classes $< \alpha$. Let $F_{n,m} = \emptyset$ if $n > m$, and let $F_{n,m} = B_{n,m}$ if $n \le m$. According to Lemma 2.2, for every $m \in \mathbb{N}$ there exists a mapping $g_m : X \to Y$ of a class $< \alpha$ such that $g_m|_{F_{n,m}} = g_{n,m}$, since the system $\{F_{n,m} : n \in \mathbb{N}\}$ is finite for every $m \in \mathbb{N}$.

It remains to prove that $g_m(x) \to f(x)$ on X. Let $x \in X$. We choose a number $n \in \mathbb{N}$ such that $x \in E_n$. Since the sequence $(F_{n,m})_{m=1}^{\infty}$ is increasing, there exists a number m_0 such that $x \in F_{n,m}$ for all $m \ge m_0$. Then $g_m(x) = g_{n,m}(x)$ for all $m \ge m_0$. Hence, $\lim\limits_{m\to\infty} g_m(x) = \lim\limits_{m\to\infty} g_{n,m}(x) = f(x)$. Therefore, $f \in B_\alpha(X, Y)$. □

Definition 5.3. Let $\{X_n : n \in \mathbb{N}\}$ be a cover of a topological space X. We say that $(X, (X_n)_{n=1}^{\infty})$ *has the property* $(*)$ if for every convergent sequence $(x_k)_{k=1}^{\infty}$ in X there exists a number n such that $\{x_k : k \in \mathbb{N}\} \subseteq X_n$.

Proposition 5.4. *Let $0 \le \alpha < \omega_1$, X a strong PP-space, Y a topological space, $(Z, (Z_n)_{n=1}^{\infty})$ have the property $(*)$, let Z_n be closed in Z (and let Z_n be a zero-set in Z if $\alpha = 0$) for every $n \in \mathbb{N}$, and $f \in C\overline{B}_\alpha(X \times Y, Z)$. Then there exists a sequence $(B_n)_{n=1}^{\infty}$ of sets of the α'th /$(\alpha+1)$'th/ functionally multiplicative class in $X \times Y$, if α is finite /infinite/, such that*

$$\bigcup_{n=1}^{\infty} B_n = X \times Y \qquad \text{and} \qquad f(B_n) \subseteq Z_n$$

for every $n \in \mathbb{N}$.

Proof. Since $X_{B_\alpha}(f)$ is dense in X, there exists a sequence $(\mathcal{U}_m = (U_{i,m} : i \in I_m))_{m=1}^{\infty}$ of locally finite co-zero covers of X and a sequence $((x_{i,m} : i \in I_m))_{m=1}^{\infty}$ of families of points of $X_{B_\alpha}(f)$ such that condition (3.2) holds.

In accordance with [16, Proposition 3.2] there exists a pseudo-metric on X such that all the set $U_{i,m}$ are co-zero with respect to this pseudo-metric. Denote by $\mathcal{T}$ the topology on X generated by the pseudo-metric. Obviously, the topology $\mathcal{T}$ is weaker than the initial one. Using the paracompactness of $(X, \mathcal{T})$, for every m we choose a locally finite open cover $\mathcal{V}_m = (V_{s,m} : s \in S_m)$ which refines $\mathcal{U}_m$. By [3, Lemma 1.5.6], for every m there exists a locally finite closed cover $(F_{s,m} : s \in S_m)$ of $(X, \mathcal{T})$ such that $F_{s,m} \subseteq V_{s,m}$ for every $s \in S_m$. Now for every $s \in S_m$ we choose $i(s) \in I_m$ such that $F_{s,m} \subseteq U_{i(s),m}$.

For all $m, n \in \mathbb{N}$ and $s \in S_m$ let

$$A_{s,m,n} = (f^{x_{i(s),m}})^{-1}(Z_n), \quad B_{m,n} = \bigcup_{s\in S_m} (F_{s,m} \times A_{s,m,n}), \quad B_n = \bigcap_{m=1}^{\infty} B_{m,n}.$$

Since f is of the α'th Baire class with respect to the second variable, for every n the set $A_{s,m,n}$ belongs to the α'th functionally multiplicative class /$\alpha+1$/ in Y for all $m \in \mathbb{N}$ and $s \in S_m$, if α is finite /infinite/ by Lemma 2.4. According to [6, Proposition 1.4] the set $B_{m,n}$ is of the α'th /$(\alpha+1)$'th/ functionally multiplicative class in $(X, \mathcal{T}) \times Y$. Then the set B_n is of the α'th $/(\alpha+1)$'th/ functionally multiplicative class in $(X, \mathcal{T}) \times Y$, and, consequently, in $X \times Y$ for every n.

We prove that $f(B_n) \subseteq Z_n$ for every n. To do this, fix $n \in \mathbb{N}$ and $(x, y) \in B_n$. We choose a sequence $(s_m)_{m=1}^{\infty}$ such that $x \in F_{m,s_m} \subseteq U_{m,i(s_m)}$ and $f(x_{m,i(s_m)}, y) \in Z_n$. Then $x_{m,i(s_m)} \underset{m\to\infty}{\to} x$. Since f is continuous with respect to the first variable, $f(x_{m,i(s_m)}, y) \underset{m\to\infty}{\to} f(x, y)$. The set Z_n is closed, then $f(x, y) \in Z_n$.

Now we show that $\bigcup\limits_{n=1}^{\infty} B_n = X \times Y$. Let $(x, y) \in X \times Y$. Then there exists a sequence $(s_m)_{m=1}^{\infty}$ such that $x \in F_{m,s_m} \subseteq U_{m,i(s_m)}$ and $f(x_{m,i(s_m)}, y) \underset{m\to\infty}{\to} f(x, y)$. Since $(Z, (Z_n)_{n=1}^{\infty})$ satisfies $(*)$, there is a number n such that $\{f(x_{m,i_m}, y) : m \in \mathbb{N}\}$ is contained in Z_n, i.e. $y \in A_{m,n,i}$ for every $m \in \mathbb{N}$. Hence, $(x, y) \in B_n$. □

Theorem 5.5. *Let X be a strong PP-space, Y a topological space, $\{Z_n : n \in \mathbb{N}\}$ a closed cover of a contractible perfectly normal space Z, let $(Z, (Z_n)_{n=1}^{\infty})$ satisfy $(*)$ and Z_n be weakly B_1-favorable for $X \times Y$ for every $n \in \mathbb{N}$. Then*

$$C\overline{C}(X \times Y, Z) \subseteq B_1(X \times Y, Z).$$

Proof. Let $f \in C\overline{C}(X \times Y, Z)$. In accordance with Theorem 4.1, $f \in H_1^*(X \times Y, Z)$. Moreover, Proposition 5.4 implies that there exists a sequence of zero-sets $B_n \subseteq X \times Y$ such that $\bigcup\limits_{n=1}^{\infty} B_n = X \times Y$ and $f(B_n) \subseteq Z_n$ for every $n \in \mathbb{N}$. Since for every n the set B_n is an F_σ^*-set and $H_1^*(X \times Y, Z_n) \subseteq B_1(X \times Y, Z_n)$, $f \in B_1(X \times Y, Z)$ by Proposition 5.2. □

Definition 5.6. A topological space X is called *strongly σ-metrizable*, if it is σ-metrizable with a stratification $(X_n)_{n=1}^{\infty}$ and $(X, (X_n)_{n=1}^{\infty})$ has the property $(*)$.

Taking into account that every regular strongly σ-metrizable space with metrizable separable stratification is perfectly normal (see [13, Corollary 4.1.6]) and every metrizable separable arcwise connected and locally arcwise connected space is weakly B_α-favorable for any topological space X for all $0 \le \alpha < \omega_1$ [7, Theorem 3.3.5], we immediately obtain the following corollary of Theorem 5.5.

Corollary 5.7. *Let X be a strong PP-space, Y a topological space and Z a contractible regular strongly σ-metrizable space with a stratification $(Z_n)_{n=1}^{\infty}$, where Z_n is a metrizable separable arcwise connected and locally arcwise connected space for every $n \in \mathbb{N}$. Then*

$$C\overline{C}(X \times Y, Z) \subseteq B_1(X \times Y, Z).$$

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(Received December 2011 – Accepted May 2012)

O. Karlova (Maslenizza.Ua@gmail.com)
Chernivtsi National University, Department of Mathematical Analysis, Kotsjubyns'koho 2, Chernivtsi 58012, Ukraine