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F-door spaces and F-submaximal spaces

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Abstract

Submaximal spaces and door spaces play an enigmatic role in topology. In this paper, reinforcing this role, we are concerned with reaching two main goals:
*The first one is to characterize topological spaces X such that $\mathbf{F}(X)$ is a submaximal space (resp., door space) for some covariant functor $\boldsymbol{F}$ from the category **Top** to itself. $\boldsymbol{T}_0$, $\boldsymbol{\rho}$ and **FH** functors are completely studied.*
*Secondly, our interest is directed towards the characterization of maps f given by a flow (X, f) in the category **Set**, such that $(X, \mathcal{P}(f))$ is submaximal (resp., door) where $\mathcal{P}(f)$ is a topology on X whose closed sets are exactly the f-invariant sets.*

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1. Introduction

Among the oldest separation axioms in topology there are some famous ones as T_0, T_1 and T_2. The T_0-, T_1- and T_2-reflections of a topological space have long been of interest to categorical topologists. The first systematic treatment of separation axioms is due to Urysohn [34].

More detailed discussion was given by Freudenthal and Van Est [35]. The first separation axiom between T_0 and T_1 was introduced by J.W. T. Youngs [39] who encountered it in the study of locally connected spaces.

In 1962, C. E. Aull and W. J. Thron were interested in separation axioms between T_0 and T_1-spaces (see [1]).

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In 2004, Karim Belaid et al in [4] gave some new separation axioms using the theory of categories and functors in the goal of studying Wallman compactification.

Definition 1.1. Let i, j be two integers such that $0 \leq i < j \leq 2$. Let us denote by $\mathbf{T}_i$ the functor from **Top** to **Top** which takes each topological space X to its $\mathbf{T}_i$-reflection (the universal T_i-space associated with X). A topological space X is said to be $T_{(i,j)}$-space if $\mathbf{T}_i(X)$ is a T_j-space (thus we have three new types of separation axioms; namely, $T_{(0,1)}$, $T_{(0,2)}$ and $T_{(1,2)}$).

Definition 1.2. Let $\mathbf{C}$ be a category and $\mathbf{F}$, $\mathbf{G}$ two (covariant) functors from $\mathbf{C}$ to itself.

(1) An object X of $\mathbf{C}$ is said to be a $T_{(\mathbf{F},\mathbf{G})}$-object if $\mathbf{G}(\mathbf{F}(X))$ is isomorphic with $\mathbf{F}(X)$.
(2) Let P be a topological property on the objects of $\mathbf{C}$. An object X of $\mathbf{C}$ is said to be a $T_{(\mathbf{F},P)}$-object if $\mathbf{F}(X)$ satisfies the property P.

One year later, H-P. Künzi and T. A. Richmond generalized the study of [4] using the T_i-ordered reflections ($i \in \{0, 1, 2\}$) of a partially ordered topological space $(X, \tau, \leq)$ and characterized ordered topological spaces whose T_0-ordered reflection is T_1-ordered (see [28]).

On the other hand, recall that a subset A of a topological space X is *locally closed* if A is open in its closure in X, or equivalently is the intersection of an open subset and a closed subset of X. The study of locally closed sets deals to important results in topology. An investigation is made in certain aspects of the most discrete case, where every subset is locally closed.

Definition 1.3 ([8, Definition 1.1])**.** A space X is called *submaximal* if every subset of X is locally closed.

One of the reasons to consider submaximal spaces is provided by the theory of *maximal* spaces. (A topological space X is said to be maximal if and only if for any point $x \in X$, $\{x\}$ is not open).

In [5], the authors give some characterizations of submaximal spaces.

Theorem 1.4. [5, Theorem 3.1] *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is submaximal;*
(ii) *$\overline{S} \setminus S$ is closed, for each $S \subseteq X$;*
(iii) *$\overline{S} \setminus S$ is closed and discrete, for each $S \subseteq X$.*

Furthermore, let a *door* space be a space in which every subset is either open or closed. Clearly, every door space is submaximal.

Recently, some authors (see [2]) have been interested in topological spaces X that have a compactification noted $\mathbf{K}(X)$ which is a door space (resp., submaximal space).

In this paper we mainly expose results concerning the following question: how can we characterize topological spaces X such that $\mathbf{F}(X)$ is a submaximal space (resp., door space) where $\mathbf{F}$ is a covariant functor from the category **Top** to itself? Recall the standard notion of reflective subcategory $\boldsymbol{\mathcal{A}}$ of $\boldsymbol{\mathcal{B}}$ that is, a full subcategory such that the embedding $\boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{B}}$ has a left adjoint $\mathbf{F} : \boldsymbol{\mathcal{B}} \longrightarrow \boldsymbol{\mathcal{A}}$ (called reflection). Further, recall that for all $i = 0, 1, 2, 3, 3.\frac{1}{2}$ the subcategory $\mathbf{Top}_i$ of T_i-spaces is reflective in **Top**, the category of all topological spaces.

The Tychonoff (resp., functionally Hausdorff) reflection of X will be denoted by $\boldsymbol{\rho}(X)$ (resp., $\mathbf{FH}(X)$).

Specifically, we are interested in $\mathbf{T}_0$, $\boldsymbol{\rho}$ and $\mathbf{FH}$ functors.

In the first section of this paper, we characterize $\mathbf{T}_0$-door spaces and $\mathbf{T}_0$-submaximal spaces.

The second section is devoted to the characterization of $\boldsymbol{\rho}$-door (resp., $\mathbf{FH}$-door) spaces and $\boldsymbol{\rho}$-submaximal (resp., $\mathbf{FH}$-submaximal) spaces.

In section three, given a flow (X, f) in **Set** we characterize maps f such that $(X, \mathcal{P}(f))$ is submaximal (resp., door).

2. $\mathbf{T}_0$-door and $\mathbf{T}_0$-submaximal spaces

First, let us recall the $\mathbf{T}_0$-reflection of a topological space. Let X be a topological space. We define the binary relation $\sim$ on X by $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. Then $\sim$ is an equivalence relation on X and the resulting quotient space $\mathbf{T}_0(X) := X/\sim$ is the $\mathbf{T}_0$-reflection of X.

The canonical surjection $\mu_X : X \longrightarrow \mathbf{T}_0(X)$ is a *quasihomeomorphism.* (A continuous map $q : X \longrightarrow Y$ is said to be a quasihomeomorphism if $U \longmapsto q^{-1}(U)$ (resp., $C \longmapsto q^{-1}(C)$) defines a bijection $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ (resp., $\mathcal{F}(Y) \longrightarrow \mathcal{F}(X)$), where $\mathcal{O}(X)$ (resp., $\mathcal{F}(X)$) is the collection of all open subsets (resp., closed subsets) of X [21]).

Let us give some straightforward remarks about quasihomeomorphisms.

Remarks 2.1.

(1) If $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ are continuous maps such that two of the three maps f, g, $g \circ f$ are a quasihomeomorphisms, then so is the third one.
(2) Let $q : X \longrightarrow Y$ be a quasihomeomorphism. Then, according to [4, Lemma 2.7], the following properties hold.
  (a) If X is a T_0-space, then q is one-one.
  (b) If Y is a T_D-space, then q is onto.
  (c) If Y is a T_D-space and X is a T_0-space, then q is a homeomorphism.
  (d) If X is sober and Y is a T_0-space, then q is a homeomorphism.

Now, we introduce some new notations.

Notations 2.2. Let X be a topological space, $a \in X$ and $A \subseteq X$. We denote by:

(1) $d_0(a) := \{x \in X : \overline{\{x\}} = \overline{\{a\}}\}$.
(2) $d_0(A) := \cup[d_0(a); a \in A]$.

The following remarks follow immediately.

Remarks 2.3. Let X be a topological space and let A be a subset of X. Then the following properties hold:

(i) $d_0(A) = \mu_X^{-1}(\mu_X(A))$.
(ii) $d_0(\varnothing) = \varnothing$, $d_0(X) = X$, $d_0(\cup[A_i : i \in I]) = \cup[d_0(A_i) : i \in I]$ and $d_0(d_0(A)) = d_0(A)$. Consequently, d_0 is a Kuratoweski closure.
(iii) $A \subseteq d_0(A) \subseteq \overline{A}$ and consequently $\overline{d_0(A)} = \overline{A}$.
(iv) In particular if A is open (resp., closed), then $d_0(A) = A$. Indeed, μ_X is an onto quasihomeomorphism and thus by [15, Lemma 1.1], it is an open map (resp., a closed map) and $\mu_X^{-1}(\mu_X(A)) = A$ for any open (resp., closed) subset A of X.

Thus, a characterization of T_0-spaces and symmetric spaces in term of d_0 will be useful.

Proposition 2.4. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a T_0-space;*
(ii) *For any subset A of X, $d_0(A) = A$;*
(iii) *For any $a \in X$, $d_0(a) = \{a\}$.*

Proof. $(i) \Longrightarrow (ii)$ If X is a T_0-space, then μ_X is an homeomorphism and thus $d_0(A) = \mu_X^{-1}(\mu_X(A)) = A$.

$(ii) \Longrightarrow (iii)$ Straightforward.

$(iii) \Longrightarrow (i)$ $\overline{\{x\}} = \overline{\{y\}}$ implies that $x \in d_0(y) = \{y\}$ and thus $x = y$. □

Recall that a *symmetric* space is a space in which for any $x, y \in X$, we have $x \in \overline{\{y\}} \Longrightarrow y \in \overline{\{x\}}$. This notion is introduced by N. A. Shanin in [30] and rediscovered by A. S. Davis in [12]. It is also studied by K. Belaid, O. Echi and S. Lazaar in [4] and called $T_{(0,1)}$-spaces.

Proposition 2.5. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $T_{(0,1)}$-space;*
(ii) *For any $a \in X$, $d_0(a) = \overline{\{a\}}$.*

Proof. $(i) \Longrightarrow (ii)$ Let X be a $T_{(0,1)}$-space and $a \in X$, then:

$d_0(a) = \{x \in X : \overline{\{x\}} = \overline{\{a\}}\} = \{x \in X : x \in \overline{\{a\}}\} = \overline{\{a\}}$.

$(ii) \Longrightarrow (i)$ $x \in \overline{\{y\}}$ implies that $x \in d_0(y)$ and thus $\overline{\{x\}} = \overline{\{y\}}$, therefore $y \in \overline{\{x\}}$. □

Proposition 2.6. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is an Alexandroff $T_{(0,1)}$-space;*
(ii) *For any subset A of X, $d_0(A) = \overline{A}$.*

Proof. $(i) \Longrightarrow (ii)$ For any subset A of X, $d_0(A) = \cup[d_0(a) : a \in A]$. Now, since X is $T_{(0,1)}$, we get $d_0(A) = \cup[\overline{\{a\}} : a \in A]$ which is closed because X is an Alexandroff space. Finally, Remarks 2.3 (iii) does the job.

$(ii) \Longrightarrow (i)$ Clearly, X is a $T_{(0,1)}$-space.

Now, let $\{F_i : i \in I\}$ be a family of closed subsets of X, then:

$$\cup[F_i : i \in I] = \cup[\overline{F_i} : i \in I] = \cup[d_0(F_i) : i \in I] = d_0(\cup[F_i : i \in I]) = \overline{\cup[F_i : i \in I]}.$$

□

Example 2.7. Let X be an infinite set equipped with the co-finite topology. Clearly X is a T_1-space and thus a $T_{(0,1)}$-space. Then for any $a \in X$ we have $d_0(a) = \{a\}$.
Now, let $m \in X$ and $A = X \setminus \{m\}$. It is easily seen that $d_0(A) = A \neq \overline{A} = X$. Therefore a $T_{(0,1)}$-space need not to be an Alexandroff $T_{(0,1)}$-space.

Now, we introduce the following definition.

Definition 2.8. Let X be a topological space. X is called a $\boldsymbol{T}_0$-*door* space if its $\mathbf{T}_0$-reflection is a door space.

Remark 2.9. Since every door space is a T_0-space, then every door space is a $\mathbf{T}_0$-door space. The converse does not hold.

Indeed, given a set $X = \{0, 1\}$ such that $\overline{\{0\}} = \overline{\{1\}}$, we can easily see that $\mathbf{T}_0(X)$ is a one point space and thus a door space.

However $\{0\}$ is not open and not closed in X.

The following result gives answer about the question mentioned in the introduction concerning door spaces.

Theorem 2.10. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $\boldsymbol{T}_0$-door space;*
(ii) *For any subset A of X, $d_0(A)$ is either open or closed.*

Proof. $(i) \Longrightarrow (ii)$ Let A be a subset of X. Since X is a $\mathbf{T}_0$-door space, then $\mu_X(A)$ is either open or closed and consequently $d_0(A) = \mu_X^{-1}(\mu_X(A))$ is either open or closed.

$(ii) \Longrightarrow (i)$ Let $\mu_X(A)$ be a subset of $\mathbf{T}_0(X)$, where $A \subseteq X$. Then, $d_0(A) = \mu_X^{-1}(\mu_X(A))$ is either open or closed in X and thus $\mu_X(A)$ is either open or closed in $\mathbf{T}_0(X)$. Therefore, X is a $\mathbf{T}_0$-door space. □

The following result is an immediate consequence of Theorem 2.10 and Proposition 2.6.

Corollary 2.11. *Every $T_{(0,1)}$ Alexandroff space is a $\boldsymbol{T}_0$-door space.*

Examples 2.12. (1) A $\mathbf{T}_0$-door space need not to be a $T_{(0,1)}$ space.

For this let X be a Sierpinski pace $\{0, 1\}$. Then X is a T_0-space which is not T_1 and thus X is not $T_{(0,1)}$.

Clearly, $\mathbf{T}_0(X) = X$ is a door space.
(2) A $\mathbf{T}_0$-door space need not to be an Alexandroff space.
It is sufficient to choose a door space which is not Alexandroff.
For this, let X be an infinite set and $m \in X$. Equip X with the topology whose closed sets are all subsets of X containing m or all finite subsets of X. Hence, if we consider a subset A of X, then two cases arise. If $m \in A$, then A is closed. If not $m \in X \backslash A$ and consequently A is open, so X is a door space. However $X \backslash \{m\} = \cup[\{x\} : x \neq m]$ is a union of closed subsets of X which is not closed.

Now, the same study will be devoted to submaximal spaces. That's why we introduce the following definition.

Definition 2.13. Let X be a topological space. X is called a *$\boldsymbol{T}_0$-submaximal* space if its $\mathbf{T}_0$-reflection is a submaximal space.

Remark 2.14. Since every submaximal space is T_0, then every submaximal space is a $\mathbf{T}_0$-submaximal space. The converse does not hold. The example in Remark 2.9 does the job.

Theorem 2.15. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $\boldsymbol{T}_0$-submaximal space;*
(ii) *For any subset A of X, we have: A dense $\Longrightarrow d_0(A)$ open;*
(iii) *For any subset A of X, $\overline{d_0(A)} \backslash d_0(A)$ is a closed set of X.*

Proof. We need a Lemma:

Lemma 2.16. *Let $f : X \longrightarrow Y$ be a quasihomeomorphism. Then the following statements are equivalent:*

(i) *f is onto;*
(ii) *For any subset A of Y, we have $f^{-1}(\overline{A}) = \overline{f^{-1}(A)}$.*

Proof of the Lemma:
$(i) \Longrightarrow (ii)$ Clearly, $f^{-1}(\overline{A}) \supseteq \overline{f^{-1}(A)}$ for any subset A of Y.
Conversely, let x be in $f^{-1}(\overline{A})$ and U be an open subset of X containing x. Since f is a quasihomeomorphism, then there exists an open subset V of Y such that $U = f^{-1}(V)$. Now, $f(x) \in V \cap \overline{A}$ and consequently $V \cap A \neq \varnothing$. Consider a point $y = f(y')$ in $V \cap A$ (since f is onto). Clearly, $y' \in f^{-1}(V) \cap f^{-1}(A) = U \cap f^{-1}(A)$ and thus $U \cap f^{-1}(A)$ is not empty which implies that $x \in \overline{f^{-1}(A)}$.
$(ii) \Longrightarrow (i)$ Let y be in Y, by (ii), $f^{-1}(\overline{\{y\}}) = \overline{f^{-1}(\{y\})}$. Since f is a quasihomeomorphism, $f^{-1}(\overline{\{y\}})$ is not empty and consequently $f^{-1}(\{y\})$ is not empty too, which implies that f is onto.
Proof of the Theorem:
$(i) \Longrightarrow (ii)$ Let $A \subseteq X$ such that $\overline{A} = X$. By Remarks 2.3 (iii), $\overline{d_0(A)} = X$ and thus $\overline{\mu_X^{-1}(\mu_X(A))} = X$.

Now, according to Lemma 2.16, $\mu_X^{-1}(\overline{\mu_X(A)}) = X$, which implies that $\overline{\mu_X(A)} = \mathbf{T}_0(X)$. Since $\mathbf{T}_0(X)$ is a submaximal space, we get $\mu_X(A)$ open and consequently $d_0(A) = \mu_X^{-1}(\mu_X(A))$ is an open set of X.

$(i) \Longrightarrow (iii)$ Let A be a subset of X, then

$$\begin{aligned}\overline{d_0(A)}\backslash d_0(A) &= \overline{\mu_X^{-1}(\mu_X(A))}\backslash \mu_X^{-1}(\mu_X(A)) \\ &= \mu_X^{-1}(\overline{\mu_X(A)})\backslash \mu_X^{-1}(\mu_X(A)) \\ &= \mu_X^{-1}(\overline{\mu_X(A)}\backslash\ \mu_X(A)).\end{aligned}$$

Now, since X is a $\mathbf{T}_0$-submaximal space, then $\overline{\mu_X(A)}\backslash\ \mu_X(A)$ is a closed subset of $\mathbf{T}_0(X)$ and thus $\mu_X^{-1}(\overline{\mu_X(A)}\backslash\ \mu_X(A))$ is a closed subset of X. Therefore, $\overline{d_0(A)}\backslash d_0(A)$ is closed.

$(ii) \Longrightarrow (i)$ Let $A \subseteq X$ such that $\mu_X(A)$ is a dense subset of $\mathbf{T}_0(X)$, that is, $\overline{\mu_X(A)} = \mathbf{T}_0(X)$, then $\mu_X^{-1}(\overline{\mu_X(A)}) = X$. Now, according to Lemma 2.16, $\overline{\mu_X^{-1}(\mu_X(A))} = X$ which means that $\overline{d_0(A)} = X$ and thus $\overline{A} = X$. By (ii), $d_0(A)$ is open and finally $\mu_X(A)$ is open.

$(iii) \Longrightarrow (i)$ Let A be a subset of X such that $\overline{d_0(A)}\backslash d_0(A)$ is closed, then $\mu_X^{-1}(\overline{\mu_X(A)}\backslash\ \mu_X(A))$ is a closed subset of X and thus $\overline{\mu_X(A)}\backslash\ \mu_X(A)$ is a closed subset of $\mathbf{T}_0(X)$. Therefore, X is a $\mathbf{T}_0$-submaximal space. □

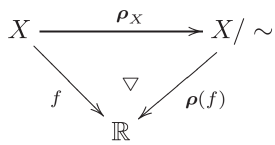
3. $\boldsymbol{\rho}$-door and $\boldsymbol{\rho}$-submaximal spaces

Let X be a topological space, F a subset of X and $x \in X$. x and F are said to be *completely separated* if there exists a continuous map $f : X \longrightarrow \mathbb{R}$ such that $f(x) = 0$ and $f(F) = \{1\}$. Now, two distinct points x and y in X are called completely separated if x and $\{y\}$ are completely separated.

A space X is said to be *completely regular* if every closed subset F of X is completely separated from any point x not in F. Recall that a topological space X is called a T_1*-space* if each singleton of X is closed. A completely regular T_1-space is called a *Tychonoff space* [33].

A *functionally Hausdorff* space is a topological space in which any two distinct points of this space are completely separated. Remark here that a Tychonoff space is a functionally Hausdorff space and consequently a Hausdorff space (T_2-space).

Now, for a given topological space X, we define the equivalence relation $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in \mathbf{C}(X)$ (where $\mathbf{C}(X)$ designates the family of all continuous maps from X to $\mathbb{R}$). Let us denote by $X/\sim$ the set of equivalence classes and let $\boldsymbol{\rho}_X : X \longrightarrow X/\sim$ be the canonical surjection map assigning to each point of X its equivalence class. Since every f in $\mathbf{C}(X)$ is constant on each equivalence class, we can define $\boldsymbol{\rho}(f) : X/\sim \longrightarrow \mathbb{R}$ by $\boldsymbol{\rho}(f)(\boldsymbol{\rho}_X(x)) = f(x)$. One may illustrate this situation by the following commutative diagram.



Now, equip $X/\sim$ with the topology whose closed sets are of the form $\cap[\boldsymbol{\rho}(f_\alpha)^{-1}(F_\alpha) : \alpha \in I]$, where $f_\alpha : X \longrightarrow \mathbb{R}$ (resp., F_α) is a continuous map (resp., a closed subset of $\mathbb{R}$). It is well known that, with this topology, $X/\sim$ is a Tychonoff space (see for instance [36]) and its denoted by $\boldsymbol{\rho}(X)$.

The construction of $\boldsymbol{\rho}(X)$ satisfies some categorical properties:

For each Tychonoff space Y and each continuous map $f : X \longrightarrow Y$, there exists a unique continuous map $\widetilde{f} : \boldsymbol{\rho}(X) \longrightarrow Y$ such that $\widetilde{f} \circ \boldsymbol{\rho}_X = f$. We will say that $\boldsymbol{\rho}(X)$ is the *$\boldsymbol{\rho}$-reflection* (or Tychonoff-reflection) of X.

From the above properties, it is clear that $\boldsymbol{\rho}$ is a covariant functor from the category of topological spaces **Top** into the full subcategory **Tych** of **Top** whose objects are Tychonoff spaces.

On the other hand, the quotient space $X/\sim$ which is denoted by $\mathbf{FH}(X)$ is a functionally Hausdorff space.

The construction $\mathbf{FH}(X)$ satisfies some categorical properties:

For each functionally Hausdorff space Y and each continuous map $f : X \longrightarrow Y$, there exists a unique continuous map $\widetilde{f} : \mathbf{FH}(X) \longrightarrow Y$ such that $\widetilde{f} \circ \boldsymbol{\rho}_X = f$. We will say that $\mathbf{FH}(X)$ is the *functionally Hausdorff-reflection* of X (or the **FH**-reflection of X).

Consequently, it is clear that **FH** is a covariant functor from the category of topological spaces **Top** into the full subcategory **FunHaus** of **Top** whose objects are functionally Hausdorff spaces.

Notations 3.1. Let X be a topological space, $a \in X$ and A a subset of X. We denote by:

(1) $d_{\boldsymbol{\rho}}(a) := \cap[f^{-1}(f(\{a\})) : f \in \mathbf{C}(X)]$.
(2) $d_{\boldsymbol{\rho}}(A) := \cup[d_{\boldsymbol{\rho}}(a) : a \in A]$.

The following results are immediate.

Proposition 3.2. *Let X be a topological space, $a \in X$ and A a subset of X. Then:*

(1) $d_{\boldsymbol{\rho}}(A) = \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A))$.
(2) *$d_{\boldsymbol{\rho}}(a)$ is a closed subset of X.*
(3) $A \subseteq d_{\boldsymbol{\rho}}(A) \subseteq \cap[f^{-1}(f(A)) : f \in \mathbf{C}(X)]$.
(4) $\forall f \in \mathbf{C}(X)$, $f(A) = f(d_{\boldsymbol{\rho}}(A))$.

Now, we give a characterization of functionally Hausdorff spaces in term of $d_{\boldsymbol{\rho}}$.

Proposition 3.3. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a functionally Hausdorff space;*
(ii) *For any subset A of X, $d_{\boldsymbol{\rho}}(A) = A$;*
(iii) *For any $a \in X$, $d_{\boldsymbol{\rho}}(a) = \{a\}$.*

Proof. $(i) \Longrightarrow (ii)$ If X is a functionally Hausdorff space, then $\mathbf{FH}(X) = X$ and μ_X is equal to 1_X and thus $d_{\boldsymbol{\rho}}(A) = A$.

$(ii) \Longrightarrow (iii)$ Straightforward.

$(iii) \Longrightarrow (i)$ First, remark that $d_{\boldsymbol{\rho}}(a) = \{a\}$ means that for any $x \in X$ such that $x \neq a$, there exists a continuous map $f : X \longrightarrow \mathbb{R}$ such that $f(x) \neq f(a)$ and thus X is a functionally Hausdorff space. □

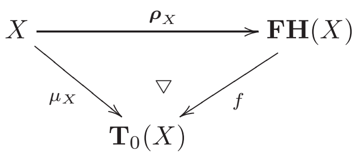
Using Defintion 1.2 for the functor $\mathbf{FH}$, one may define an other separation axiom: A space X is called $T_{(0,\mathbf{FH})}$ if its $\mathbf{T}_0$-reflection is functionally Hausdorff.

The following result characterize when $\boldsymbol{\rho}_X : X \longrightarrow \mathbf{FH}(X)$ is a quasihomeomorphism.

Proposition 3.4. *Let X be a topological space. Then the following statements are equivalent:*

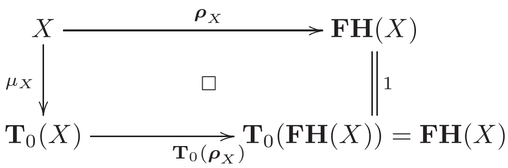
(a) *X is a $T_{(0,\boldsymbol{FH})}$-space;*
(b) *The canonical surjection $\boldsymbol{\rho}_X : X \longrightarrow \boldsymbol{FH}(X)$ is a quasihomeomorphism.*

Proof. $(a) \Longrightarrow (b)$ Since X is a $T_{(0,\mathbf{FH})}$-space, then $\mathbf{T}_0(X)$ is a functionally Hausdorff space and consequently there exists a unique continuous map $f : \mathbf{FH}(X) \longrightarrow \mathbf{T}_0(X)$ making commutative the following diagram



That is $f \circ \boldsymbol{\rho}_X = \mu_X$. On the other hand, since $\mathbf{FH}(X)$ is a T_0-space, there is a unique continuous map $g : \mathbf{T}_0(X) \longrightarrow \mathbf{FH}(X)$ such that $g \circ \mu_X = \boldsymbol{\rho}_X$. Now, combining the previous equalities we get easily $f \circ g = 1_{\mathbf{T}_0(X)}$ and $g \circ f = 1_{\mathbf{FH}(X)}$ which means that f and g are homeomorphisms and finally $\boldsymbol{\rho}_X$ is a quasihomeomorphism.

$(b) \Longrightarrow (a)$ Consider the following commutative diagram



Clearly, $\mathbf{T}_0(\boldsymbol{\rho}_X)$ is a quasihomeomorphism between a T_0-space and a functionally Hausdorff space. Now, since $\mathbf{FH}(X)$ is a T_D-space, then according to Remarks 2.1 (2.c), $\mathbf{T}_0(\boldsymbol{\rho}_X)$ is a homeomorphism which implies that $\mathbf{T}_0(X)$ is a functionally Hausdorff space. □

Now, we give a characterization of $T_{(0,\mathbf{FH})}$-spaces in term of $d_{\boldsymbol{\rho}}$.

Proposition 3.5. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $T_{(0,\boldsymbol{FH})}$-space;*
(ii) *For any $a \in X$, $d_{\boldsymbol{\rho}}(a) = d_{\boldsymbol{\rho}}(\overline{\{a\}}) = \overline{\{a\}}$.*

Proof. $(i) \Longrightarrow (ii)$ Clearly, $d_{\boldsymbol{\rho}}(a)$ is a closed subset of X containing a and thus $\overline{\{a\}} \subset d_{\boldsymbol{\rho}}(a)$.

Conversely, let $x \in d_{\boldsymbol{\rho}}(a)$, then $f(x) = f(a)$ for any $f \in \mathbf{C}(X)$. Now, suppose that $\mu_X(x) \neq \mu_X(a)$, then since $\mathbf{T}_0(X)$ is a functionally Hausdorff space, there exists a continuous map g from $\mathbf{T}_0(X)$ to $\mathbb{R}$ satisfying $g(\mu_X(x)) \neq g(\mu_X(a))$ and thus $g \circ \mu_X$ is a continuous map of $\mathbf{C}(X)$ separating x and y, contradiction. Finally, $\mu_X(x) = \mu_X(a)$, that is, $\overline{\{x\}} = \overline{\{a\}}$ and consequently $x \in \overline{\{a\}}$.

On the other hand, since X is a $T_{(0,\mathbf{FH})}$-space, then by Proposition 3.4 $\boldsymbol{\rho}_X : X \longrightarrow \mathbf{FH}(X)$ is an onto quasihomeomorphism and thus by [15, Lemma 1.1] $d_{\boldsymbol{\rho}}(\overline{\{a\}}) = \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(\overline{\{a\}})) = \overline{\{a\}}$.

$(ii) \Longrightarrow (i)$ Let $\mu_X(x)$ and $\mu_X(a)$ be two distinct points in $\mathbf{T}_0(X)$, that is, $\overline{\{a\}} \neq \overline{\{x\}}$. Then $x \notin \overline{\{a\}}$ or $a \notin \overline{\{x\}}$ which means that $x \notin d_{\boldsymbol{\rho}}(a)$ or $a \notin d_{\boldsymbol{\rho}}(x)$ and consequently there exists a continuous map f from X to $\mathbb{R}$ separating a and x. Now, by universality of $\mathbf{T}_0$, let $\widetilde{f}$ be the unique continuous map from $\mathbf{T}_0(X)$ to $\mathbb{R}$ such that $\widetilde{f} \circ \mu_X = f$. Clearly, $\widetilde{f}$ is a continuous map separating $\mu_X(x)$ and $\mu_X(a)$. □

Proposition 3.6. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is an Alexandroff $T_{(0,\boldsymbol{FH})}$-space;*
(ii) *For any subset A of X, $d_{\boldsymbol{\rho}}(A) = \overline{A}$.*

Proof. The same proof as in Proposition 2.6. □

Example 3.7. Let $\mathbb{R}$ be the real line equipped with usual topology. Clearly $\mathbb{R}$ is a $T_{(0,FH)}$-space which is not an Alexandroff space. Hence, $d_\rho(a) = \overline{\{a\}} = \{a\}$ for any $a \in \mathbb{R}$ but $d_\rho(\mathbb{Q}) = \mathbb{Q} \neq \overline{\mathbb{Q}} = \mathbb{R}$, where $\mathbb{Q}$ is the set of rational numbers.

Let us introduce the following definition.

Definition 3.8. Let X be a topological space. X is called a $\boldsymbol{\rho}$-door (resp., **FH**-door) space if its $\boldsymbol{\rho}$-reflection (resp., **FH**-reflection) is a door space.

By the same way as in Theorem 2.10, the following result gives immediately.

Theorem 3.9. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is an $\boldsymbol{FH}$-door space;*
(ii) *For any subset A of X, $d_{\boldsymbol{\rho}}(A)$ is either open or closed.*

Before giving a characterization of $\boldsymbol{\rho}$-door spaces, let us recall an interesting result which characterizes Tychonoff spaces in term of *zero-sets* (resp., *cozero-sets*). Let X be a topological space and $A \subseteq X$. A is called a zero-set if there exists $f \in \mathbf{C}(X)$ such that $A = f^{-1}(\{0\})$. The complement of a zero-set is called a cozero-set.

Proposition 3.10 ([36, Proposition 1.7])**.** *A space is Tychonoff if and only if the family of zero-sets of the space is a base for the closed sets (or equivalently, the family of cozero-sets of the space is a base for the open sets).*

Let us state a useful remark.

Remark 3.11. A closed (resp., open) subset of $\boldsymbol{\rho}(X)$ is of the form $\cap[\boldsymbol{\rho}(f)^{-1}(\{0\}) : f \in H]$ (resp., $\cup[\boldsymbol{\rho}(f)^{-1}(\mathbb{R}^\star) : f \in H]$) , where H is a collection of continuous maps $f : X \longrightarrow \mathbb{R}$.

Indeed, $\boldsymbol{\rho}(X)$ is a Tychonoff space, then the collection $\{g^{-1}\{0\}; g : \boldsymbol{\rho}(X) \longrightarrow \mathbb{R}$ continuous$\}$ (resp., $\{g^{-1}(\mathbb{R}^\star); g : \boldsymbol{\rho}(X) \longrightarrow \mathbb{R}$ continuous$\}$) is a basis of closed (resp., open) subsets of $\boldsymbol{\rho}(X)$.

According to the universal property of $\boldsymbol{\rho}(X)$, each continuous map $g : \boldsymbol{\rho}(X) \longrightarrow \mathbb{R}$ may be written as $g = \boldsymbol{\rho}(f)$ with $f = g \circ \boldsymbol{\rho}_X$.

Theorem 3.12. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $\boldsymbol{\rho}$-door space;*
(ii) *For any subset A of X, $d_{\boldsymbol{\rho}}(A)$ is either an intersection of zero-sets or a union of cozero-sets of X.*

Proof. $(i) \Longrightarrow (ii)$ Let A be a subset of X. Since $\boldsymbol{\rho}(X)$ is a door space, then $\boldsymbol{\rho}_X(A) \subseteq \boldsymbol{\rho}(X)$ is either open or closed.

• If $\boldsymbol{\rho}_X(A)$ is closed, then it is equal to $\bigcap[\boldsymbol{\rho}(f_i)^{-1}(\{0\}) : \ i \in I]$ (where $\{f_i : \ i \in I\}$ is a family of continuous maps from X to $\mathbb{R}$) and consequently $\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A)) = \bigcap[\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}(f_i)^{-1}(\{0\})) : \ i \in I] = \bigcap[f_i^{-1}(\{0\}) : \ i \in I]$. Therefore, $d_{\boldsymbol{\rho}}(A)$ is an intersection of zero-sets of X.

• If $\boldsymbol{\rho}_X(A)$ is open, then it is equal to $\bigcup[\boldsymbol{\rho}(g_i)^{-1}(\mathbb{R}^\star) : \ i \in J]$ (where $\{g_i : i \in J\}$ is a family of continuous maps from X to $\mathbb{R}$) and thus $\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A)) = \bigcup[g_i^{-1}(\mathbb{R}^\star) : \ i \in J]$. Therefore, $d_{\boldsymbol{\rho}}(A)$ is a union of cozero-sets of X.

$(ii) \Longrightarrow (i)$ Conversely, let $\boldsymbol{\rho}_X(A) \subseteq \boldsymbol{\rho}(X)$, where A is a subset of X.

• If $d_{\boldsymbol{\rho}}(A) = \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A))$ is an intersection of zero-sets of X, then let $\{f_i : i \in I\}$ be a family of continuous maps from X to $\mathbb{R}$ such that $\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A)) = \bigcap[\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}(f_i)^{-1}(\{0\})) : \ i \in I]$. Now, since $\boldsymbol{\rho}_X$ is onto, then:

$$\begin{aligned}\boldsymbol{\rho}_X(A) &= \boldsymbol{\rho}_X(\bigcap[f_i^{-1}(\{0\}) :\ i\in I]) \\ &= \boldsymbol{\rho}_X(\bigcap[\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}(f_i)^{-1}(\{0\}) :\ i\in I]) \\ &= \boldsymbol{\rho}_X(\boldsymbol{\rho}_X^{-1}(\bigcap[(\boldsymbol{\rho}(f_i)^{-1}(\{0\}) :\ i\in I])) \\ &= \bigcap[\boldsymbol{\rho}(f_i)^{-1}(\{0\}) :\ i\in I]\end{aligned}$$

Consequently, $\boldsymbol{\rho}_X(A)$ is a closed subset of $\boldsymbol{\rho}_X$.

• If $d_{\boldsymbol{\rho}}(A) = \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A))$ is a union of cozero-sets of X, then let $\{g_i :\ i \in J\}$ be a family of continuous maps from X to $\mathbb{R}$ such that $\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A)) = \bigcup[\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}(g_i)^{-1}(\mathbb{R}^\star)) :\ i \in J]$.

It is clearly seen, by the same way as in the first case, that $\boldsymbol{\rho}_X(A) = \bigcup[\boldsymbol{\rho}(g_i)^{-1}(\mathbb{R}^\star) :\ i \in J]$ and thus $\boldsymbol{\rho}_X(A)$ is an open subset of $\boldsymbol{\rho}(X)$.

Finally, $\boldsymbol{\rho}_X(A)$ is either open or closed for every subset A of X which means that $\boldsymbol{\rho}(X)$ is a door space. □

Definition 3.13. Let X be a topological space. X is said to be a $\boldsymbol{\rho}$-submaximal (resp., **FH**-submaximal) space if its $\boldsymbol{\rho}$-reflection (resp., **FH**-reflection) is submaximal.

Now, in order to characterize $\boldsymbol{\rho}$-submaximal spaces and **FH**-submaximal spaces, we introduce the following definitions:

Definition 3.14. Let X be a topological space.

(1) A subset V of X is called a *functionally open* subset of X (for short F-open) if and only if $d_{\boldsymbol{\rho}}(V)$ is open in X.
(2) A subset V of X is called a *functionally dense* subset of X (for short F-dense) if and only if for any F-open subset W of X, $d_{\boldsymbol{\rho}}(V)$ meets $d_{\boldsymbol{\rho}}(W)$.
(3) A nonempty subset V of X is said to be a $\boldsymbol{\rho}$-dense if $g(V) \neq \{0\}$ for every nonzero continuous map g from X to $\mathbb{R}$.

Remarks 3.15. (1) V is an F-open subset of X if and only if $\boldsymbol{\rho}_X(V)$ is an open subset of $\mathbf{FH}(X)$.

(2) Clearly, a dense subset is a $\boldsymbol{\rho}$-dense subset. The converse does not hold. Indeed, let $X := \{0, 1\}$ be the Sierpinski space, then it is easily seen that $\boldsymbol{\rho}(X)$ is a one point space, that is, any continuous map f from X to $\mathbb{R}$ is constant. Hence, any nonempty subset A of X is $\boldsymbol{\rho}$-dense. Now, to conclude choose $A = \{1\}$.

(3) Every F-dense subset of X is $\boldsymbol{\rho}$-dense. Indeed, let U be an open subset of $\boldsymbol{\rho}(X)$ and A an F-dense subset of X, that is $\boldsymbol{\rho}_X(A)$ dense in $\mathbf{FH}(X)$. Since $U = \cup[\boldsymbol{\rho}(f)^{-1}(\mathbb{R}^*) : f \in H]$, where H is a collection of continuous maps $f : X \longrightarrow \mathbb{R}$ and $\boldsymbol{\rho}_X^{-1}(U) = \boldsymbol{\rho}_X^{-1}(\cup[\boldsymbol{\rho}(f)^{-1}(\mathbb{R}^*) : f \in H]) = \cup[\boldsymbol{\rho}_X^{-1}\left(\boldsymbol{\rho}(f)^{-1}(\mathbb{R}^*)\right) : f \in H] = \cup[f^{-1}(\mathbb{R}^*) : f \in H]$, then U is an open subset of $\mathbf{FH}(X)$. Thus $\boldsymbol{\rho}_X(A) \cap U \neq \varnothing$. Therefore A is $\boldsymbol{\rho}$-dense.

(4) An F-dense subset of X is not necessary dense. Indeed, let $X = \{0, 1\}$ be the Sierpinski space. Clearly $\{1\}$ is F-dense but not dense.

Proposition 3.16. *Let X be a topological space and A a subset of X. Then the following statements are equivalent:*

(i) *A is a $\boldsymbol{\rho}$-dense subset of X;*
(ii) *$\boldsymbol{\rho}_X(A)$ is a dense subset of $\boldsymbol{\rho}(X)$.*

Proof. $(i) \Longrightarrow (ii)$ Let A be a $\boldsymbol{\rho}$-dense subset of X. Then for any nonzero continuous map g from X to $\mathbb{R}$, we have $A \cap g^{-1}(\mathbb{R}^\star) \neq \varnothing$. So, let a be in A such that $g(a) \neq 0$, then $\boldsymbol{\rho}(g)(\boldsymbol{\rho}_X(a)) = g(a) \neq 0$ and thus $\boldsymbol{\rho}_X(A) \cap \boldsymbol{\rho}(g)^{-1}(\mathbb{R}^\star) \neq \varnothing$. Now, by Remark 3.11, $\boldsymbol{\rho}_X(A)$ meets every nonempty open subset of $\boldsymbol{\rho}(X)$.

$(ii) \Longrightarrow (i)$ Let A be a subset of X. Since $\boldsymbol{\rho}_X(A)$ is dense, then for any nonzero continuous map g from X to $\mathbb{R}$, we have $\boldsymbol{\rho}_X(A) \cap \boldsymbol{\rho}(g)^{-1}(\mathbb{R}^\star) \neq \varnothing$ which means that there exists $a \in A$ satisfying $\boldsymbol{\rho}(g)(\boldsymbol{\rho}_X(a)) \neq 0$ or equivalently $g(a) \neq 0$. Therefore, A is a $\boldsymbol{\rho}$-dense subset of X. □

We are now in a position to give the characterization of $\boldsymbol{\rho}$-submaximal spaces.

Theorem 3.17. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $\boldsymbol{\rho}$-submaximal space;*
(ii) *For any subset A of X, we have: A $\boldsymbol{\rho}$-dense $\Longrightarrow d_{\boldsymbol{\rho}}(A)$ is a union of cozero-sets of X.*

Proof. $(i) \Longrightarrow (ii)$ Let A be a $\boldsymbol{\rho}$-dense subset of X. According to Proposition 3.16, $\boldsymbol{\rho}_X(A)$ is a dense subset of $\boldsymbol{\rho}(X)$. Since X is a $\boldsymbol{\rho}$-submaximal space, then $\boldsymbol{\rho}_X(A)$ is an open subset of $\boldsymbol{\rho}(X)$ and thus $\boldsymbol{\rho}_X(A) = \bigcup[\boldsymbol{\rho}(f)^{-1}(\mathbb{R}^\star):\ f \in H]$ (where H is a subfamily of $\mathbf{C}(X)$). So that $\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A)) = \bigcup[\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}(f)^{-1}(\mathbb{R}^\star)):\ f \in H]$. Therefore, $d_{\boldsymbol{\rho}}(A) = \bigcup[f^{-1}(\mathbb{R}^\star):\ f \in H]$ is a union of cozero-sets of X.

$(ii) \Longrightarrow (i)$ Conversely, let A be a subset of X such that $\overline{\boldsymbol{\rho}_X(A)} = \boldsymbol{\rho}(X)$. Then, by Proposition 3.16, A is a $\boldsymbol{\rho}$-dense subset of X and consequently $d_{\boldsymbol{\rho}}(A)$ is a union of cozero-sets of X. Hence there exists a subfamily $\{f_i : i \in I\}$ of $\mathbf{C}(X)$ satisfying $\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A)) = \bigcup[f_i^{-1}(\mathbb{R}^\star):\ i \in I]$. Then:

$$\begin{aligned}
\boldsymbol{\rho}_X(A) &= \boldsymbol{\rho}_X(\bigcup[f_i^{-1}(\mathbb{R}^\star):\ i \in I]) \\
&= \boldsymbol{\rho}_X(\bigcup[\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}(f_i)^{-1}(\mathbb{R}^\star)):\ i \in I]) \\
&= \boldsymbol{\rho}_X(\boldsymbol{\rho}_X^{-1}(\bigcup[\boldsymbol{\rho}(f_i)^{-1}(\mathbb{R}^\star):\ i \in I])) \\
&= \bigcup[\boldsymbol{\rho}(f_i)^{-1}(\mathbb{R}^\star):\ i \in I]
\end{aligned}$$

Finally, $\boldsymbol{\rho}_X(A)$ is an open subset of $\boldsymbol{\rho}(X)$. □

Theorem 3.18. *Let X be a topological space. Then the following statements are equivalent:*

*(i) X is **FH**-submaximal;*
(ii) For any F-dense subset A of X, $d_{\boldsymbol{\rho}}(A)$ is open.

Proof. $(i) \Longrightarrow (ii)$

Let A be an F-dense subset of X. First, let us show that $\boldsymbol{\rho}_X(A)$ is a dense subset of $\mathbf{FH}(X)$. Indeed, consider $\boldsymbol{\rho}_X(U)$ an open subset of $\mathbf{FH}(X)$. Then

$d_{\boldsymbol{\rho}}(U) = \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(U))$ is open in X and consequently U is an F-open subset of X. Since A is F-dense, $d_{\boldsymbol{\rho}}(U) \cap d_{\boldsymbol{\rho}}(A) \neq \varnothing$ and thus $\boldsymbol{\rho}_X(U) \cap \boldsymbol{\rho}_X(A) \neq \varnothing$.

Now, since X is **FH**-submaximal, then $\boldsymbol{\rho}_X(A)$ is open in $\mathbf{FH}(X)$ and $d_{\boldsymbol{\rho}}(A) = \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A))$ is open in X.

$(ii) \Longrightarrow (i)$

Let $\boldsymbol{\rho}_X(A)$ be a dense subset of $\mathbf{FH}(X)$, where A is a subset of X, and V an F-open subset of X, that is $d_{\boldsymbol{\rho}}(V)$ is open in X and thus $\boldsymbol{\rho}_X(V)$ is open in $\mathbf{FH}(X)$. Since $\boldsymbol{\rho}_X(A)$ is dense in $\mathbf{FH}(X)$, then $\boldsymbol{\rho}_X(V) \cap \boldsymbol{\rho}_X(A) \neq \varnothing$. Thus $\boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(V)) \cap \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A)) \neq \varnothing$. Hence $d_{\boldsymbol{\rho}}(V) \cap d_{\boldsymbol{\rho}}(A) \neq \varnothing$. Therefore, A is an F-dense subset of X.

Now, by (ii), $d_{\boldsymbol{\rho}}(A)$ is open in X and consequently $\boldsymbol{\rho}_X(A)$ is open in $\mathbf{FH}(X)$. Therefore, $\mathbf{FH}(X)$ is a submaximal space. □

4. Alexandroff Topology

According to Kennisson, a flow in a category $\mathbf{C}$ is a couple (X, f), where X is an object of $\mathbf{C}$ and $f : X \longrightarrow X$ is a morphism, called the iterator (see [25] and [26]).

Now, let (X, f) be a flow in the category **Set**. In [16], the author define the topology $\mathcal{P}(f)$ on X with closed sets are exactly those A which are f-invariant (i.e., $f(A) \subseteq A$). It is clearly seen that for any subset A of X, the topological closure $\overline{A}$ is exactly $\cup[f^n(A) : n \in \mathbb{N}]$. In particular for any point $x \in X$, $\overline{\{x\}} = \mathcal{O}_f(x) = \{f^n(x) : n \in \mathbb{N}\}$ called the orbit of x by f. One can see easily that the family $\{V_f(x) : x \in X\}$ is a basis of open sets of $\mathcal{P}(f)$, where $V_f(x) := \{y \in X : f^n(y) = x, \text{ for some } n \text{ in } \mathbb{N}\}$.

Clearly, $\mathcal{P}(f)$ is an Alexandroff topology on X.

Characterizing maps f such that $(X, \mathcal{P}(f))$ is submaximal, which is one of our main goals, is given by the following result.

Proposition 4.1. *Let (X, f) be a flow in* **Set**. *Then the following statements are equivalent:*

(i) *$(X, \mathcal{P}(f))$ is a submaximal space;*
(ii) *$f^2 = f$.*

Proof. $(i) \Longrightarrow (ii)$ let $x \in X$. Two cases arise.

• If $f(x) = x$, then $f^2(x) = f(x)$.

• If $f(x) \neq x$, then $x \in \{f(x)\}^c$ and thus $f(x) \in \overline{\{x\}} \subseteq \overline{\{f(x)\}^c}$, consequently $\overline{\{f(x)\}^c} = X$. Now, since $(X, \mathcal{P}(f))$ is a submaximal space, then $\{f(x)\}^c$ is open, equivalently $\{f(x)\}$ is closed and finally $f^2(x) = f(x)$.

$(ii) \Longrightarrow (i)$ Let A be a dense subset of X. Since $f^2 = f$, then any point in $f(X)$ is closed and thus, since the topology is principal, every subset of $f(X)$ is closed. In particular every subset of $f(A)$ is closed. On the other hand $\overline{A} = A \cup f(A) = X$, then A^c is closed ($A^c \subseteq f(A)$), so A is open. □

Example 4.2. Consider the map $f\colon \begin{array}{l}\mathbb{N} \longrightarrow \mathbb{N} \\ n \longmapsto n+1\end{array}$ where $\mathbb{N}$ is the set of all natural numbers including 0.
It is clearly seen that $f^2 \neq f$. Now, consider the topological space $(\mathbb{N}, \mathcal{P}(f))$ and set $A = 2\mathbb{N}$. A is a dense subset of $(\mathbb{N}, \mathcal{P}(f))$ which is not open since for each $n \in \mathbb{N} \setminus \{0\}$, we have $2n-1 \in V_f(2n)$.

Before giving a characterization of maps f such that $(X, \mathcal{P}(f))$ is door, let us recall that a point $x \in X$ is called a fixed point if $f(x) = x$ and we denote by $Fix(f)$ the family of all fixed points of X.

Proposition 4.3. *Let (X, f) be a flow in* **Set**. *Then the following statements are equivalent:*

(i) $(X, \mathcal{P}(f))$ *is a door space;*
(ii) $|f(Fix(f)^c)| \leq 1$.

Proof. $(i) \Longrightarrow (ii)$ Suppose that $|f(Fix(f)^c)| \geq 2$. Then there exist two distinct points x and y in $Fix(f)^c$ such that $f(x) \neq f(y)$. Set $A = \{x, f(y)\}$. Clearly A is neither closed ($f(x) \notin A$) nor open ($f(y) \notin A^c$).

$(ii) \Longrightarrow (i)$

• If $|f(Fix(f)^c)| = 0$, then $Fix(f) = X$ and thus $(X, \mathcal{P}(f))$ is the discrete topology which is door.

• If $|f(Fix(f)^c)| = 1$. Let x_0 such that $f(Fix(f)^c) = \{x_0\}$ and let us show that for any point x in X distinct from x_0, $\{x\}$ is open.

Indeed, set $A = f^{-1}(\{x\}) \setminus \{x\}$. Assume that $A \neq \varnothing$ and let $y \in X$ such that $y \neq x$ and $f(y) = x$. Then $y \in Fix(f)^c$ and thus $f(y) = x_0$, contradiction. Hence A is empty and consequently for any point $x \in X$ distinct from x_0, $V_f(x) = \{x\}$ is open. Now, consider a subset C of X, then it is open if $x_0 \notin C$ and it is closed when it contains x_0. Finally $(X, \mathcal{P}(f))$ is a door space. □

Example 4.4. Let $\mathbb{Z}$ be the set of all integers and $f\colon \begin{array}{l}\mathbb{Z} \longrightarrow \mathbb{Z} \\ n \longmapsto |n|\end{array}$ where $|n|$ denotes the absolute value of the integer n.
Then, we have $Fix(f) = \mathbb{N}$ and thus $|f(Fix(f)^c)| = |\mathbb{N} \setminus \{0\}| > 1$.
Now, consider the topological space $(\mathbb{Z}, \mathcal{P}(f))$ and set $A = \{-1, 2\}$. Clearly A is neither closed ($\overline{A} = \{-1, 1, 2\}$) nor open ($V_f(2) = \{2, -2\}$).

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